C.V.Raman Polytechnic, BBSR

# Lecture Notes

Subject-Strength of Materials (Th-2)


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CHAPTER 1.0

# SIMPLE STRESS AND STRAIN

## 1.1 - Types of Load

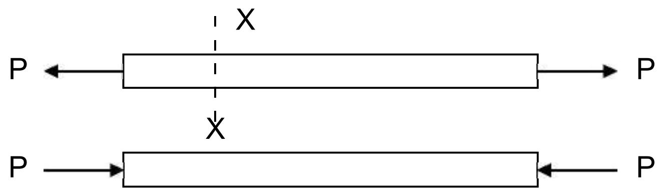
Load is an external force. Hydraulic force, steam pressure, tensile force, compressive force, shear force, spring force and different types of load. Again load may be classified as live load, dead load.

**Definition**

Strength of material is the study of the behaviour of structural and machine members under the action of external loads, taking into account the internal forces created and resulting deformation.

**Types of load**

The simplest type of load (P) is a direct pull or push, known technically as tension or compression.

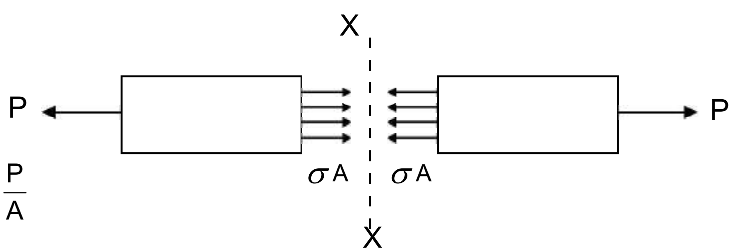



If a member is in motion the load may be caused partly by dynamic or inertia forces. For instance, the connecting Rod of a reciprocating engine, load on a fly wheel.

**STRESS**

**Definition**

The Force transmitted across any section, divided by the area of that section, is called intensity of stress or stress.




$$\sigma = \frac{P}{A}$$

Where

$\sigma$ - Stress

P - Load

A - Area

$\sigma$A - Internal forces of cohesion

**Direct stress (Tensile / compressive)**

Stresses which are normal to the plane on which they act are called direct stresses and either tensile or compressive.

Unit - N / m2

**STRAIN**

Stain is a measure of the measure of the deformation produced in the member by the load.

If a rod of length L is in tension and the elongation produced is L, then the direct

strain= $\frac{\text{Elongation}}{\text{Original length}}$ $\varepsilon = \frac{X}{L}$

Tensile strain will be positive compressive strain will be negative.

**Hooke's Law**

This states that strain is proportional to the stress producing it.

A material is said to be elastic if all the deformations are proportional to the load.

**Principle of superposition**

It states that the resultant strain will be the sum of the individual strains caused by each load acting separately.

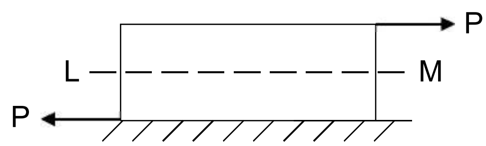
**Young's Modules**

Within the limits for which Hooke's law is obeyed, the ratio of the direct stress to the strain produced is called young's modules or the modules of Elasticity, i.e. E = E= $\frac{\sigma}{\varepsilon}$

For a bar of uniform cross-section A and length L this can be written as E= $\frac{PL}{AX}$ or $\frac{PL}{AE} = X$

**Tangential Stress**

If the applied load persists of two equal and opposite parallel forces not in the same line, then there is a tendency for one part of the body to slide over or shear from the other part across any section LM.



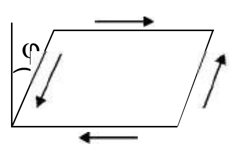

Area of gross section is parallel to load

$$\tau = \frac{P}{A}$$

Shear stress is tangential to the area over which it acts.

Every shear stress is accompanied by an equal complementary shear stress.

**Shear Strain**




The shear strain or slide is $\varphi$, and can be defined as the change in the right angle. It is measured in radians.

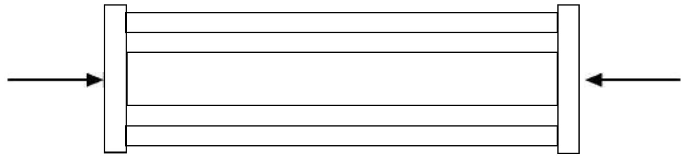
**Modules of rigidity**

For elastic material shear strain is proportional to the shear stress.

Ratio $\frac{\text{Shear Stress}}{\text{Shear Strain}} = \text{Modules of rigidity}$

Ratio $G = \frac{\tau}{\varphi}$ N/mm$^2$

## 1.2 Stresses in composite section



Any tensile or compressive member which consists of two or more bars or tubes in parallel, usually of different materials in called compound bars.

**Analysis**

A compound bar is made up of a rod of area A, and modules E1 and a tube of equal length of area A2 and modules E2. If a compressive load P is applied to the compound bar find how the load is shared. Since the road and tube are of the same initial length and must remain together then the strain in each part must be the same. The total load carried is P and let if be shared W1 and W2,

$\varepsilon_1 = \varepsilon_2$ ,L1=L2

compatibility equation : $\frac{W_1}{A_1 E_1} = \frac{W_1}{A_2 E_2}$

Equilibrium equation : $W_1 + W_2 = P$

Substituting, $W_2 = \frac{A_2 E_2}{A_1 E_1} \times W_1$

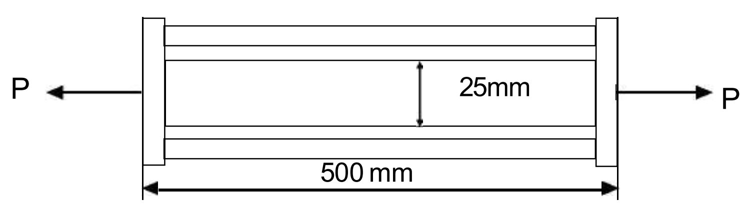
from (i) & (ii) given $W_1 \left( 1 + \frac{A_2 E_2}{A_1 E_1} \right) = P$ or

$$W_1 = \frac{P A_1 E_1}{A_1 E_1 + A_2 E_2}$$

Then $W_2 = \frac{P A_2 E_2}{A_1 E_1 + A_2 E_2}$

**Example**

A composite bar is made up of a brass rod of 25m diameter enclosed in a steel tube, being co-axial of 40mm external diameters and 30mm internal diameter as shown below. They are securely fixed at each end. If the stress in brass and steel are not to exceed 70MPa and 120 MPa respectively find the load (P) the composite bar can safely carry.




Also find the change in length, if the composite bar is 500mm long. Take E for steel Tube as 200 GPa and brass rod as 80 GPa respectively.

**Data Given**

Let steel tube denoted as 1 and brass rod denoted as 2

| | |
|---|---|
| d10= 40mm | E1 = 200GPa |
| d1i = 30mm | E2 = 80 GPa |
| d2 = 25mm | |
| $\sigma$ 1= 120 MPa | W1 - Load carried by tube |
| $\sigma$ 1= 70 MPa | W2 - Load carried by rod. |

From compatibility equation :

$$\frac{W_1}{A_1 E_1} = \frac{W_2}{A_2 E_2}$$

$$A_1 = \frac{\pi}{4}(d_{1_O}^2 - d_{1_i}^2) = \frac{\pi}{4}(40^2 - 30^2)$$

$$\Rightarrow A_1 = 500mm^2$$

$$\text{and } A_2 = \frac{\pi}{4}\, 25^2 = 491mm^2$$

Now putting in equation – (1)

$$\Rightarrow W_1 = W_2 \text{ x } \frac{550 \text{ x } 200}{491 \text{ x } 80}$$

$$\Rightarrow W_1 = 2.8 W_2$$

$$W_1 = \sigma_1 A_1 = 120 \text{ x } 550 = 66000N$$

$$\text{and } W_2 = \frac{W_1}{2.8} = \frac{66000}{2.8} = 2357\,N$$

From equlibrium equation

$$\Rightarrow P = W_1 + W_2$$

$$= 66000 + 2357 = 89.57\,KW \text{ (Ans)}$$

Change in length

$$\delta\ell_1 = \delta\ell_2 = \frac{W_1 \ell_1}{A_1 E_1} = \frac{66000 \text{ x } 500}{550 \text{ x } 200 \text{ x } 10^3} = 0.3mm$$

**Poisson's Ratio**

The ratio between lateral strain to the liner strain is a constant which is known as poisson's ratio.

The symbol is '$\mu$'.

**Bulk Modules**

When a body is subjected to three mutually perpendicular stresses of equal intensity the ratio of direct stress to the corresponding volumetric strain is known as bulk modules.

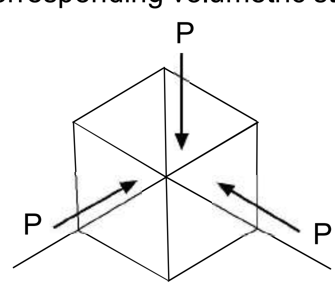



Fig. $K = \frac{-P}{\delta V / V}$

P - hydrostatic pressure

(-) - negative sign taking account of the reduction in volume.

**Relation between K and E**

The above figure represents a unit cube of material under the action of a uniform pressure P. It is clear that the principle stresses are -P, -P and -P and the linear strain in each direction is

$$-P/E + \mu P/E + \mu P/E = \frac{-P}{A}(1-2\mu)$$

But we know

Volumetric strain = sum of linear strain

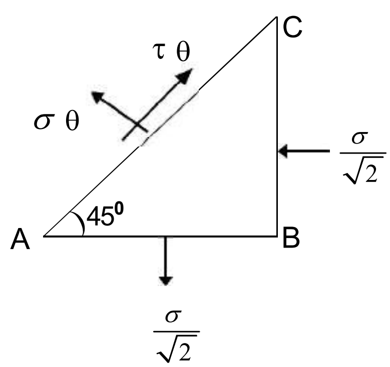
$$\text{By defination } K = \frac{-P}{\delta V / V}$$

$$\text{or } K = \frac{-P}{\frac{-3P}{E}(1-2\mu)}$$

$$\text{or } K = \frac{E}{3(1-2\mu)}$$

or $\mathbf{E = 3K(1-2\mu)}$

**Relation between E and G**




It is necessary first of all to establish the relation between a pure shear and pure normal stress system at a point in an elastic material.

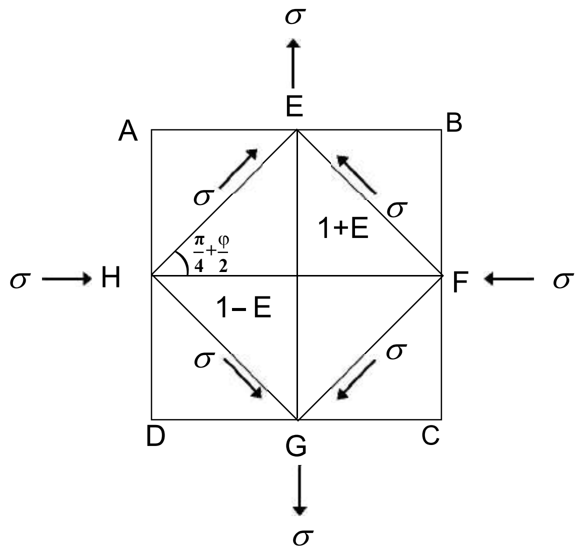
In the above figure the applied stresses are $\sigma$ tensile on AB and $\sigma$ compressive on BC. If the stress components on a plane AC at $45^0$ to AB are $\sigma_\theta$ and $\tau_\theta$ Then the forces acting are as shown taking the area on AC as units.

Resolving along and at right angle to AC

$$\tau_\theta = \frac{\sigma}{\sqrt{2}}\operatorname{Sin}45 + \frac{\sigma}{\sqrt{2}}\operatorname{Cos}45 = \sigma$$

$$\text{and } \sigma_\theta = \frac{\sigma}{\sqrt{2}}\operatorname{Cos}45 - \frac{\sigma}{\sqrt{2}}\operatorname{Sin}45 = 0$$

So a pure shear on planes at $45^0$ to AB and BC.




This figure shows a square element ABCD, sides of unstrained length 2 units under the action of equal normal stresses, $\sigma$ tension & compression. then it has been shown that the element EFGH is in pure shear of equal magnitude $\sigma$.

Liner strain in direction $EG = \frac{\sigma}{E} + \frac{\mu\sigma}{E}$

Say $\varepsilon = \frac{\mu}{E}(1+\mu)$

Liner strain in direction $HF = -\frac{\sigma}{E} - \frac{\mu\sigma}{E} = -\varepsilon$

Hence the strained lengths of EO and HO are $l + \varepsilon$ and $l - \varepsilon$ respectively.

The shear strain $\varphi = \frac{\sigma}{G}$

on one element EFGH and the angle EHG will increase by to $\frac{\pi}{4}+\varphi$ and angle EHO = $\frac{\pi}{4}+\frac{\varphi}{2}$

Considering the triangle tan EHO = $\frac{E0}{H0}$

$$\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) = \frac{1+\varepsilon}{1-\varepsilon}$$

$$\frac{1+\varepsilon}{1-\varepsilon} = \tan\frac{\tan\frac{\pi}{4}+\tan\frac{\varphi}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{\varphi}{2}}$$

$$= \frac{1+\frac{\varphi}{2}}{1-\frac{\varphi}{2}}$$

$$\varepsilon = \frac{\varphi}{2}$$

$$(1+\mu)\frac{\sigma}{\varepsilon} = \frac{\sigma}{2G}$$

then rearranging $E = 2G(1+\mu)$

by removing $\mu$, $E = \frac{9GK}{G+3K}$

## 1.3 Temperature stress

Determination of temperature stress in composite bar (single core).

**Temperature stresses in Composite Bar**

If a compound bar made up of several materials is subjected to a change in temperature there will be tendency for the components parts to expand different amounts due to the unequal co-efficient of thermal expansion. If the parts are constrained to remain together then the actual change in length must be the same for each. This change is the resultant of the effects due to temperature and stresses condition.

Now let $\sigma_1$ = Stress in brass

$\varepsilon_1$ = Strain in brass

$\alpha_1$ = Coefficient of liner expausion for brass

$A_1$ = Cross sectional area of brass bar

and $\sigma_2$, $\varepsilon_2$, $\alpha_2$, $A_2$ = Corresponding values for steel.

$\varepsilon$ = Actual strain of the composite bar per unit length.

As compressive load on the brass in equal to the tensile load on the steel, therefore

$$\sigma_1 . A_1 = \sigma_2 . A_2$$

strain in brass $\varepsilon_1 = \alpha_1 t - \varepsilon$

$$\varepsilon_2 = \varepsilon - \alpha_2 \Delta t_2$$

$$\varepsilon_1 + \varepsilon_2 = \alpha_1 \Delta t_1 + \alpha_2 \Delta t_2 = \Delta t (\alpha_1 - \alpha_2)_1$$

**Thermal stresses in simple bar**

Let L = original length of the body

$\Delta t$ = Increase in temperature

$\alpha$ = Coefficient of liner expansion.

We know that the increase in length due to increase of temperature

$$\delta L = L \alpha \Delta t$$

$$\varepsilon = \frac{\delta L}{L} = \frac{L \alpha \Delta t}{L} = \alpha \Delta t$$

$$\text{Stress } \sigma = \varepsilon E$$

**Example -1**

An aluminium alloy bar fixed at its both ends is heated through 20K find the stress developed in the bar. Take modules of elasticity and coefficient of linear expansion for the bar material as 80 GPa and 24 X $10^{-6}$/K respectively.

**Data Given**

$\Delta t$ = 20K

E = 80GPa = 80 X $10^3$ N/mm$^2$

$\alpha$ = 24 X $10^{-6}$/K

**Solution**

Then the thermal stress

$\sigma t = \alpha \Delta t E = 24 x 10^{-6} x 20 x 80 x 10^{3}$

$= 38.4 \, N/mm^{2} = 38.4 \, mPa$

**Example - 2**

A flat steel bar 200mm X 20mm X 8mm is placed between two aluminium bars 200mm X 20mm X 6mm. So as to form a composite bar. All the three bars are fastened together at room temperature. Find the stresses in each bar where the temperature of the whole assembly in raised through $50^0$c, Assume $E_s$=200GPa, $E_a$=80GPa, $\alpha_s = 12x10^{-6}/^0c$, $\alpha_a = 24x10^{-6}/^0c$

**Data given**

| | |
|---|---|
| Aluminium | 6mm |
| Steel | 8mm |
| Aluminium | 6mm |

$\Delta t$ = $50^0$c, Es = 200GPa = 200 x $10^3$ N/mm²

$\varepsilon a$=80GPa = 80 x $10^3$ N/mm²

$\alpha_s = 12x10^{-6}/^0c, \ \alpha_a = 24x10^{-6}/^0c$

**Solution**

As = 20 x 8 = 160 mm²

Aa = 2 x 20 x 6 = 240 mm²

$$\alpha_s = \frac{Aa}{As} x \sigma A = \frac{240}{160} x \sigma A = 1.5 \sigma A$$

$$\varepsilon_s = \frac{\sigma s}{\varepsilon s} = \frac{\sigma s}{200 x 10^3}$$

$$\varepsilon a = \frac{\sigma a}{\varepsilon a} = \frac{\sigma a}{80 x 10^3}$$

$$\varepsilon_s + \varepsilon_a = t(\alpha_a - \alpha_s)$$

$$\frac{\sigma s}{200x10^3} + \frac{\sigma a}{80 x 10^3}$$

$$= 50(24 x 10^{-6} - 12 x 10^{-6})$$

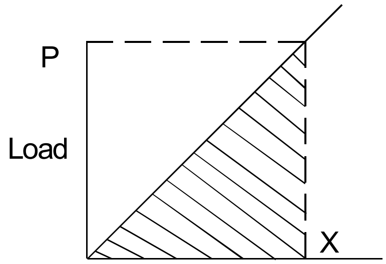
$$\text{or, } \frac{1.5 \sigma a}{200x10^3} + \frac{\sigma a}{80 x 10^3}$$

$$= 50 x 12 x 10^{-6}$$

$$\Rightarrow \sigma a = 30 N/mm^2 = 30 MPa$$

$$\sigma a = 1.5 \sigma a = 1.5 x 30 = 45 N/mm^2 = 45 MPa$$

**1.4. Strain energy resilience stress due to gradually applied load and compact load.**




**Strain Energy**

The strain energy (U) of the bar is defined as the work done by the load in strain it.

For a gradually applied load or static load the work done is represented by the shaded area in above figure.

$$U = \frac{1}{2} P . X$$

$$U = \frac{1}{2} \sigma A \frac{\sigma}{E} L$$

$$= \frac{1}{2E} \sigma^2 A L = \frac{1\sigma}{2E} Vol.$$

**Resilience**

The strain energy per unit volume usually called as resilience in simple tension or compression is $\frac{\sigma^2}{2E}$.

**Proof resilience**

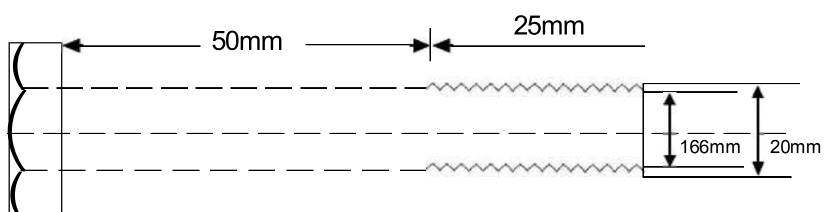
It is the value at the elastic limit or at the proof stress for non-ferrous materials.

Strain energy is always a positive quantity and being work units will be expressed as Nm (i.e. joules)

**Example 1**

Calculate the strain energy of the bolt as shown below under a tensile load of 10 KN. Show that the strain energy is increased for the same max stress by turning down the same of the bolt to the root diameter of the turned, E=20500 N/mm$^2$

**Data Given**




P= 10 KN, E= 205,000 N/mm$^2$

**Solution**

It is a normal practice to assume that the load is distributed events over the core.

$$A_c = \frac{\pi}{4} 16.6^2 = 217 \, mm^2$$

Stress in screwed portion = $\frac{P}{A_c} = \frac{10,000}{217} = 46 N/mm^2$

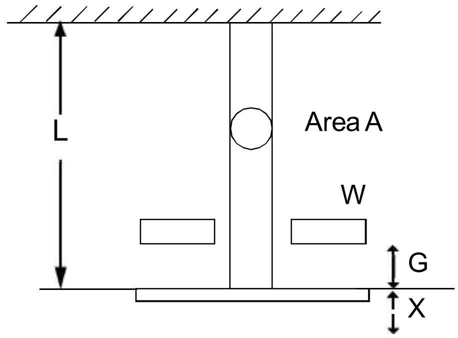
Stress in shank = $\frac{P}{A_C} = \frac{10,000}{\frac{\pi}{4} \times 20^2} = 31.8 N/mm^2$

$$\text{Total strain Energy} = \frac{1}{2 \times 205000}(46^2 x210x25 + 31.8^2 x314x50) = 67N/mm^2$$

If turned to 16.6mm

$$S.E = \frac{1}{2 \times 205000}(46^2 x217x75) = 84N/mm$$

**Impact load**




Supposing a weight W falls through a height 'h' on to 'a' collar attached to one end of a uniform bar, the other end being fined. Then an extension will be caused which is greater than that due to one application of the same load gradually applied.

Let X is the maximum extension, set up and the corresponding strain is $\sigma$.

Let P be the equivalent static load which would produced the same extension X.

Then the strain energy at this instant = $E1 = \frac{1}{E}(\sigma_1 - \mu\sigma_2)$

$$\text{or } E1 = \frac{Pd}{4t_1 E}(2-\mu)$$

Neglecting loss of energy at compact loss of PE of weight = Gain of strain energy.

$$w(h+x) = \frac{1}{2}Px$$

$$\text{or } w(h + \frac{PL}{AE}) = \frac{1}{2}P^2L/AE$$

Rearranging and multiplying through AE/L

$$P^2/2 - WP - WhAE/L = 0$$

Solving and discarding the negative root

$$P = W + \sqrt{W^2 + 2WGAE/L}$$

$$= W[1 + \sqrt{1 + 2hAE/WL}]$$

$$\text{From which } X = \frac{PL}{AE}, \sigma = \frac{P}{A} \text{ can be found}$$

$$\text{When } h = 0, P = 2W$$

i.e. the stress produced by a suddenly applied load is twice the static stress. Ex- Referring figure-1, let a mass of 100Kg falls 4cm on to a collar attached to a bar of 2 cm dia, 3mm long find max stress, E= 205,000N/mm²

$$\sigma = \frac{P}{A} = \frac{W}{A}[1 + \sqrt{1 + 2hAE/WL}]$$

$$= \frac{981}{100\pi}, [1 + \sqrt{1 + \frac{2x40x\pi 100x205000}{981x3x1000}}]$$

$$= 134N/mm^2$$

CHAPTER 2.0.

# THIN CYLINDER AND SPHERICAL SHELL UNDER INTERNAL PRESSURE

## 2.1. Definition of hoop stress

By symmetry the three principal stresses in the shell will be the

(i) circumferential or hoop stress

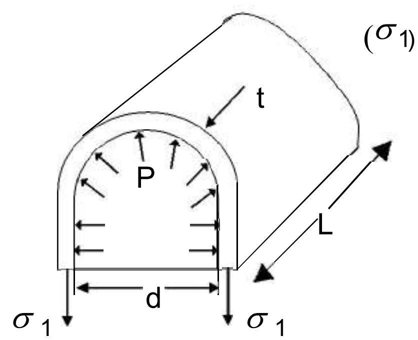
(ii) longitudinal stress

(iii) radial stress.

**Thin cylinder :**

If the ratio of thickness to internal diamer is less than about 1/20, then the hoop stress and longitudinal stress are constant over the thickness and the radial stress is small and can be neglected.

## 2.2 Hoop stress or circumferential stress derivation




Let d - internal diameter

l - length of cylinder

t - thickness

p - pressure

consider the equilibrium of a half cylinder of length L.

section through a diameteral plane, $\sigma_1$ acts on an area 2tL and the resultant vertical pressure force is found from the projected area horizontal d x L

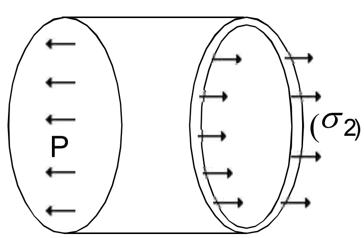
**Equating forces**

$\sigma_1$ x 2 x tL = P x d x L

$$= \sigma_1 = \frac{PD}{2t}$$

hoop stress in a tensile stress acts circumferentially on the cylinder.

**Longitudinal stress $\sigma_2$ Derivation**




Consider the equilibrium of a section cut by a transverse plane, $\sigma_2$ acts on an area $\pi_2$, dt (d should be the main diameter) and pacts on a projected area of $\frac{\pi}{4}d^2$ equating the forces.

Equating the forces

$$\sigma_2 \times dt = P \times \frac{\pi}{4} d^2$$

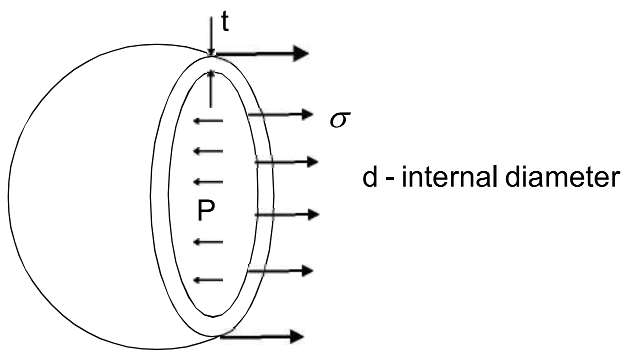
Whatever the actual shape of the end

$$\text{i.e. } \sigma_2 = \frac{Pd}{4t}$$

In case of long cylinder or tubes this stress may be neglected.

**Thin spherical shell under internal pressure derivation**

Again the radial stress will be neglected and the circumferential or hoop stress will be neglected and by symmetry the two principal stresses are equal, in fact the stress in any tangential direction is equal to $\sigma$.

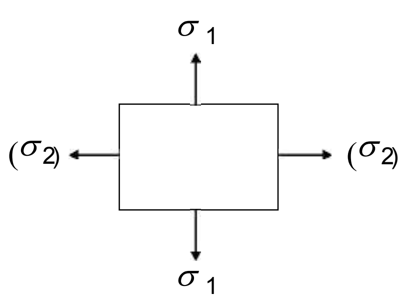



From above figure it is seen that

$$\sigma \pi dt = P \frac{\pi}{4} d^2$$

$$\text{i.e. } \sigma = \frac{Pd}{4t}$$

**Volumetric strain**




**Hoop Strain**

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \mu\sigma_2)$$

$$\text{or } \varepsilon_1 = \frac{Pd}{4t_1E}(2-\mu)$$

**Longitudinal Strain**

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \mu\sigma_1)$$

**Volumetric Strain on capacity**

The capacity of a cylinder $\frac{\pi}{4}d^2L$ If the dimension is increased by $\delta d$ and $\delta L$, the volumetric strain

$$= \frac{(d+\delta d)(L+\delta L) - d^2L}{d^2L}$$

$$= \frac{[d^2L + d^2\delta L + 2\delta d.dL + 2\delta d.d.\delta L + \delta d^2L + \delta d^2\delta L d^2L]}{d^2L}$$

$$= (d^2\delta L + 2\delta d.\,dL)/d^2L$$

$$= 2.\delta d/d + \delta L/L$$

$$= 2 \text{ x diameteral strain} + \text{longitudinal strain}$$

$$= 2 \text{ x hoop strain} + \text{longitudinal strain}$$

Change in volume = $(2\varepsilon_1 + \varepsilon_2)$ volume

For spherical shell, volume strain = 3 x hoop strain

Change in diameter = $\varepsilon_1.d$

Change in length = $\varepsilon_2.L$

**Example – 1**

A gas cylinder of internal diameter 40mm is 5mm thick, if the tensile stress in the material is not to exceed 30 MPa, find the maximum pressure which can be allowed in the cylinder.

Data given

D = 40mm, t = 5m

$\sigma_1$= 30MPa = 30 N/mm2

**Solution**

$$\text{we know}, \sigma_1 = \frac{Pd}{2t}$$

$$\text{or, } 30 = \frac{P \text{ x } 40}{2 \text{ x } 5}$$

$$= P = 7.5MPa$$

**Example – 2**

A cylindrical thin drum 80mm diameter and 4m long is made 10mm thick plates. If the drum is subjected to an internal pressure of 2.5MPa determine its changes is diameter and length. E = 200GPa.

Data given

d = 80 mm

L = 4m

T = 10mm

P = 2.5 N/mm2

E = 200 x$10^3 N/mm^2$

**Solution**

$$\varepsilon_1 = \frac{Pd}{4tE}(2-\mu)$$

$$\varepsilon_1 = \frac{2.5\,x\,800}{4\,x\,10\,x\,200\,x\,10^3}(2-0.25)$$

$$\delta d = \varepsilon_1\,x\,d = \frac{2.5\,x\,800^2}{4\,x\,200\,x\,10^3}\,x\,1.75$$

$$= 0.35mm \;\; (Ans)$$

**Change in length**

$$\varepsilon_2 = \frac{Pd}{2tE}(\frac{1}{2}-\mu)$$

$$\delta L = \varepsilon_2 L$$

$$= \frac{PdL}{2tE}(\frac{1}{2}-\mu)$$

$$= \frac{2.5\,x\,800\,x\,4\,x\,10^3}{4\,x\,10\,x\,200\,x\,10^3}(\frac{1}{2}-0.25)$$

$$= 0.5mm(Ans)$$

**Example – 3**

A cylindrical vessel 2m long and 500mm dia with 10mm thick plates in subjected to an internal pressure of 3MPa, calculate the change in volume of the vessel.

E= 200GPa, $\mu$ = 0.3

**Data given**

L = 2 x $10^3$ mm

d = 500 mm

t = 10mm

P = 3MPa

E = 200 x $10^3 N/mm^2$

$$\varepsilon_2 = \frac{Pd}{2tE}(\frac{1}{2}-\mu)$$

$$= \frac{3\,x\,500}{2\,x\,10\,x\,200\,x\,10^3}(\frac{1}{2}-0.3)$$

$$= 0.075\,x\,10^{-3}$$

$$V = \frac{\pi}{4}d^2L = \frac{\pi}{4}x\,500^2\,x\,2\,x\,10^3$$

$$= 392.2\,x\,10^6\,mm^3$$

Change in Volume

= V $(2\varepsilon_1 - \varepsilon_2)$

= 392.7 (2x.32x$10^3$ + .075 x $10^{-3}$)

= 185 x $10^{-3}$mm $^3$

**CHAPATER. 3.0**

# TWO DIMENSION STRESS SYSTEMS

### 3.1 Determination of normal stress, shear stress and resultant stress on oblique plane.

In many instances, however, both direct and shear stresses are brought into play, and the resultants stress across any section will be neither normal nor tangential to the plane.

If $\sigma_r$ Is the resultants stress making an angle $\gamma$ with the normal to the plane on which of acts.

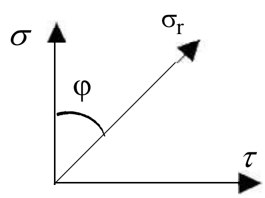



Fig 3.1

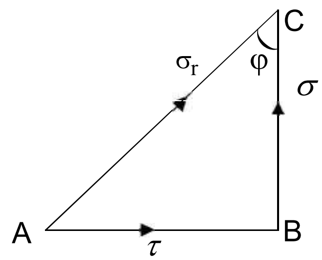



Fig 3.2

$$\varphi = \tan\frac{\tau}{\sigma}$$

$$\sigma_r = \sqrt{\sigma^2 + \tau^2}$$

**<u>Stress on oblique plane</u>**

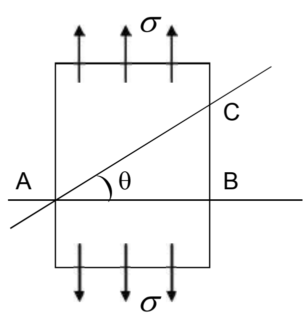



Fig 3.3

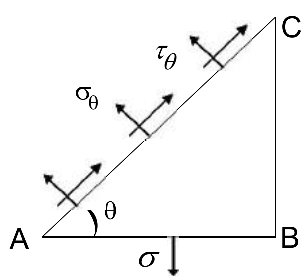



Fig 3.4

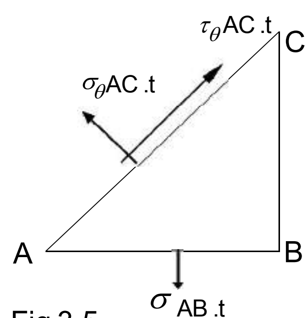



Fig 3.5

The problem is to find the stress acting on any plane AC at an angle $\theta$ to AB. This stress will not be normal to the plane, and may be resolved into two components $\sigma_\theta$ and $\tau_\theta$ .

As per Figure 3.4 show the stresses acting on the three planes of the triangular prism ABC. There can be no stress on the plane BC, which is a longitudinal plane of the bar, the stress $\tau_\theta$ must be up the plane for equilibrium.

Figure 3.5 shows the forces acting on the prism, taking a thickness t perpendicular the figure.

The equations of equilibrium resolve in the direction of $\sigma_\theta$.

$$\sigma_\theta . AC.t = \sigma\, AB.t\, Cos\,\theta$$

$$= \sigma_\theta = \sigma\left(\frac{AB}{AC}\right) Cos\,\theta$$

$$= \sigma\, Cos^2\,\theta$$

Resolve in the direction $\tau_\theta$

$$\tau_\theta.AC.t = \sigma AB.t\sin\theta$$

$$\Rightarrow \tau_\theta = \sigma\left(\frac{AB}{AC}\right)\sin\theta$$

$$\Rightarrow \tau_\theta = \sigma\cos^2.\theta\sin\theta$$

$$\Rightarrow \tau_\theta = \frac{1}{2}\sigma\sin 2\theta$$

$$\Rightarrow \sigma_r = \sqrt{(\sigma_\theta^2 + \tau_\theta^2)}$$

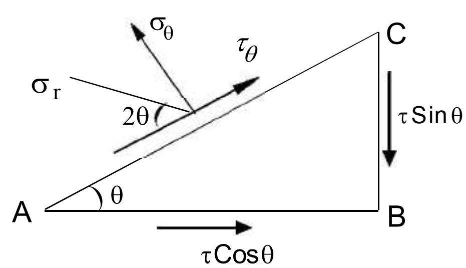
$$\Rightarrow \sigma\sqrt{\cos^4\theta + \cos^2\theta.\sin^2\theta}$$

$$\therefore \sigma_r = \sigma\cos\theta$$

It is seen that maximum shear stress is equal to one-half the applied stress and acts on planes at $45^0$ to it.

**Pure Shear**

As the figures will always be right-angled triangles there will be no loss of generality by assuming the hypotenuse to be of unit length. By making use of these specification it will be found that the area on which the stresses act are proportional to 1 (for AC), Sin $\theta$ (for BC) and Sin $\theta$ (for AB) and future figures will show the forces acting on such an element.




Let tue $\tau$ act on a plane AB and there is an equal complementary shear stress on plane BC. The aim is to find $\sigma\theta$ & $\tau\theta$ acting on AC at[a] angle $\theta$ to AB.

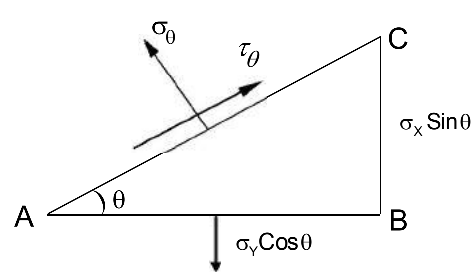
Resolving in the direction of $\sigma_\theta$

$$\sigma_\theta x 1 = (\tau\cos\theta)\sin\theta + (\tau\sin\theta).\cos\theta$$
$$= \tau\sin 2\theta$$

Resolving in the direction of $\tau_\theta$

$$\tau_\theta x 1 = (\tau\sin\theta)\sin\theta - (\tau\cos\theta).\cos\theta$$
$$= -\tau\cos 2\theta(\theta \langle 45) \text{down to plane}$$
$$\sigma_r = \sqrt{\sigma^2{}_\theta + \tau^2{}_\theta} = \tau \text{ at } 2\theta \text{ to } \tau_\theta$$

**Pure Normal stresses on give planes**




Let the known stresses be $\sigma_x$ on BC and $\sigma_Y$ on AB, then the forces on the element are proportional to those shown.

Resolving in the direction of $\sigma_\theta$

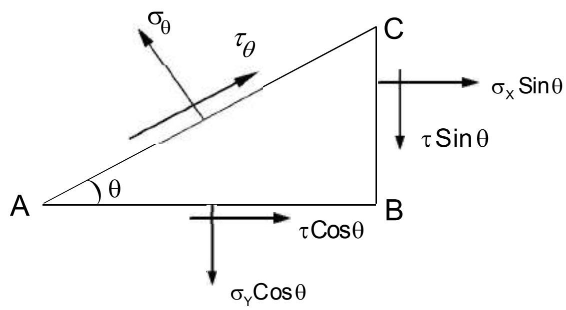
$$\therefore \sigma_\theta = \sigma_Y \cos^2\theta + \sigma_X \sin^2\theta$$

Resolving in the direction of $\tau_\theta$

$$\tau_\theta = \sigma_Y \cos\theta \sin\theta - \sigma_X \sin\theta \cos\theta$$

$$\therefore \tau_\theta = \frac{1}{2}(\sigma_Y - \sigma_X)\sin 2\theta$$

## General two dimensional Stress system




Resolving in the direction of $\sigma_\theta$

$$\sigma_\theta = \sigma_Y \cos\theta\cos\theta + \sigma_X \sin\theta\sin\theta + \tau\cos\theta\sin\theta + \tau\sin\theta\cos\theta$$

$$= \sigma_Y\left(\frac{1+\cos^2\theta}{2}\right) + \sigma_X\left(\frac{1-\cos^2\theta}{2}\right) + \tau\sin^2\theta$$

$$= \frac{1}{2}(\sigma_Y + \sigma_X) + \frac{1}{2}(\sigma_Y - \sigma_X)\tau\cos^2\theta + \tau\sin^2\theta$$

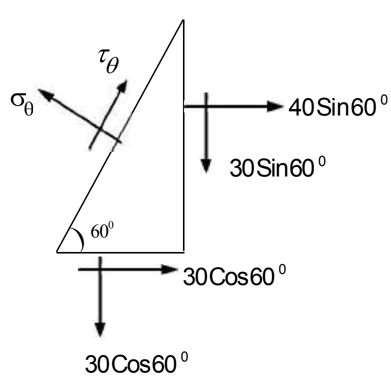
Resolving in the direction of $\tau_\theta$

$$\tau_\theta = \sigma_Y \cos\theta\sin\theta - \sigma_X \sin\theta\cos\theta$$
$$- \tau\cos\theta\cos\theta + \tau\sin\theta\sin\theta$$

$$\therefore \tau_\theta = \frac{1}{2}(\sigma_Y - \sigma_X)\sin 2\theta - \tau\cos 2\theta$$

**Example – 1**

If the stress on two perpendicular planes through a point are 60 N/mm2 tension, 40 N/mm2 compression and 30 N/mm2 shear find the stress components and resultant stress on a plane at $60^{\mathbf{0}}$ to that of the tensile stresses.



**Resolving**

$$\sigma_\theta = 60\,Cos60^0 .\ Cos60^0 - 40\,Sin60^0 .Sin60^0 + 30\,Cos60^0\,Sin60^0 + 30\,Sin60^0\,Cos60^0$$

$$= 60\,x\frac{1}{2}\,x\frac{1}{2} - 40\,x\frac{\sqrt{3}}{2}\,x\frac{\sqrt{3}}{2} + 30\frac{1}{2}\,x\frac{\sqrt{3}}{2} + 30\,x\frac{\sqrt{3}}{2}\,x\frac{1}{2}$$

$$= 15 - 30 + 7.5\sqrt{3} + 7.5\sqrt{3}$$

$$\boxed{= \sigma_\theta = 11N/mm^2}$$

and

$$\tau_\theta = 60\,Cos60^0 .\ Sin60^0 + 40\,Sin60^0 .Cos60^0 - 30\,Cos60^0\,Cos60^0 + 30\,Sin60^0\,Sin60^0$$

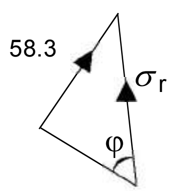
$$= 15\sqrt{3} + 10\sqrt{3} - 7.5 + 22.5$$

$$= 58.3\ N/mm^2$$

$$= \sigma_r = \sqrt{(112 + 58.32)} = 59.3N/mm^2$$

at angle to the

$$\gamma = \tan^{-1}\frac{58.3}{11} = 80^0\,15^0$$




($20^0$ to the 60 $N/mm^2$)

**Principal Planes**

From equation

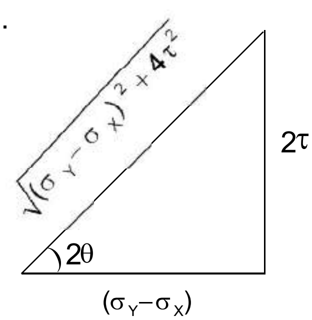
$$\tau_\theta = \frac{1}{2}(\sigma_Y - \sigma_X)\,Sin\,2\theta - \tau Cos2\theta$$

There are values of 0 for which $\tau_\theta$ is zero and the plane on which the shear component is zero are called principal planes.

From equation above.

$$\tan 2_\theta = \frac{2\tau}{(\sigma_Y - \sigma_X)} \quad (\text{when } -\tau_\theta = 0)$$

This gives two values of 2θ differing by $180^0$ and hence two values of θ differing by $90^0$ i.e. the principle planes are two planes at right angles.




$$Sin2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}}$$

$$Cos2\theta = \pm\frac{\sigma_Y - \sigma_X}{\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}}$$

## Principal Stresses

The stresses on the principal planes will be pure normal (tension or compression) and their values are called the principal stresses.

We know,

$$\sigma_\theta = \frac{1}{2}(\sigma_Y + \sigma_X) + \frac{1}{2}(\sigma_Y - \sigma_X) x \ Cos2\theta + \tau Sin2\theta$$

Principal stresses =

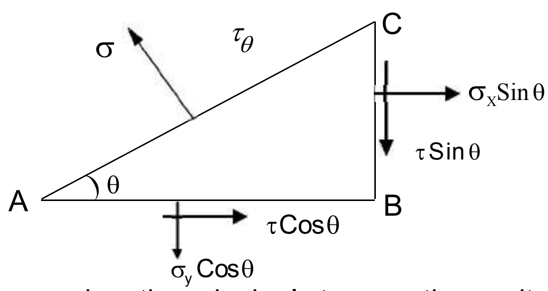
$$\frac{1}{2} x (\sigma_Y + \sigma_X) \pm \frac{\frac{1}{2}(\sigma_Y - \sigma_X)^2}{\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}}$$

$$\pm \frac{\tau . 2\tau}{\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}}$$

$$= \frac{1}{2} = (\sigma_Y + \sigma_X) \pm \frac{\frac{1}{2}[(\sigma_Y - \sigma_X)^2 + 4\tau^2]}{\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}}$$

$$= \frac{1}{2} x (\sigma_Y + \sigma_X) \pm \frac{1}{2}\sqrt{(\sigma_Y - \sigma_X)^2 + 4\tau^2}$$

### Shorter method for principal stresses




Let AC be a principal plane and $\sigma$ the principal stress acting on it $\sigma_X$, $\sigma_y$ and $\tau$ are the known stress on planes BC and AB as before.

Resolve in the direction of $\sigma_X$

$\sigma Sin\theta = \sigma_X Sin\theta + \tau Cos\theta$

or $\sigma - \sigma_x = \tau Cos\theta$ ......(1)

Resolve in the direction of $\sigma_y$

$\sigma Cos\theta = \sigma_y Cos\theta + \tau Sin\theta$

or $\sigma - \sigma_y = \tau tan\theta$ ......(2)

Multiply corresponding sides of equations (1) and (2) i.e.

$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau^2$

or $\sigma^2 - (\sigma_x + \sigma_y)\sigma + \sigma_x\sigma_y - \tau^2 = 0$

Solving

$ax^2 + bx + c = 1$

$$x = \frac{-1b \pm \sqrt{b^2 - 4ca}}{2a}$$

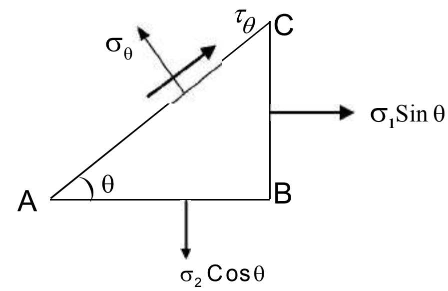
Here

$$\sigma = \frac{(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x + \sigma_y)^2 - 4\sigma_x\sigma_y + 4\tau^2}}{2}$$

$$\text{or } \sigma = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

The values of 0 for the principal planes are of course found by substitution of the principal stresses values in equation (1) & (2).

**Maximum shear stress**




Let AB and BC be the principal planes and $\sigma$ 1 and $\sigma$ 2 the principal stresses.

Then resolve

$$\tau_\theta = \sigma_2 \text{Cos}\theta.\ \text{Sin}\theta - \sigma_1 \text{Sin}\theta.\text{Cos}\theta$$

$$= \frac{1}{2}(\sigma_2 - \sigma_1)\text{Sin}2\theta$$

Hence the maximum shear stress occurs when 2 0= $90^0$ i.e. on planes at $45^0$ to the principal planes and its magnitude is

$$\tau_{max} = \frac{1}{2}(\sigma_2 - \sigma_1)$$

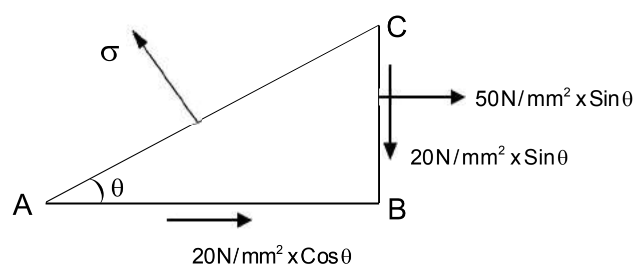
$$= \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

In words : The maximum shear stress is one-half the algebraic difference between the principal stresses.

**Example – 2**

At a section in abeam the tensile stress due to bending is 50 N/mm² and there is a shear stress of 20 N/mm². Determine from first principles the magnitude and direction of the principal stresses and calculate the maximum shear stress.

**Solution**




Resolve in the direction AB :

$\sigma \text{Sin}\,\theta = 50\,\text{Sin}\,\theta + 20\,\text{Cos}\,\theta$

$\sigma - 50 = 20\,\text{cot}\,\theta$ ......(1)

Resolve in the direction BC :

$\sigma \text{Cos}\,\theta = 20\,\text{Sin}\,\theta$......(2)

$\sigma = 20\tan\theta$

Multiplying corresponding sides of equations (i) and (ii)

$\sigma(\sigma - 50) = 20^2$

$\sigma^2 - 50\sigma - 400 = 0$

$$\sigma = \frac{50 \pm 10\sqrt{(25-16)}}{2}$$

$$= \frac{50 \pm 64}{2} = 57 \text{ or } -7$$

i.e. the principal stresses are 57 N/mm² tension, 7 N/mm² compression,

$$\tan\theta = \frac{\sigma}{20} = \frac{57}{20} \text{ or } \frac{-7}{20}$$

Giving 0=70⁰ and 160⁰, being the directions of the principal planes.

Max shear stress =

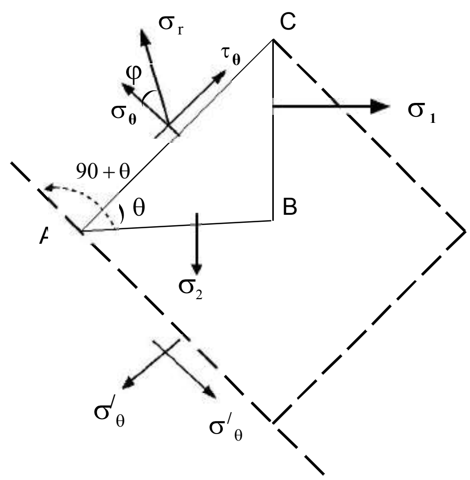
$$= \frac{1}{2}(\sigma_2 - \sigma_1)$$

$$= \frac{1}{2}[57 - (-7)]$$

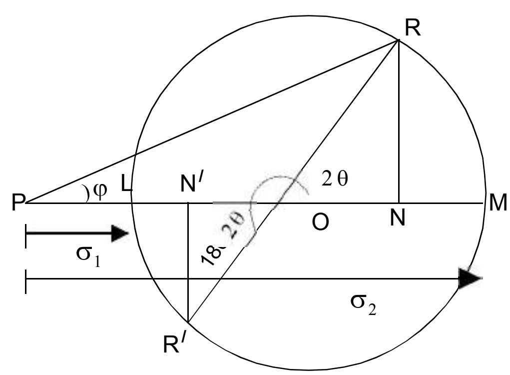
$$= 32 N/mm^2$$

and the planes of maximum shear are at $45^0$ to be principle planes i.e. 0=25⁰ and 115⁰. (Ans)

**Maximum shear stress using Mohr's Circle**




The stress circle will be developed to find the stress components on any plane AC which makes an angle $\theta$ with AB.




**Construction**

Mark off PL = $\sigma_1$ and PM = $\sigma_2$ (positive direction to the right). It is shown here for $\sigma_2 > \sigma_1$, but this is not a necessary condition. On LM as diameter describes a circle center O.

Then the radius OL represents the plane of $\sigma_1$ (BC) and OM represents the plane of $\sigma_2$(AB) plane AC is obtained by rotating. AB through $\theta$ anticlockwise, and if OM on the stress circle is rotated through 2$\theta$ in the same direction, the radius OR in obtained which will be shown to represent the plane AC.

OR could equally will be obtained by rotating OL clockwise through $180^0 - 2\theta$, corresponding to rotating BC clockwise through $90^0 - \theta$.

Draw RN⊥r to PM

Then PN = PO + ON

$$=\frac{1}{2}(\sigma_1+\sigma_2)+\frac{1}{2}(\sigma_2-\sigma_1)\text{Cos}2\theta$$
$$=\sigma_1\frac{(1-\text{Cos}2\theta)}{2}+\sigma_2\frac{(1+\text{Cos}2\theta)}{2}$$
$$=\sigma_1\,\text{Sin}^2\theta+\sigma_2\text{Cos}^2\theta)=\sigma_{\theta},\text{ the normal stress component on AC}$$

and $RN=\frac{1}{2}(\sigma_2-\sigma_1)\text{Sin}2\theta$

$=\tau_{\theta}$, the shear stress component on AC

Also the resultant stress

$$=\sigma_r=\sqrt{(\sigma^2_{\theta}+\tau^2_{\theta})}=PR$$

And its inclination to the normal of the plane is given $\varphi=\langle RPN$

$\sigma_\theta$ is found to be a tensile stress and $\tau_\theta$ is considered positive if R is above PM,

The stresses on the plane AD, at right angles for AC, are obtained from the radius $OR^{/}$, at $180^0$ to OR

$$\text{i.e. } \sigma^1_{\theta}=PN^1,\ \tau^1_{\theta}=R^1N^1$$

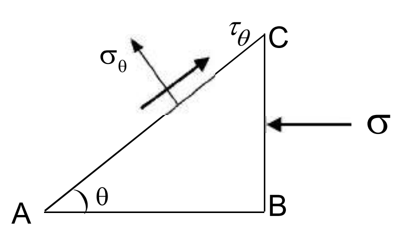
and $\tau_\theta=\tau^1_\theta$ but of opposite type, tending to give an anticlockwise rotation.

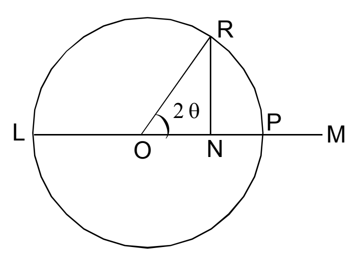
The maximum shear stress occurs when RN=OR , i.e. $\theta=45^{\mathbf{0}}$ and is equal in magnitude to $OR=\frac{1}{2}(\sigma_2-\sigma_1)$ The maximum value of $\varphi$ is obtained when PR is a tangent to the stress circle.

Two particular cases which have previously been treated analytically will be dealt with by this method.

**1. Pure compression**

IF $\sigma$ is the compressive stress the other principal stress is zero.



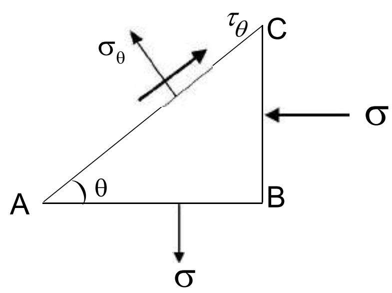



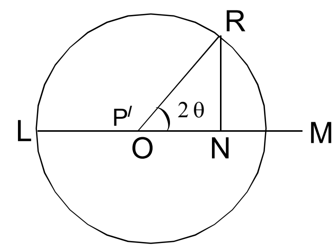

PL = $\sigma$ numerically, measured to the left for compression, PM = 0

$$\text{Hence, } OR=\frac{1}{2}\sigma$$
$$\sigma_\theta=PN,\text{Compressive}$$
$$\tau_\theta=PN,\text{Positive}$$

$$\text{Maximum shear stress}=OR=\frac{1}{2}\sigma\ \text{ occuring when } \theta=45^0.$$

**2. Principal stresses equal tension and compression**




PM = $\sigma$ to the right

PL = $\sigma$ to the left

Here O coincides with P

$\sigma_\theta = PN$, is tensile for

$\theta$ between $\pm 45^0$, compressive for

$\theta$ between $45^0$ and $135^0$

$\tau_\theta = RN$, when $\theta = 45^0$

$\tau_\theta$ reach maximum $= \sigma$, on planes when the normal stress is zero (Pure shear)

**Example -3**

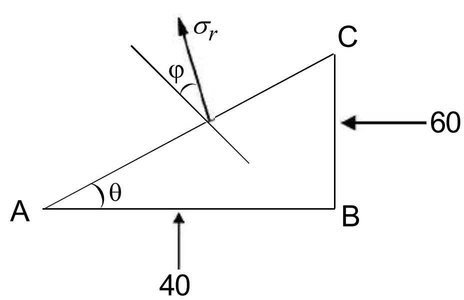
A piece of materials is subjected to two compressive stresses at right angles, their values being 40 N/mm2 and 60 N/mm2. Find the position of the plane across which the resultant stress in most inclined to the normal and determine the value of this resultant stress.

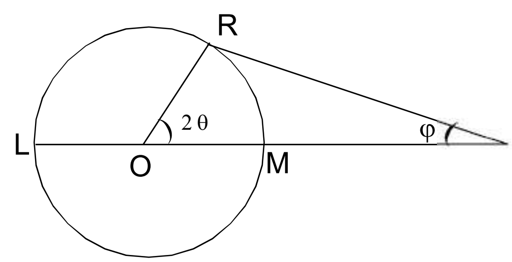
**Solution**

$\sigma_1 = 60 N/mm^2$ (Compressure)

$\sigma_2 = 40 N/mm^2$ (Compressure)

In the figure, the angle $\theta$ is inclined to the plane of the 40 tons N/m2 compression.




In above figure PL =60, PM=40, The maximum angle $\varphi$ is obtained when PR is a tangent to the stress circle.

OR = 10, PO = 50

$$\text{Then } \varphi = \sin^{-1}\frac{1}{5} = 11^0 30^/$$

$$\sigma_r = PR = -\sqrt{(50^2 - 10^2)} = -49 N/mm^2$$

$$2\theta = 90 - \varphi$$

$$\theta = 39^0 15^/$$

which gives the plane required

**Example -4**

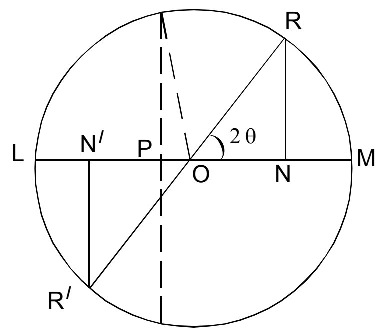
At a point in a piece of elastic material there are three mutually perpendicular planes on which the stresses are as follows : tensile stress 50 N/mm2, shear stress 40 N/mm2 on plane, compressive stress 35 N/mm2 and complementary shear stress 40 N/mm2 on the second plane, no stress on the third plane. Find (a) the principal stresses and the positions of the plane on which they act (b) the position of the planes on which there is no normal stress.

Solution

Mark off PN = 50, NR = 40

$PN' = -35$, $N'R' = -40$

Join RR', Cutting NN' at 0, Draw circle centre O, radius OR




Then ON = $\frac{1}{2}$ NN'

= 42.5

$OR = \sqrt{42.5^2 + 40^2} = 58.4$

$PO = PN - ON = 7.5$

(a) The Principal stresses are

PM = PO + OM = 6.5 N/mm$^2$ (tensile)

PL = OL - OP = 50.9 N/mm$^2$ (compressure)

or, $2\theta$ = tan-1$\frac{40}{42.5} = 43^0\ 20'$

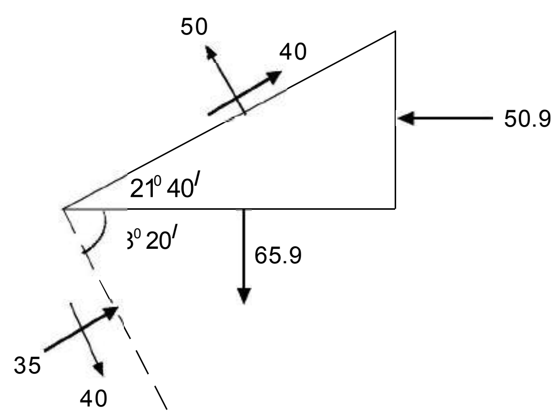
$\Rightarrow \theta = 21^0\ 40'$

(b) If there is no normal stress, then for that plane N and P coincides and

$2\theta = 180 - \text{Cos}' \frac{7.5}{58.4}$

$2\theta = 97^0\ 24'$

$\theta = 48^0\ 42'$ to the principal plane



CHAPTER 4.0

# SHEAR FORCE & BENDING MOMENT

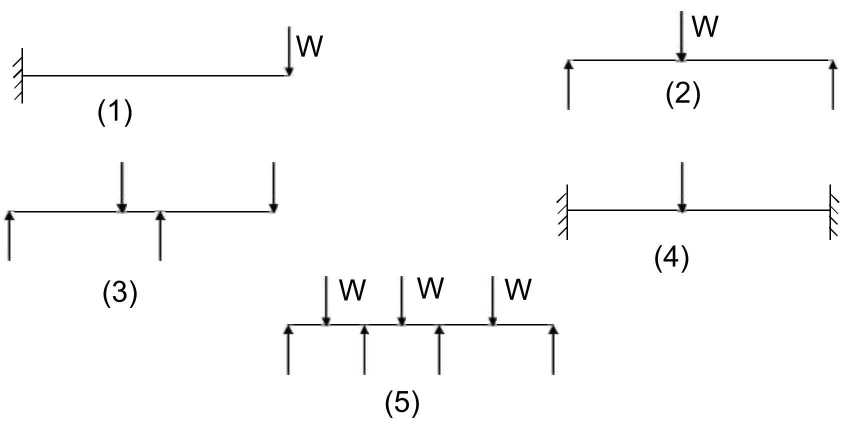
## 4.1 – Types of beam and load

### Beam

A structural member which is acted upon by a system of external loads at right angles to its axis is known as beam.

### Types of Beam

1. Cantilever beam
2. Simply supported beam
3. Over hanging beam
4. Rigidity fixedor built in beams
5. Contimous beam



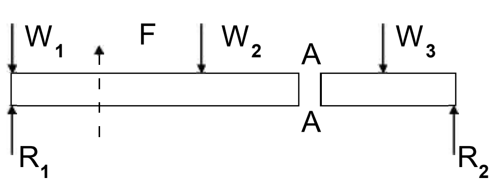

### Types of load

1. Concentrated or point load
2. Uniformly distributed load
3. Uniformly varying load


## 4.2. Concepts of share force and bending moment

### Shear force

The shearing force at any section of beam represents the tendency for the portion of beam to one side of the section of slide or shear laterally relative to the other portion.

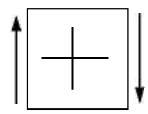



The resultant of the loads and reactions to the left of A is vertically upwards and the since the whole became is in equilibrium, the resultant of the forces to the right of AA must also be F acting down ward. F is called the shearing force.

**Definition**

The shearing force at any section of a beam is the algebraic sum of the lateral component of the forces on either side of the section.

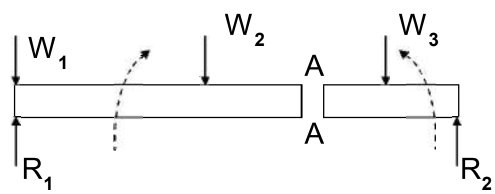
Shearing force will be considered positive when the resultant of the forces to the left is upwards or to the right in downward.



A shear force diagram is one which shows the variation of shearing force along the length of the beam.

**Concepts of Bending Moment**

In a small manner it can be argued that if the moment about the section AA of the forces to the left is M clockwise then the moment of the forces to the right of AA must be anticlockwise. M is called the bending moment.




**Definition**

The algebraic sum of the moments about the section of all the forces acting on other side of the section.

Bending moment will be considered positive when the moment on the left of section is clockwise and on the right portion anticlockwise. This is referred as sagging the beam because concave upwards. Negative B.M is termed as hogging. A BMD is one which shows the variation of bending moment along the length of the beam.

**4.3 Shear force and bending moment diagram and its silent features.**

i. Illustration in cantilever beam

ii. Illustration in simply supported beam

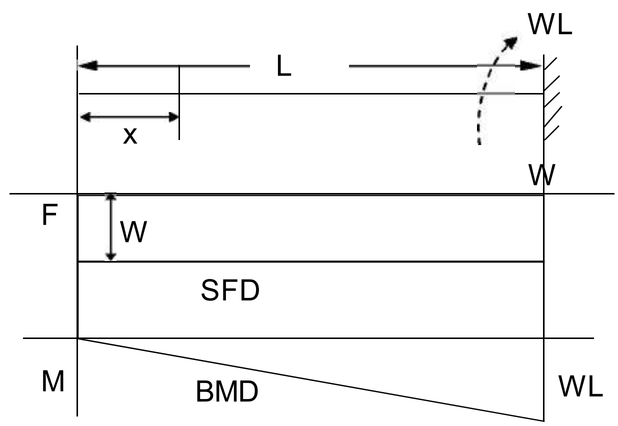
iii. Illustration in overhang beam

Carrying point load and u.d.L.

**Concentrated loads**

**Example -1**

A cantilever of length L carries a concentrated load W at its free end, draw the SF & BM diagram.



**Solution**

At a section a distance x from the free end, consider the forces to the left.

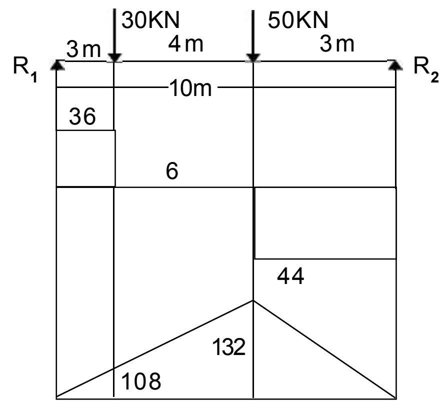
Then F = – W, and in constant along the whole beam for all values of x. Taking moments about the section given M = – Wx

A x = 0, M = 0, At –x = L, M = – WL

At end from equilibrium condition the fixing moment is WL and reactions W.

**Example – 2**

A beam 10m long is simply supported at its ends and carries concentrated loads of 30 KN and 50 KN at distance of 3m from each and. Draw the SF & BM diagram.




**Solution**

First calculate R1 and R2 at support

R1 x 10 = 30 x 7 + 50 x 3

= R1 = 36KN

and R2 = 30+50 – 36 = 44KN

Let x be the distance of the section from the left hand end.

**Shearing force**

O < x < 3m, F= 36KN

3 < x < 7, F= 36 -30 = 6 KN

7 < x < 10, F = 36-30-50= -44 KN.

**Bending moment**

0 < X , 3 M = R1 X = 36 x KNM

3 < X , 7, M = R1 X – 30 (X-3) = 6X +90 KNM

Kx < 10 , 7, M = R1 X – 30 (X-3) – 50 (X-7) = 44 X + 440 KNM

Principal values of M are

at X = 3m, m = 108 KNM

at x = 7m, M = 132 KNM

at x = 10, M = 0.