DIGITAL SIGNAL PROCESSING

6th Sem ETC

by

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Introduction

Signal:

A *signal* is defined as any physical quantity th a t varies with time, space, or any other in dependent variable or variables. Mathematically, we describe a signal as a function ofone or mo re independent variables. For example, the functions

$$s(t)= 5t$$

describe a signal,one that varies linearly with the ind ep end ent variable*t* (time).

$$s(x, y) = 3x + 2xy + 10y^2$$

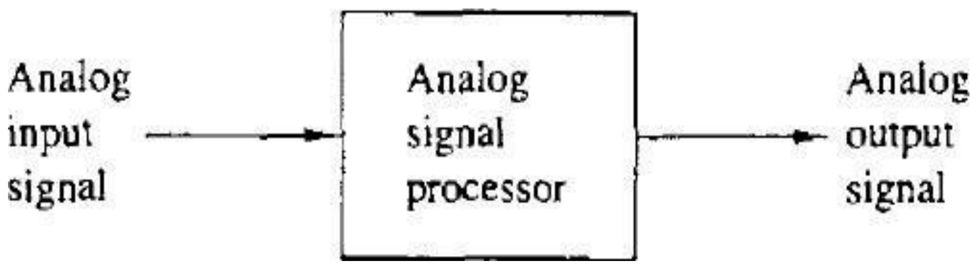
This functiondescribes a signaloftwo independent variables *x* and ythat could represent the two spatial coordinates in a p lane.

System:

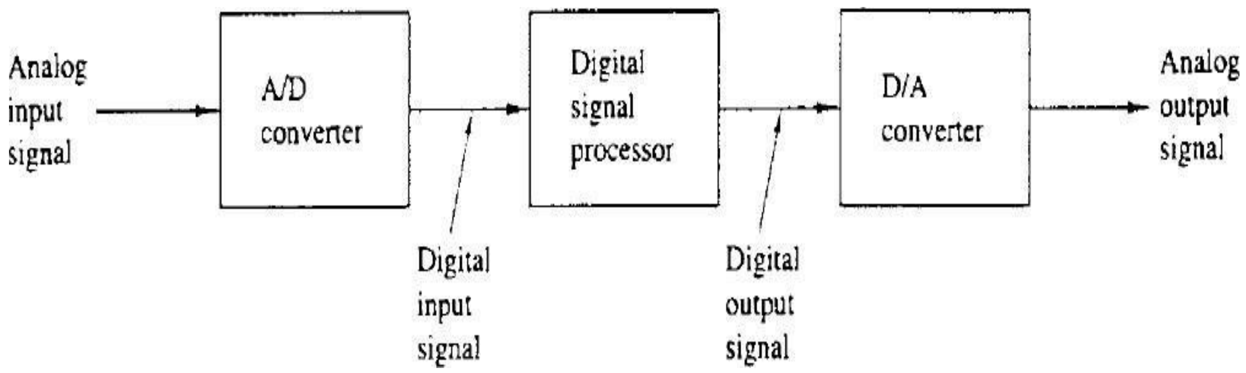
A*system* may also be defined as a physicaldevice th a t performs an operatioon a signal. For ex ample, a filter used to reduce the noise and interference corrupting desired in formation bearing signal is called a system .

signal processing:

W h en we pass a signal thrugh a system , as in filtering, we say that we have processed the signal. In this case the processing of the signal involves filtering the noise and interference from the desired signal. If the operation on the signal is n o n linear, the system is said to be non linear, and so forth . Such operations are usually referred to as *signal rocessing.* **Analog signal processing:**



Digitalsignalprocessing:



AdvantagesofDigitaloverAnalog SignalProcessing:

1- a digitalprogrammable systemallow s flexibility in re configuring the digital signalprocessing operations simply by changing the program .

2- adigitalsystemprovidesmuchbettercontrolofaccuracy.

3- Digital signals are easily stored on magnetic media (tape or disk) without deterioration or loss of signal fidelity beyond that introduced in the A/D conversion.

4- digitalimplementationofthesignalprocessingsystemischeaperthananalogsignal processing.

Limitations:

One practical limitation is the speed of operation of A /D converters and digital signal processors. We shall see that signals having extremely wide band widths require fast-sampling -rate A /D converters and fast digital signal processors. Hence there are analog signals with large bandwidths for which a digital processing approach is beyond the state of the art of digital hardware.

Discretetimesignalsandsystems

CLASSIFICATIONOFSIGNALS: Thereare3typesofsignals

Continuous-timesignals:*Continuous-timesignals*or*analogsignals*aredefined foreveryvalue of time.

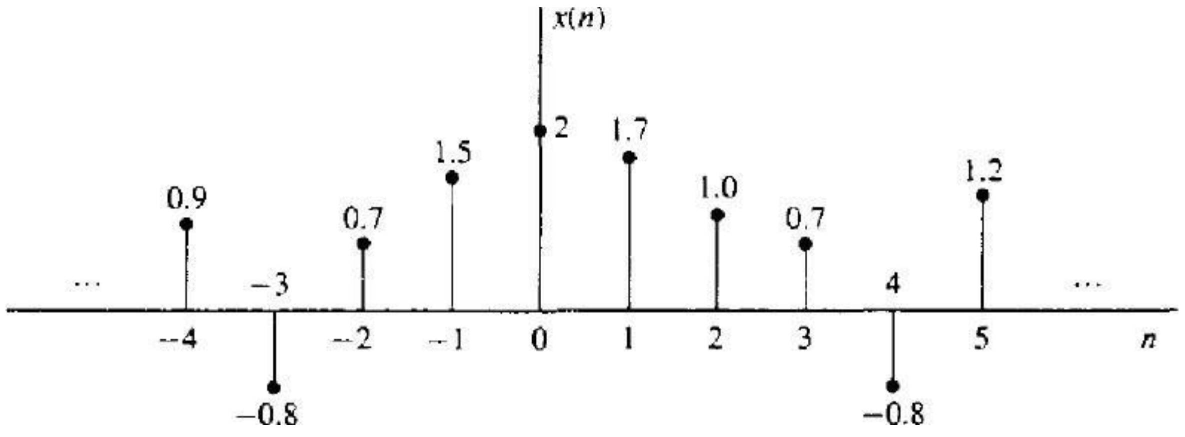
Discrete-timesignals: Discrete-timesignalsaredefinedonlyatcertainspecificvaluesoftime.

Digital Signals: digital signal is defined as a function ofan integer independent variable and its values are taken from a finite set of possible values, which are represented by a string of 0's and l's .

DISCRETE-TIME SIGNALS : Adisc rete-time signal*x{n)* is a functionofan independent variable that is an integer. discrete-time signal is *n o t defined* at instants between two successive samples. Simply, the signal*x (n)* is not defined for no ninteger valueso f *n*. So x(n) was obtained fromsampling an analog signal x a(t), then .i(n) = x a(nT) , where Tis the sampling period (i.e., the time between successive samples).

Representationofdiscrete-timesignal:

Adiscrete-timesignalcanbe representedinvariousway.Butallcanberepresentedgraphically.



Graphicalrepresentationofadiscrete-timesignal.

Besides the graphical representation of a discrete-time signal or sequence as illustrated in aboveFig. there are some alternative representations that are often more convenient to use. These are:

1. **Functionalrepresentation**:

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. **Tabularrepresentation:**

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

3. **Sequencerepresentation:**Aninfinite-durationsignalorsequencewiththetimeorigin*(n* =

0)indicatedbythe symbol↑isrepresentedas

$$x(n) = \{\ldots 0, 0, \underset{\uparrow}{1}, 4, 1, 0, 0, \ldots\}$$

Afinite-durationsequencecanberepresentedas

$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\}$$

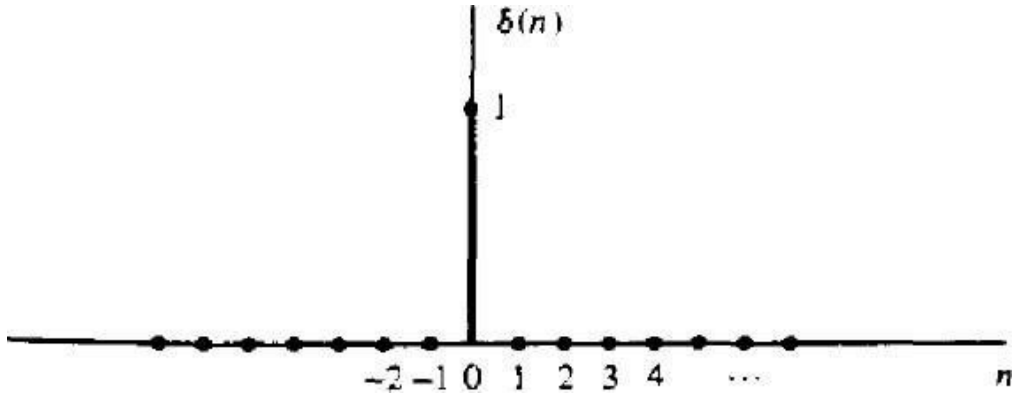
SomeElementaryDiscrete-TimeSignals:

Indiscrete-time signals and systems there are a number ofbasic signals that appearoftenand play an important role. These signals are defined below .

1. Unitsamplesequence/unitimpulse:Itisdenotedasδ(n)andisdefinedas

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

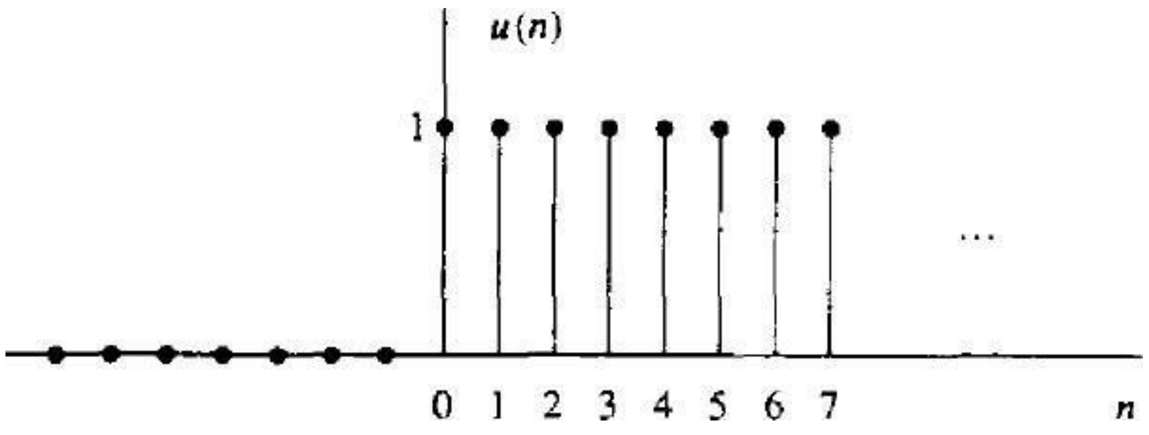
theunitimpulsesequenceisasignal thatis zeroevery where,exceptatn =0whereitsvalueis unity. The graphical representation of *δ(n)* is



2. **Unitstepsignal:**Itisdenotedasu(n)andisdefined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

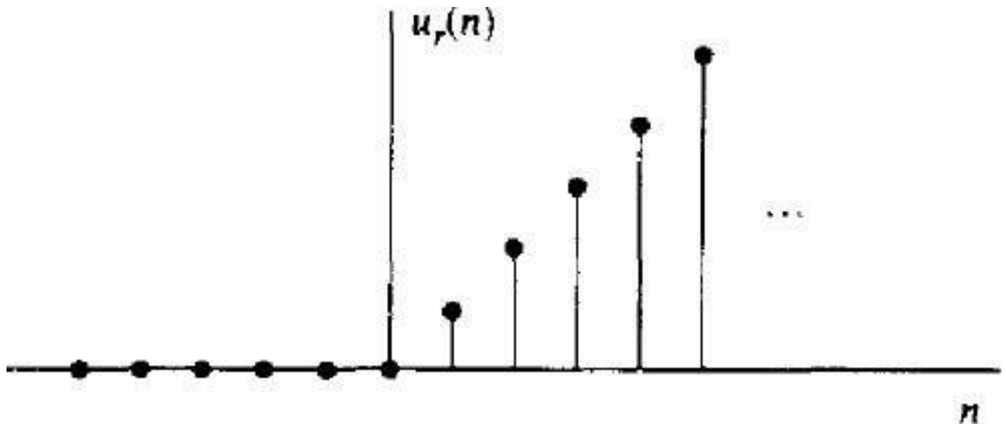
Thegraphicalre presentationof*u(n)*is



3. **Unitrampsignal***:* Itisdenotedas $u_r(n)$ andisdefinedas

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

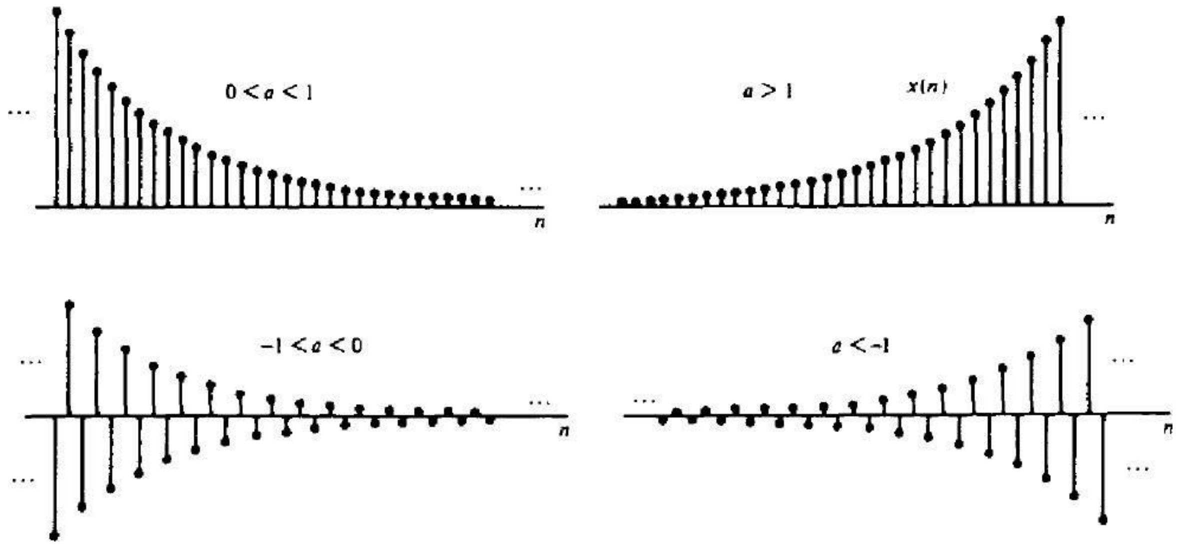
Thegraphicalrepresentationofu_r(n)is



4-Exponentialsignal: It isasequenceoftheform

$$x(n) = a^n \qquad \text{for all } n$$

Ifthe parameter *a* is real, then x(n) is a realsignal. illustratationof *x(n)* for various values ofthe parameter *a is*



Whentheparameter*a*iscomplexvalued, itcanbeexpressedas

$$a \equiv re^{j\theta}$$

where *r*and *Ɵ* arenowtheparameters.Hence wecanexpress*x(n)*as

$$x(n) = r^n e^{j\theta n}$$
$$= r^n(\cos\theta n + j\sin\theta n)$$

ClassificationofDiscrete-TimeSignals:

1- **Energysignalsand powersignals:**TheenergyEofasignal *x(n)*isdefined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

If*E*isfinite(i.e.,0<*E*<∞),if*E*isfinite,*P*=0.thenx(n) iscalledan*energysignal.*

Manysignals that possess infinite energy, have a finite average power. The average powerofa d iscrete-time signal *x(n)* is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Ifwedefine thesignalenergyof*x(n)*overthefiniteinterval—*N*<*n*<*N*as

$$E_N \equiv \sum_{n=-N}^{N} |x(n)|^2$$

theaverage powerofthe signal*x(n)*as

$$P \equiv \lim_{N\to\infty} \frac{1}{2N+1} E_N$$

if*E*is infiniteand *P*is finite.thesignaliscalleda *powersignal.*

2- **Periodicsignalsandaperiodic signals:**

signal*x(n)* isperiodicwithperiod *N(N*>0) ifandonlyif the

$$x(n+N) = x(n) \text{ for all } n$$

sinusoidal signal of the form

$$x(n) = A\sin 2\pi f_0 n$$

isperiodicwhenf$_0$,isarationalnumber,thatis,iff$_0$canbeexpressedas

$$f_0 = \frac{k}{N}$$

where*k*and*N*areintegers.

3- **Symmetric(even)andantisymmetric(odd)signals:**

Arealvalued signal*x(n*)iscalled symmetric (even)if

$$x(-n) = x(n)$$

Ontheotherhand ,asignal*x(n)*iscalledantisymmetric(odd)if

$$x(-n) = -x(n)$$

We can illustrate that any arbitrary signal can be expressed as the sum of two signal components, oneofwhichisevenandtheotherodd. Theevensignalcomponent is formedby adding x(n) to *x (—n)* and dividing by 2. that is.

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

Similarly,we formanoddsignalcomponent *$x_0(n)$*accordingtotherelation So

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

we obtain x(n),that is,

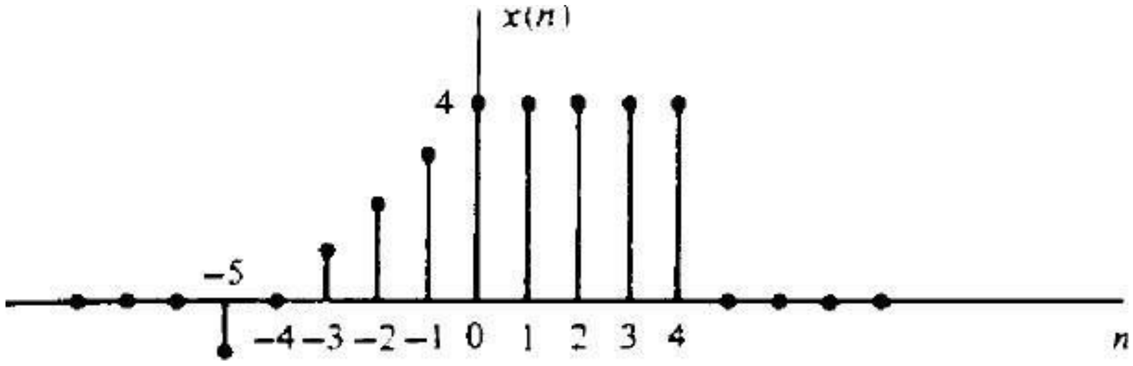
$$x(n) = x_e(n) + x_o(n)$$

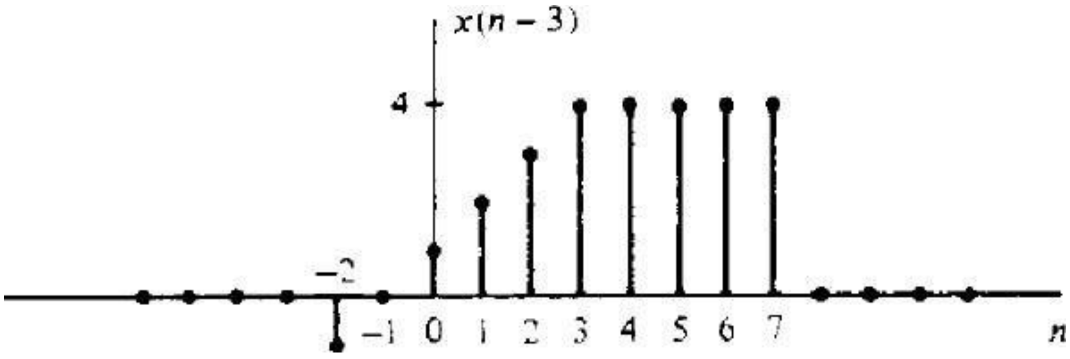
SimpleManipulationsofDiscrete-TimeSignals:

Timeshifting:

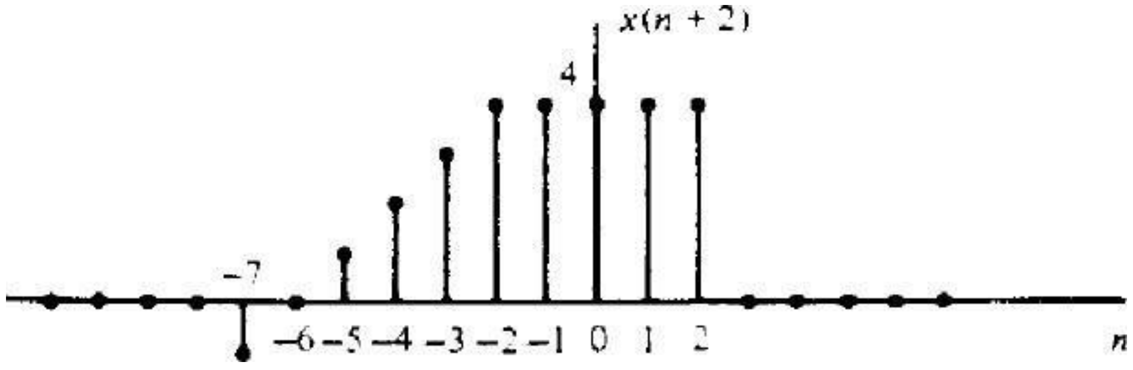
A signal *x (n)* may be shifted in time by replacing the independent variable *n* by *n — k,* w here *k* isan integer.Ifk isa positive integer,the time shift results ina delayofthe signalby *k* unitso ftime. If*k* is a negative integer, thetime shift results inanadvanceofthe signalby \k\\ units in time.

Ex- Asignal *x(n)* is graphically illustrated in Fig. below. Show a graphicalrepresentationofthe signals *x(n* — 3) and *x(n* + 2).





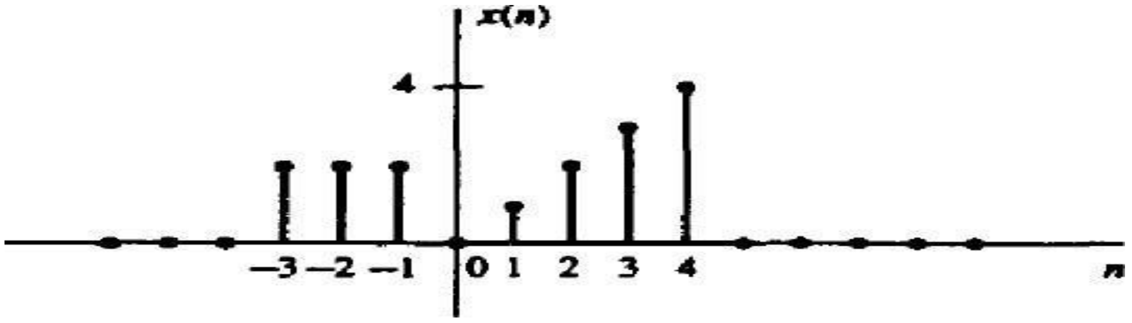
x(n − 3)
4
−2
−1
0
1
2
3
4
5
6
7
n

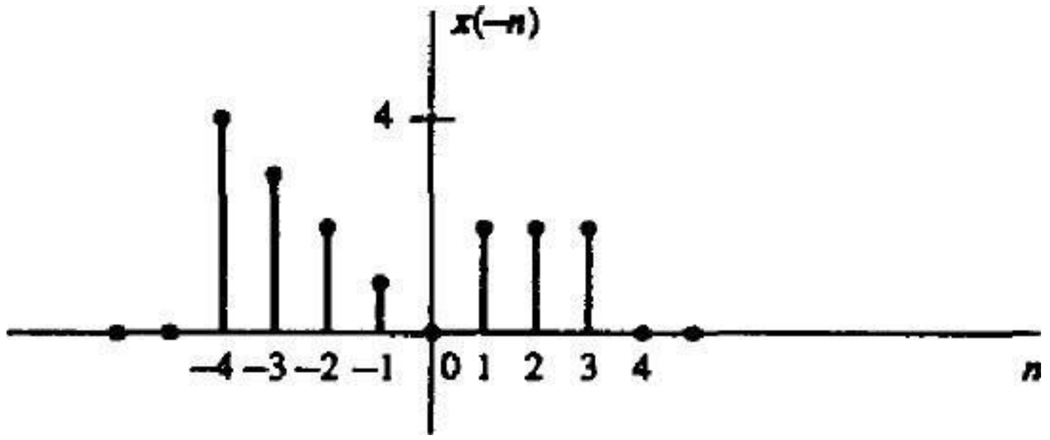


The signal *x* (n — 3) is obtained by delaying x(n) by three units in time. On the other hand, the signal *x(n* + 2) is obtained by advancing *x (n)* by two units in time. Note that delay correspondsto shiftingasignaltotheright, whereasadvance implies shiftingthesignaltothe left on the time axis.

TimeFolding:Theoperationsoffoldingisdefinedby FD[x(n)] = *x (— n)*

Example:





Addition,multiplication, andscalingofsequences:

Amplitude modifications include *addition, multiplication,* and *scaling* o f discrete-time signals. *Amplitude scaling* o f a signal by a constant *A* is accomplished by multiplying the value o f every signal sample by *A*.

$$y(n) = Ax(n) \qquad -\infty < n < \infty$$

The sumoftwo signals x1(n) an d x2(n) is a signal y(n), whose value at any instant is equal to the sum of the values of these two signals at that instant, that is.

$$y(n) = x_1(n) + x_2(n) \qquad -\infty < n < \infty$$

Theproductoftwosignals issimilarlydefinedonasample-to-samplebasisas

$$y(n) = x_1(n)x_2(n) \qquad -\infty < n < \infty$$

DISCRETE-TIMESYSTEMS:

A*discrete-time system* is a device or algorithmthat operates on a discrete -time signal, called the *input*o r*excitation,* according to some w ell-defined rule,to produce another discrete-time signal called the *output* or *response* of the system .

We say that the input signal *x(n)* is *Transformed* by the system in to a signal y(n), and the general relationship Between x(n) and *y(n*) as

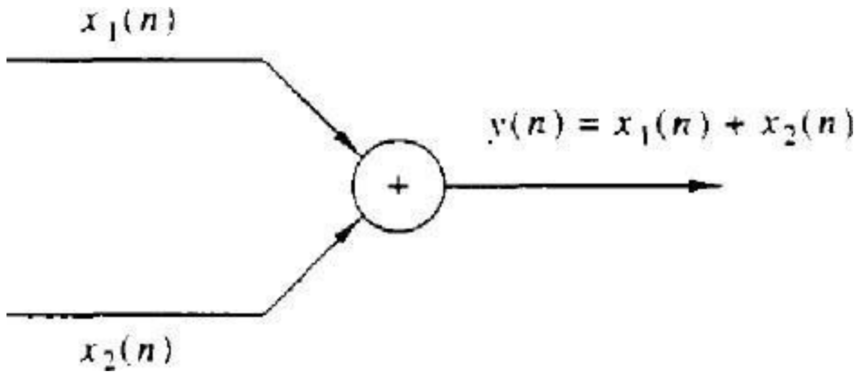
$$y(n) \equiv \mathcal{T}[x(n)]$$

where the symbol*T* denotes the transformation (also called an operator), or processing performed by the system on x(n) to produce y(n).

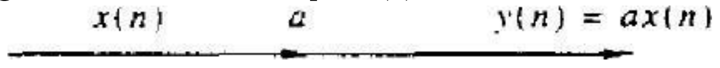
RepresentationofDiscrete-TimeSystems:

It is useful at this point to introduce a block diagram representation of discrete time systems. For this purpose we need to define some basic building blocks that can be interconnected to form complex systems.

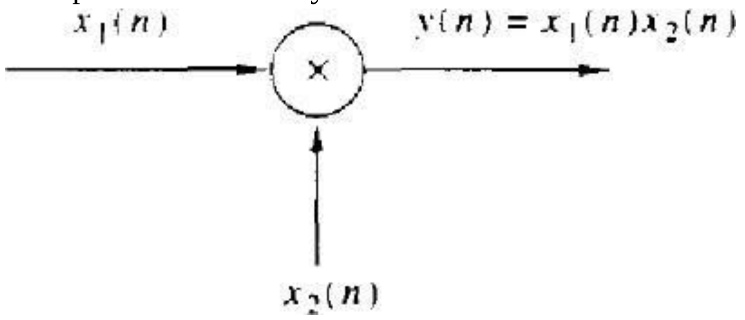
An adder: Figure below illustratesa system(adder)that performs the additiono ftwo signal sequences to form another (the sum) sequence, which we denote as y(n).



A constant multiplier: This operation is depicted by below Fig., and simply represents applying a scale factor on the input *x (n).*



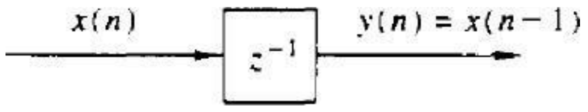
A signal multiplier: Figure below illustrates the multiplication of two signal sequences to form another (the product) sequence, denoted in the figure as y(n). we can view the multiplicationoperation as memory less.



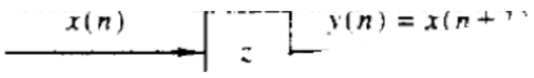
Aunit delay element: Theunit delayisaspecialsystemthat simplydelaysthesignalpassing th rough it byone sample. Fig. below illustrates such a system.If the input signal is *x(n),* the output is *x(n* — 1). In fact, the sample *x{n* — 1) is stored in memoryat time *n* — 1 an d it is recalled fro m memory at time *n* to form y(n),

$$y(n) = x(n-1)$$

The useofthe symbolz^{-1}todenotethe unitofdelay



Aunit advance element: In contrast to the unit delay, a unit advance moves the input *x (n*) ahead byone sample in time to yield *x(n* + 1). Fig. below illustrates this operation , with the operator z being used to denote the unit advance.



ClassificationofDiscrete-TimeSystems:

TherearevarioustypesofDiscrete-TimeSystemssuch as

1- Staticversusdynamicsystems:
Adiscrete-time systemis called *static* ormemoryless ifits output at anyinstant ndepends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be *dynamic* or to have memory.T h e systems described by the following input-output equations are both static or memory less

$$y(n)=ax\{n)$$
$$y(n)=nx(n)+bx^3(n)$$

Ontheother hand, thesystemsdescribed bythe following input-output relationsaredynamic systems or systems with memory.

$$y(n) = x(n) + 3x(n-1)$$

$$y(n) = \sum_{k=0}^{n} x(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

Time-invariant versustime-variant systems: Wecansubdividethegeneralclassofsystems in tothe two broad categories, time -invariant systems and time -variant systems. Asystemis called time-in variant if its input-output characteristics do not change with time. A relaxed system *T* is *time invariant* o r*shift invariant* if and only if

$$x(n) \xrightarrow{T} y(n)$$

impliesthatforeveryinp ut signalx(n)a nd everytimeshift*k.*

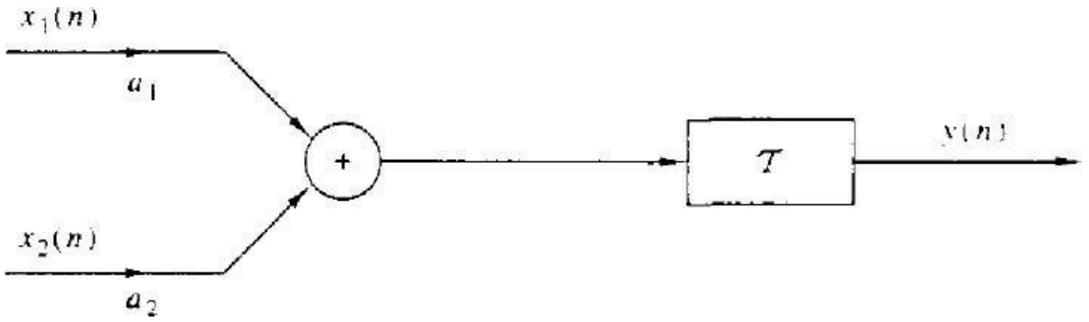
$$x(n-k) \xrightarrow{T} y(n-k)$$

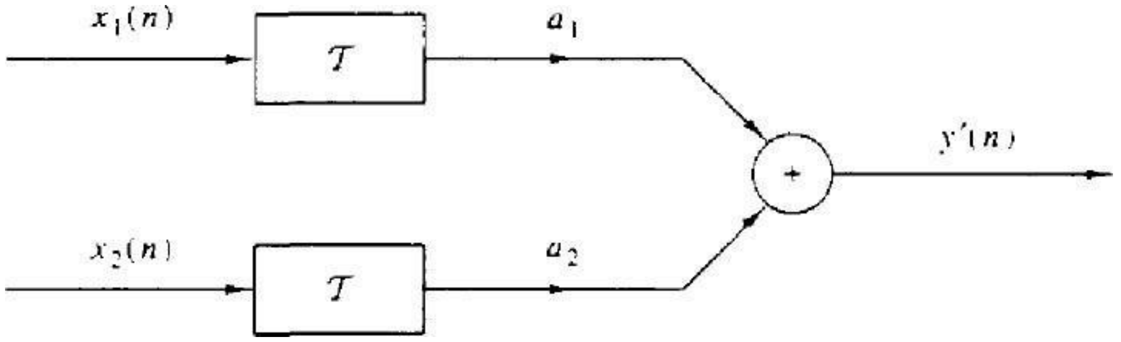
Now if this output y{n, k) = y{n — k), for all possible values o f k, the system is time invariant. O nthe other hand , ifthe output y(n, k) ≠ y(n — k), even for one value o fk, the system is time variant.

Linearversus nonlinearsystems: The generalclass o fsystems canalso be subdivided into linear systems and nonlinear systems. Alinear system is one that satisfies the *superposition principle.* Simply stated, the principle o f superposition requires that the response o f the system to a weighted sum o f signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual inputsignals**.**Arelaxed T system is linear if and only if

$$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$$

foranyarbitraryinputsequences $x_\backslash (n)$and $x_2(n)$,andanyarbitraryconstants$a_1$and a_2.





Causalversusnoncausalsystems:

Asystemissaid to be *causal* iftheoutputofthesystemat anytime *n* [i.e., *y(n)]* dependsonly on presentand past inputs [i.e.,*x { n*), *x(n* - 1),*x(n* — 2) , . . .] , but does not depend on future inputs [i.e., *x(n* + 1), *x(n* + 2) , . . .] . In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

Ifa systemdoes not satisfythis definition, it is called *noncausal.* Sucha systemhas anoutput tha t depends not onlyon present and past inputs but also on future inputs.

Stableversusunstablesystems:

Anarbitraryrelaxed systemis said to be stable ifand only ifeverybounded input produces a bounded output (i:e; BIBO).

The conditions that the input sequence *x{n)* and the output sequence *y(n)* are bounded is transla ted mathematically to mean that there exist some finite numbers, say *M x* and *M y.* such that

$$|x(n)| \le M_x < \infty \qquad |y(n)| \le M_y < \infty$$

for all *n.* If. for some bounded input sequence ,x(n), the output is unbounded (infinite), the system is classified as unstable .

DISCRETE-TIMELINEARTIME-INVARIANTSYSTEMS:

The linearity and time-invariance properties of the system , the response of the system to any arb itrary input signal can be expressed in terms of the unit sample response of the system . The gen eral form of the expression thatrelates the unitsample response of the system and the arbitrary input signal to the output signal, called the convolution sum or the convolution formula,isalsoderived.Thusweareabletodeterminetheoutputofanylinear,timeinvariantsystemtoanyarbitraryinputsignal.

ResponseofLTISystemstoArbitraryInputs:

TheConvolutionSum:

An arbitrary input signal *x(n)* in to a weighted sum of impulses, We are now ready to determine the response of any relaxed linear system to any Input signal. First, we denote the response y(n, k) of the system to the input unit Sample sequence at*n* = k by the special symbol *h(n, k)*, -∞<*k* < ∞. T h a t is,

$$y(n,k) \equiv h(n.k) = \mathcal{T}[\delta(n-k)]$$

iftheinputisthearbitrarysignalx(n) thatisexpressed asasumofweightedimpulses,that is.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

thenthe responseofthe systemto*x(n)*isthe corresponding sumofweighted outputs,thatis,

$$\begin{aligned} y(n) = \mathcal{T}[x(n)] &= \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n.k) \end{aligned}$$

Clearly, the above equation follows from the superposition property of linear systems, and is know n as the *superposition summation.*th en by the time-invariance property , the responseof the system to the delayed unit sample sequence δ(n - *k)* is

$$h(n-k) = \mathcal{T}[\delta(n-k)]$$

Consequently, the *superpositionsummation*formulainreducesto

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Theaboveformulagivestheresponse*y(n)*oftheLTIsystemasafunction oftheinputsignal *x(n)*andtheunitsample(impulse)response*h(n)* iscalleda*convolutionsum.*

To summarize,theprocessofcomputingtheconvolutionbetween*x (k)*and*h(k)* involvesthe following four steps**.**

1. *Folding*.**Fold***h(k)***about***k***=0 toobtain***h(-k).*
2. *Shifting*,Shift*h(—k)*byn$_0$totheright(left) ifn$_0$ispositive(negative),toobtain*h(*n$_0$— k).
3. *Multiplication.* Multiply*x(k)* by*h(*n$_0$—k)toobtaintheproductsequence$v_{n0}(k) = x(k)\,h($n$_0$— k).
4. *Summation.*Sumallthevaluesoftheproductsequence$v_{n0}(k)$toobtainthevalueofthe output at time n = n$_0$.

Example:

Theimpulseresponseofalineartime-invariantsystemis

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response ofthe systemto the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Solution : We shall compute the convolution according to its formula. But we shall usegraphs of the sequences to aid us in the computation. In Fig. below we illustrate the input signal sequence *x(k)* and the impulse response *h{k)* of the system, using *k* as the time index. The first step in the computation of the convolution sum is to fold *h(k).* The folded sequence *h(-k)* is illustrated inconsequent figs . Now we can compute the output at *n* = 0. according to the convolution formula which is

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

Sincetheshift *n*=0,weuse*h(—k)*directlywithoutshifting it.Theproduct sequence We

$$v_0(k) \equiv x(k)h(-k)$$

continue the computation by evaluating the response of the system at *n* = 1.

$$y(1) = \sum_{h=-\infty}^{\infty} x(k)h(1-k)$$

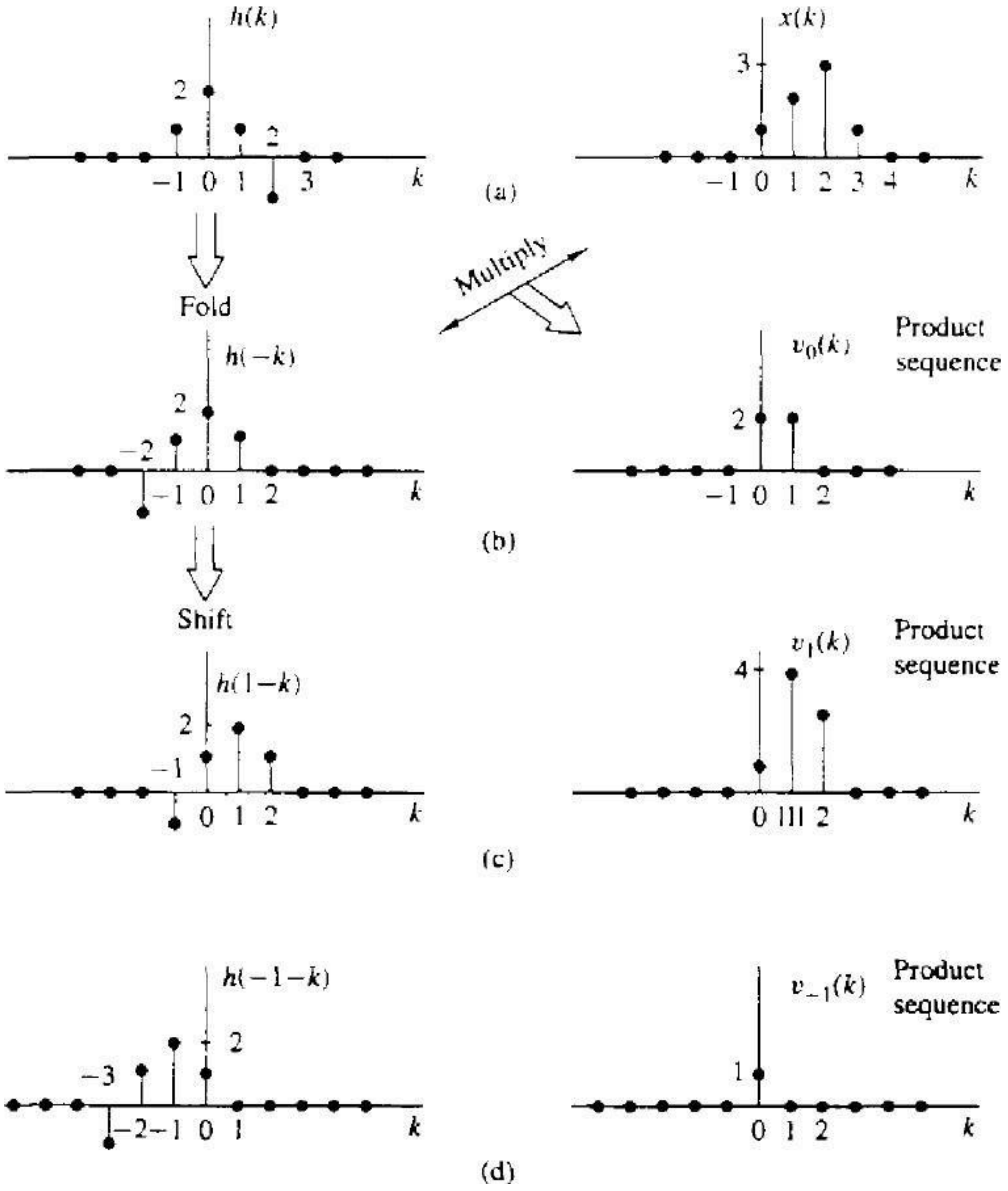
Finally, thesumofallthevaluesintheproductsequenceyields

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

In a similarmanner, we canobtain y(2) byshifting*h (- k)* two units to the right. And y(2) = 8.
Theny(3)=3.y(4)=-2,y(5)=-1.For*n*>5,wefindthaty(n)=0becausetheproduct sequences contain all zeros.

Nextwewishtoevaluate y(n)for*n*<0.Webeginwith*n*=*-1.Then*

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$



$$y(0) = \sum_{h=-\infty}^{\infty} v_0(k) = 4$$

Finally,summingoverthevaluesoftheproductsequence,weobtain then

$$y(-1) = 1$$

$$y(n) = 0 \qquad \text{for } n \le -2$$

Nowwehavetheentire responseofthesystemfor-∞<*n* <∞.whichwesummarizebelow as

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

PropertiesofConvolution:1- Commutative law :

$$x(n) * h(n) = h(n) * x(n)$$

2- Associativelaw:

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

3-Distributivelaw:

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Finite-DurationandInfinite-DurationImpulseResponsesystem:

Linear time-invariant system s into two types, those that have a finite-duration Impulse response (FIR) and those that have an infinite-duration impulse response(IIR). Thus an fir systemhas an impulse response that is zero outside o f some Finite time interval.

StabilityandunstableLinearTime-InvariantSystems:

We defined an arbitraryrelaxed systemas BIBO stable if and only if its output sequence y(n) is bounded for every bounded input *x(n).*

Theoutputisboundediftheimpulseresponseofthesystemsatisfiesthecondition

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

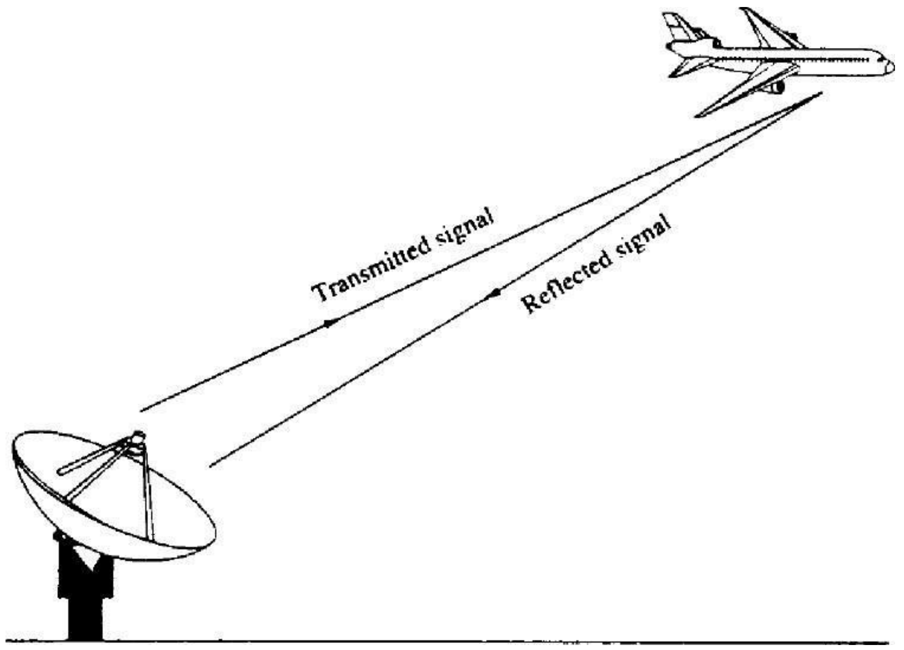
That is,*alineartime-invariantsystemisstableif itsimpulseresponseisabsolutely summable*

.

CORRELATIONOFDISCRETE-TIMESIGNALS:

Amathematicaloperationthat closelyresembles convolution is correlation.Just as inthe case ofconvolution,two signalsequences are involved incorrelation. correlationbetweenthetwo signals is to measure the degreeto which the two signals are similar and thus to extract some in formation that depends to a large extent on the application. Correlation o f signals is often encountered inradar, sonar, digitalcommunications, geology, ando the rare asin science and engineering .

Let us suppose that we have two signal sequences x(n) and y(n) that we wish to compare. In radar and active sonar applications. x(n) can represent the sampled version of thetransmitted signaland y{n) canrepresent the sampled versionofthe received signalat the output of the analog -to -digital (A /D) converter. If a target is p resent in the space being searched by the radar or sonar, the received signal y(n) consists of a delayed version of the transmitted signal, reflected from the target.



This comparison process is performed by means of the correlation operation of 2 different types.

Cross-correlationandAutocorrelationSequences:
Suppose that we have two real signal sequences *x(n)* and *y(n)* each of which has finite energy. T he*cross-correlation* o f*x(n)* and y(n) is a sequence $r_{xy}(l)$, which is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \qquad l = 0, \pm1, \pm2, \ldots$$

or,equivalently,as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n) \qquad l = 0, \pm1, \pm2, \ldots$$

The indexlisthe(time) shift (or *lag)*parameter andthesubscripts *xy*onthecross-correlation se quencer$r_{xy}(l)$, indicate the sequences being correlated .If we reverse the roles of x(n) an d y(n) and there fore reverse the order of the indices xy. we obtain the cross-correlation sequence

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l)$$

or,equivalently,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n)$$

Bycomparing theabove4 equationsweconcludethat

$$r_{xy}(l) = r_{yx}(-l)$$

Hence , $r_{yx}(l)$ provides exactlythe same informationas $r_{xy}(l)$,withrespecttothe similarityof $x(n)$ to y(n).

Example:
Determinethecross-correlationsequence $r_{xy}(l)$ ofthe sequences

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Solution: Letususethedefinitionofcross-correlationtocompute$r_{xy}(l)$.For$l$=0wehave

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

Theproduct sequence$v_0(n)=x(n)y(n)$is

$$v_0(n) = \{\ldots, 0, 0, 2, 1, 6, -14, \underset{\uparrow}{4}, 2, 6, 0, 0, \ldots\}$$

andhence thesumoverall valuesof$n$is

$$r_{xy}(0) = 7$$

For $l > 0$, we simply shift y(n) to the right relative to x(n) hy l units, compute the product sequence $v_l(n)$ = x(n)y(n— l), and finally, sumover all valueso fthe product sequence. Thus we obtain

$$r_{xy}(1) = 13, \quad r_{xy}(2) = -18, \quad r_{xy}(3) = 16, \quad r_{xy}(4) = -7$$
$$r_{xy}(5) = 5, \quad r_{xy}(6) = -3, \quad r_{xy}(l) = 0, \quad l \geq 7$$

For l< 0, we shift$y(n)$ to the left relative to x(n) by l units, compute the product sequence $v_l(n)$ = x(n)y(n — l), and sum over all values of the product sequence. Thus we obtain the values of the cross-correlation sequence

$$r_{xy}(-1) = 0, \quad r_{xy}(-2) = 33, \quad r_{xy}(-3) = -14, \quad r_{xy}(-4) = 36$$
$$r_{xy}(-5) = 19, \quad r_{xy}(-6) = -9, \quad r_{xy}(-7) = 10, \quad r_{xy}(l) = 0, \ l \leq -8$$

Therefore,thecross-correlationsequenceof$x(n)$ andy(n)is

$$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, 0, \underset{\uparrow}{7}, 13, -18, 16, -7, 5, -3\}$$

Thentheconvolutionof$x(n)$with y(—n)yieldsthecross-correlation$r_{xy}(l)$that is,

$$r_{xy}(l) = x(l) * y(-l)$$

Autocorrelation*n:*

when*y(n) = x(n),*wehavethe*autocorrelation* ofx(n),whichisdefinedasthesequence

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

or, equivalently, as

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n)$$

For finite-durationsequences,

$$r_{xy}(l) = \sum_{n=i}^{N-|k|-1} x(n)y(n-l)$$

and

$$r_{xx}(l) = \sum_{n=i}^{N-|k|-1} x(n)x(n-l)$$

where *i =l*,*k*=0 forl>0,andi=0,*k=l*forl<0.

PropertiesoftheAutocorrelationandCrosscorrelationSequences:

1- Thecross-correlationsequencesatisfiestheconditionthat

$$|r_{xy}(l)| \le \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_x E_y}$$

when y(n)=*x (n)*,reducesto

$$|r_{xx}(l)| \le r_{xx}(0) = E_x$$

2- Thenormalizedautocorrelationsequenceisdefinedas

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}$$

Similarly,wedefinethenormalizedcross-correlationsequence

$$\rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$$

Now$\backslash\rho_{xx}\{l)\backslash$<1and$\backslash\rho_{xy}\{l)\backslash$<1,andhencethesesequencesare independent ofsignalscaling. 3-the

cross-correlation sequence satisfies the property

$$r_{xy}(l) = r_{yx}(-l)$$

theautocorrelationsequencesatisfiesthe property

$$r_{xx}(l) = r_{xx}(-l)$$

Hencetheautocorrelationfunctionisanevenfunction.

<u>MODULE-2</u>

<u>TheOne-sidedz-Transform:</u>

Theone-sided orunilateralz-transformofa signalx(n)isdefinedby

$$X^+(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1.1)$$

Properties:

1. Itdoesnotcontaininformationaboutthesignalx(n)fornegativevaluesof time.
2. Itisuniqueonlyfor causalsignals.
3. Theone-sidedz-transform$X^+(z)$ofx(n)isidenticaltothetwo-sidedz-transform of the signal x(n)u(n). **Shifting Property:**

❖ Timedelay:

If $x(n) \overset{z^+}{\leftrightarrow} X^+(z)$

$\overset{+}{z}$then $x(n-k) \leftrightarrow z^{-k}[X^+(z)+\sum^{k}$

] k>0 ……(1.2)

$(-n)z^{n}{}_{n=1}$

Incasex(n)isacausalsignal

then $x(n-k) \overset{z^+}{\leftrightarrow} z^{-k}X^+(z)$ k>0 ....(1.3)

❖ Timeadvance:

$$x(n+k) \overset{z^+}{\leftrightarrow} z^{k}[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}] \qquad k>0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

FinalValueTheorem:

If $x(n) \overset{z^+}{\leftrightarrow} X^+(z)$

then $\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\, X^+(z)$...........................(1.5)

Thelimitexists iftheROCof$(z-1)^{+}(z)$includestheunitcircle.

AnalysisofLTISysteminz-domain:

ResponseofSystemswith RationalSystem:

Weconsideralinearconstantcoefficientdifferenceequation:

$$(n)=-\sum_{k=1}^{N} a_k y(n-k)+\sum_{k=0}^{M} b_k x(n-k)$$ ………………………. (2.1)

correspondingsystemfunctionH(z)isgivenby

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$ …………………………………………..(2.2)

weapplyaninput signalx(n)whosez-transformisX(z).For $X(z) = \frac{N(z)}{Q(z)}$, $H(z) = \frac{B(z)}{A(z)}$and zero initial conditions, the z-transform of the output of the system has the form

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$ ………………………………….... (2.3)

Supposethesystemcontainssimple poles $p_1,p_2,\ldots\ldots\ldots,p_N$andX(z)containspoles $q_1,q_2,\ldots\ldots,q_L$,where$p_k\neq q_m$forallk=1,2,……,Nandm =1,2,……,L. Assuming no pole-zero cancellation the partial fraction expansion ofY(z) yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1-q_k z^{-1}}$$ ………….......................... (2.4)

TheinversetransformofY(z)istheoutputsignaly(n)fromthe system:

$$y(n) = \sum_{k=1}^{N} A_k(p_k)^n u(n) + \sum_{k=1}^{L} Q_k(q_k)^n u(n)$$ …………………... (2.5)

wherescalefactors{A_k}and{Q_k}arefunctionsofbothsetsofpoles{p_k}and {q_k}.

ResponseofPole-ZeroSystemswithNon-zeroInitialConditions:

Weconsidertheinputsignalx(n)tobeacausalsignalappliedatn=0.Theeffectsofall previousinputsignalstothesystemarereflectedintheinitialconditionsy(-1),y(-2),......................... , y(-N).Weareinterestedindeterminingtheoutputy(n)for$n\geq0$.

$$Y^{+}(z)=-\sum^{N} a_k z^{-k}[Y^{+}(z) +\sum^{k} y(-n)z^{n}]+\sum^{M} b_k z^{-k}X^{+}(z)$$

$k=1$ $n=1$ $k=0$

CausalityandStability:

A causallineartimeinvariantsystemisonewhoseunitsampleresponseh(n)satisfiesthe condition

$$h(n) = 0 \qquad n<0$$

AnLTIsystemiscausalifandonlyiftheROCofthesystemfunctionistheexteriorofa circle of radius $r < \infty$, including the point $z = \infty$.

Anecessaryandsufficient conditionforanLTIsystemtobeBIBOstable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

An LTI systemisBIBO stable ifand only ifthe ROCofthe systemfunction includesthe unit circle.

Consequently,acausalandstablesystemmusthaveasystemfunctionthatconvergesfor$|z| > r < 1$.Since the ROC cannot contain any poles of H(z) , it follows that ***a causal linear time-invariant system is BIBO stable if and only if all the poles of H(z) are inside the unit circle***.

<u>The DFT as a Linear Transformation:</u>

The formulasfortheDFTandIDFTmaybeexpressedas

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \quad , \qquad k=0,1,\dots,N-1 \ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad , \qquad n = 0,1,\dots,N-1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.2)$$

where $\qquad W_N = e^{\frac{-j2\pi}{N}}$

whichis anNthrootofunity.

Thecomputationofeachpoint oftheDFTcanbe accomplished byNcomplex multiplications and (N-1)complexadditions. Hence the N-point DFT values canbe computed inatotalofN^2c omplexmultiplicationsandN(N-1)complexadditions.

LetusdefineanN-pointvector$\mathbf{x}_N$ofthesignalsequencex(n),n=0,1,…,N-1,anN-point vector $\mathbf{X}_N$ of frequency samples, and an $N \times N$ matrix $\boldsymbol{W}_N$as

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\boldsymbol{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \quad \dots\dots\dots\dots\dots\dots..(3.3)$$

Withthesedefinitions,theN-pointDFTmaybeexpressedinthematrixformas

$$N \qquad x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0, 0, \dots, 0}_{M-1\ zeros}\} \quad \dots\dots\dots\dots(5.5.1)$$

where $\mathbf{W}_N$is the matrix of the linear transformation. $\mathbf{W}_N$is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$exists, then we also write

$$\mathbf{x}_N = \mathbf{W}_N^{-1}\mathbf{X}_N \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.(3.5)$$

IDFTcanalsobeexpressedas

$$N = L + M - 1 \qquad y(n) = \{y_1(0), y_1(1), \dots, y_1(L-1), y_1(L)+y_2(0), y_1(L+1)+y_2(1), \dots, y_1(N-1)+y_2(M-1), y_2(M), \dots\} \quad \dots\dots\dots\dots\dots..(5.5.5)$$

where$\mathbf{W}_N^*$denotesthecomplexconjugateofthe matrix$\mathbf{W}_N$.Comparisonofequations3.5and 3.6leadsustoconcludethat

$$x(n) = \frac{1}{N}\left\{\frac{V(0)}{2} + \sum_{k=1}^{N-1} V(k)\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right]\right\}, \quad 0 \le n \le N-1 \quad \dots\dots(6.3) \qquad N \times N$$

whichinturnimplies

$$C_N \qquad x(n), 0 \le n \le N-1$$

where$\mathbf{I}_N$isa$N \times N$identitymatrix.

CircularConvolution:

Supposethatwehavetwofinite-durationsequencesoflengthN,$x_1(n)$and$x_2(n)$.Their respective N point DFTs are

$$= 4\left(\frac{N}{2}\log_2 N\right) = 2N\log_2 N$$

$$2 \text{ to } \frac{N}{2}\log_2 N$$

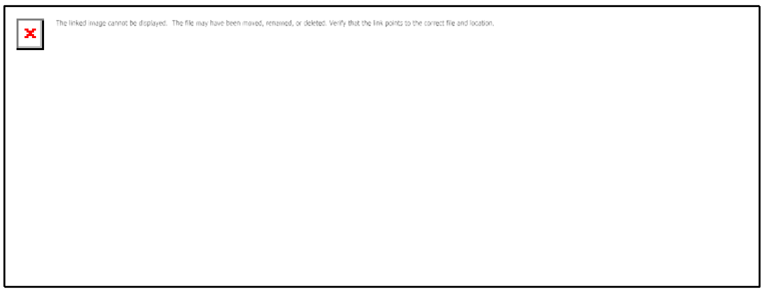
$$= 2N\log_2 N + 2\left(\frac{N}{2}\log_2 N\right) = 3N\log_2 N$$

$$12 \left(= \frac{8}{2}\log_2 8 = 4 \times 3\right)$$

MultiplyingtheabovetwoDFTswe get:

$$X_3(k) = X_1(k)X_2(k) \quad , \qquad k = 0,1,\dots,N-1 \quad \dots\dots\dots\dots\dots\dots\dots (4.2)$$

IDFTof$\{X_3(k)\}$is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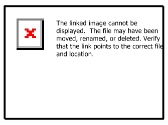
Substitutingfor$X_1(k)$and$X_2(k)$in(4.3)usingDFTsgivenin(4.1)and(4.2),weobtain

$$W_N^k \;)$$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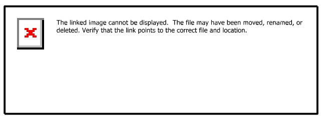
Solution The twiddle factors are

$W_8^0 = 1$ $\qquad$ $W_8^1 = e^{-j2\pi/8} = e^{-j\pi/4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$ $\qquad$ $W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

Theinner suminthebracketsin(4.4) hastheform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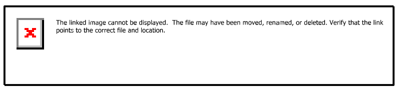
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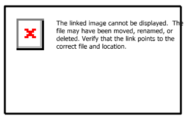
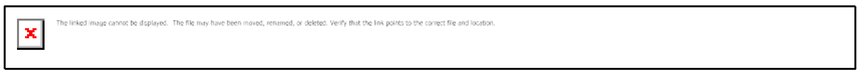
wh

ere$a$isdefinedas

$$a = e^{j2\pi(m-n-l)/N}$$

Consequently,





Ifwesubstitutetheresultin(4.6)into(4.4),weobtain

$$\boldsymbol{x_3(m)} = \sum_{\boldsymbol{n=0}}^{\boldsymbol{N-1}} \boldsymbol{x_1(n)x_2((m-n))_N} \quad , \qquad m = 0,1,\dots,N-1 \quad \dots\dots\dots..(4.7)$$

The above convolution sum is called ***circular convolution***. Thus we conclude that ***multiplicationof the DFTsof two sequencesis equivalent to the circular convolution of the two sequences in the time domain.***

<u>LinearFilteringMethodsBasedontheDFT</u>:

UseoftheDFTinLinearFiltering:

Suppose we have a finite-duration sequence x(n) of length L which excites an FIR filter of length M. Let

$$x(n) = 0, \quad n < 0 \; and \; n \geq L$$

$$h(n) = 0, \quad n < 0 \; and \; n \geq M$$

where$h(n)$istheimpulseresponseoftheFIRfilter. The

output sequence $y(n)$of the FIR filter:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.1)$$

The duration of$y(n)$is$L + M - 1$.

Thefrequency-domainequivalentto(5.1)is

$$Y(\omega) = X(\omega)H(\omega) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.2)$$

If the sequence $y(n)$is to be represented uniquely in the frequency domain by samples of its spectrum $Y(\omega)$at a set of discrete frequencies, the number of distinct samples must equal or exceed $L + M - 1$.Therefore, a DFT of size $N \geq L + M - 1$is required to represent {y(n)} in the frequency domain.

Nowif

$$X(z_k) = V^{-k^2/2}y(k) = \frac{y(k)}{h(k)} \qquad k = 0,1,\ldots,L-1$$

then

$$Y(k) = X(k)H(k)\,, \qquad k = 0,1,\ldots,N-1 \qquad \ldots\ldots\ldots\ldots\ldots.. (5.3)$$

where $\{X(k)\}$and $\{H(k)\}$are the N-point DFTs of the corresponding sequences x(n) and h(n), respectively. Since the sequences x(n) and h(n) have a duration less than N, we simply pad these sequences with zeros to increase their length to N.
Since the ($N = L + M - 1$)-point DFT of the output sequence y(n) is sufficient to represent y(n) inthe frequencydomain, it follows that the multiplicationofthe N-point DFTs X(k) and H

(k)followedbythecomputationoftheN-pointIDFT,mustyieldsequence{y(n)}.

Thus, ***the N-point circular convolution of x(n) with h(n) must be equivalent to the linear convolution of x(n) with h(n)***. *Thus with zero padding, the DFT can be used to performlinear filtering.* **<u>Filtering of Long Data Sequences:</u>**

LettheFIRfilterhasdurationM.TheinputdatasequenceissegmentedintoblocksofL points, where , by assumption, $L \gg M$. **Overlap-save method:**

Sizeofinputdatablocks,$N = L + N - 1$DFTs

and IDFTs are of length N.

Eachdatablock consists ofthe last $M - 1$data pointsofthe previous data block followed by L new data points to form a data sequence of length $N = L + N - 1$. An N-point DFT is computed for each data block.

The impulse response ofthe FIR filter is increased in length by appending$L - 1$zeros and an N-point DFT of the sequence is computed once and stored. The multiplication of the two N-point DFTs {$H(k)$} and {$X_m(k)$} for the mth block of data yields

$$\hat{Y}_m(k) = H(k)X_m(k), \qquad k = 0,1,\dots,N-1 \qquad \dots\dots\dots\dots (5.4.1)$$

ThentheN-pointIDFTyieldsthe result

Since the data record is of length N, the first $M - 1$ points of $y_m(n)$are corrupted byaliasing and must be discarded. The last Lpoints of $y_m(n)$ are exactly same as the result from linear convolution and, as a consequence,

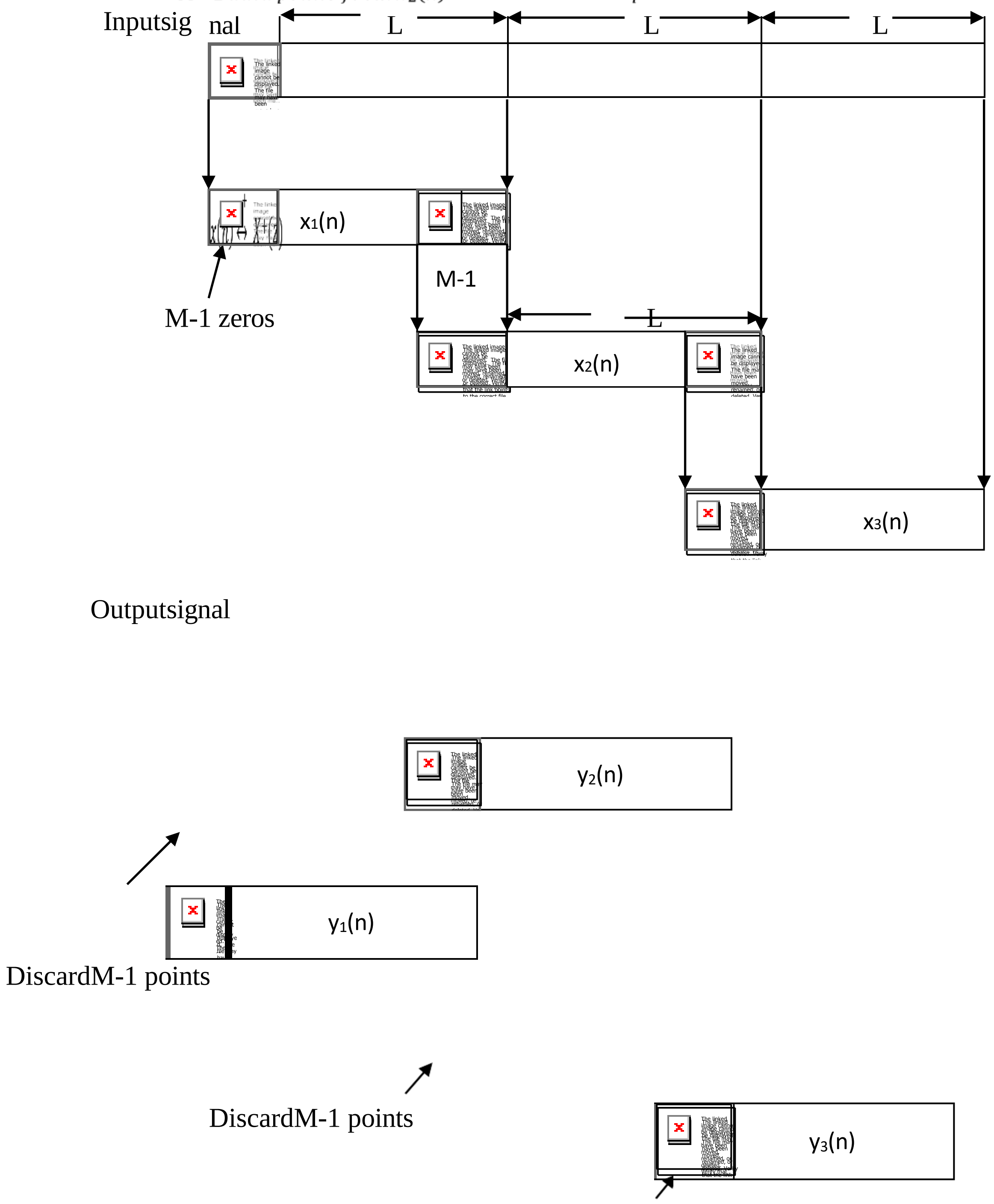
$$\hat{y}_m(n) = y_m(n)\,, \qquad n = M, M+1, \dots, N-1 \quad \dots\dots\dots\dots (5.4.3)$$

To avoid loss of data due to aliasing, the last $M - 1$points of each data record are saved and these points become the first $M - 1$points ofthe subsequent record. To begin the processing, the first $M - 1$points of the first record are set to zero. Thus blocks of data sequences are:

$$x_1(n) = \{\underbrace{0, 0, \dots, 0}_{M-1\ points}, x(0), x(1), \dots, x(L-1)\} \qquad \dots\dots\dots (5.4.4)$$

$$x_2(n) = \{x\,(\underbrace{L-M+1),\ldots,x(L-1}_{M-1\ data\ points\ from\ x_1(n)}),\underbrace{x(L),\ldots,x(2L-1}_{L\ new\ data\ points})\} \quad \ldots\ldots\ldots(5.4.5)$$

$$\square_3(n) = \{x\,(\underbrace{2L-M+1),\ldots,x(2L-1}_{M-1\ data\ points\ from\ x_2(n)}),\underbrace{x(2L),\ldots,x(3L-1}_{L\ new\ data\ points})\} \quad \ldots\ldots\ldots(5.4.6)$$



DiscardM-1points (Linear

FIR filtering by the overlap-save method)

Overlap-addmethod:

Size of input block = L

SizeoftheDFTsandIDFTis$N = L + M - 1$.

To each data block we append $M - 1$zeros and compute the N-point DFT. The data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0,0,\dots,0}_{M-1\ zeros}\} \quad \dots\dots\dots\dots (5.5.1)$$

$$x_2(n) = \{x(L), x(L-1), \dots, x(2L-1), \underbrace{0,0,\dots,0}_{M-1\ zeros}\} \quad \dots\dots\dots\dots (5.5.2)$$

$$x_3(n) = \{x(2L), \dots, x(3L-1), \underbrace{0,0,\dots,0}_{M-1\ zeros}\} \quad \dots\dots\dots\dots (5.5.3)$$

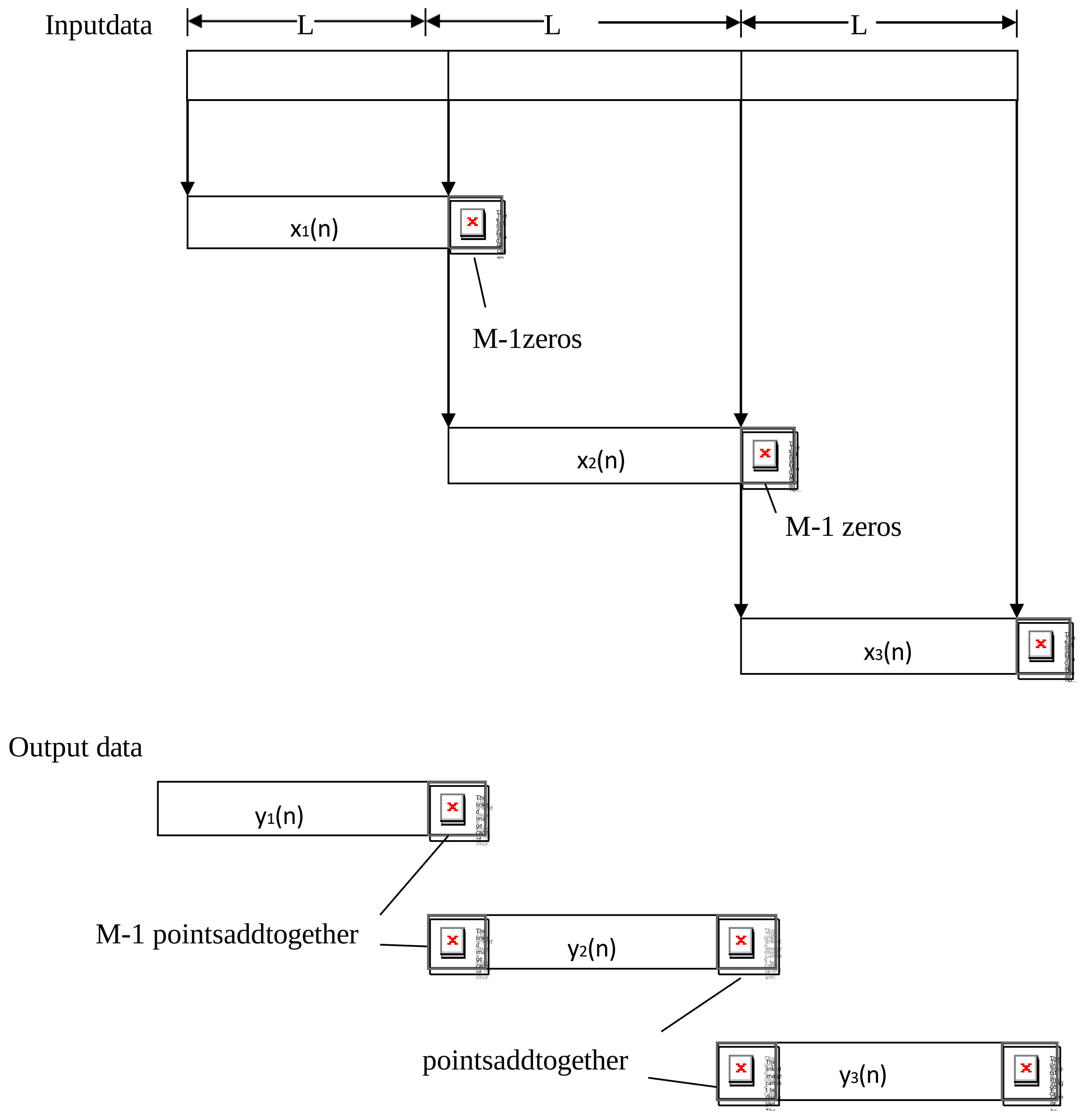
andso on.Thetwo$N$-pointDFTsaremultipliedtogethertoform

$$Y_m(k) = H(k)X_m(k), \qquad k = 0,1,\dots,N-1 \quad \dots\dots..(5.5.4)$$

The IDFT yields data blocks of length Nthat are free of aliasing, since the size of the DFTs and IDFT is $N = L + M - 1$and the sequences are increased to N-points by appending zeros to each block.

Since each data block is terminated with M-1 zeros , the last M-1 points from each output block must be overlapped and added to the first M-1 points of the succeeding block. Hence this method is called the overlap-add method. The output sequence is:

$$y(n) = \{y_1(0), y_1(1), \dots, y_1(L-1), y_1(L)+y_2(0), y_1(L+1)+y_2(1), \dots, y_1(N-1)+y_2(M-1), y_2(M), \dots\} \quad \dots\dots\dots\dots\dots..(5.5.5)$$



(LinearFIRfilteringbytheoverlap-addmethod)

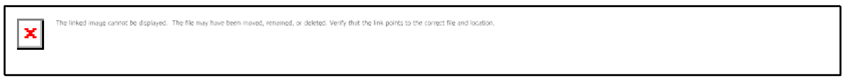
__TheDiscreteCosineTransform__:

ForwardDCT:

LetanN-pointsequencex(n)whichisrealandeven, thatis,

$(n) = x(N-n), 0 \leq n \leq N-1$

Let s(n) bea2N-pointevensymmetricextensionofx(n)definedby



TheDCT of x(n)can becomputedby takingthe2N-pointDFT of s(n)andmultiplyingthe result by $W_{2N}^{k/2}$. The forward DCT is defined by

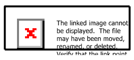
$$V(k) = 2\sum_{n=0}^{N-1} x(n)\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad 0 \le k \le N-1 \quad \ldots\ldots\ldots\ldots\ldots\ldots..(6.2)$$

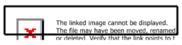
InverseDCT

$$x(n) = \frac{1}{N}\left\{\frac{V(0)}{2} + \sum_{k=1}^{N-1} V(k)\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right]\right\}, \quad 0 \le n \le N-1 \quad \ldots\ldots..(6.3)$$

DCTasan OrthogonalTransform

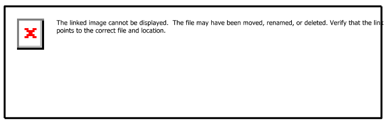
The $N \times N$DCTmatrix $C_N$ofthesequence $x(n), 0 \le n \le N-1$isarealorthogonalmatrix, that is, it satisfies

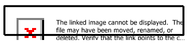




Orthogonality simplifies the computation of the inverse transform because it replaces matrix inversion by matrix transposition. **Circular Correlation :**

If x(n) and y(n) are two periodic sequences, each with period N, then their cross correlationsequence is defined as





Module-III

FastFourierTransformAlgorithms:

1.Introduction

For afinite-durationsequence*x(n)*oflength*N*,theDFTsummaybewritten as

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, k = 0,1,\ldots,N-1$$

Where $W_N = e^{-j2\pi/N}$. There are a total of *N* values of *X(.)* ranging from *X(0)* to *X(N–1)*. The calculationof*X(0)* involvesno multiplicationsat allsinceeveryproducttermTinvolves $W_N^0 = e^{-j0} = 1$. Further, the first term in the sum always involves W_N^0 or $e^{-j0} = 1$ and therefore does not require a multiplication. Each *X(.)* calculation other than *X(0)* thus involves (*N-1*) complex multiplications. And each *X(.)* involves (*N–1*) complex additions. Since there are *N* values of *X(.)* theoverallDFTrequires$(N\text{-}1)^2$complexmultiplicationsand *N(N-1)*complexadditions.Forlarge *N* we mayround these off to N^2 complex multiplications and the same number ofcomplex additions.

Eachcomplexmultiplication isoftheform

(A+jB)(C+jD) = (AC–BD)+j(BC+AD)

and therefore requires four real multiplications and two realadditions. Each complex addition is of the form

(A+jB)+(C+jD)=(A+C)+j(B+D)

and requires two real additions. Thus the computation of all N values of the DFT requires $4N^2$ real multiplicationsand$4N^2$ $(=2N^2+2N^2)$realadditions. Efficient algorithmswhichreducethenumber of multiply-and-add operations are known by the name of **fast Fourier transform** (FFT). The Cooley-Tukey and Sande-Tukey FFT algorithms exploit the following properties of the **twiddle factor**(phase factor), $W_N = e^{-j2\pi/}$(the factor $e^{-j2\pi/N}$is called the *N*th principal root of 1):

1. Symmetry property $W_N^{k+N/2} = -W_N^k$
2. Periodicity property $W_N^{k+N} = W_N^k$

Toillustrate,forthecaseof$N$=8,thesepropertiesresultinthe following relations:

$$W_8^0 = -W_8^4 = 1 \quad W_8^1 = -W_8^5 = \frac{1-j}{\sqrt{2}}$$

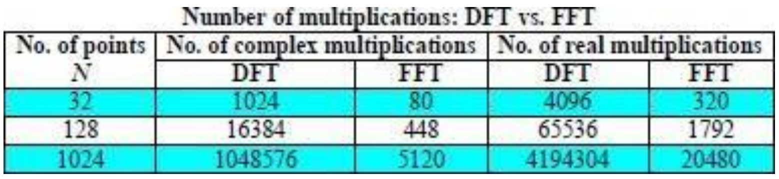
$$W_8^2 = -W_8^6 = -j \quad W_8^3 = -W_8^7 = -\frac{1+j}{\sqrt{2}}$$

Theuseofthesepropertiesreducesthenumber ofcomplexmultiplicationsfromN^2 to $\frac{N}{2}\log_2 N$ (actuallythenumberofmultiplicationsislessthanthisbecauseseveralofthemultiplicationsby $W_N$are really multiplications by ±1 or ±*j* and don't count); and the number of complex additionsare reduced from $N^2$to $N \log_2 N$. Thus, with each complex multiplication requiring four real multiplications and two real additions and each complex addition requiring two real additions, the computation of all *N* values of the DFT requires

Number of real multiplications $= 4\left(\frac{N}{2}\log_2 N\right) = 2N\log_2 N$

Numberofrealadditions $= 2N\log_2 N + 2\left(\frac{N}{2}\log_2 N\right) = 3N\log_2 N$

WecangetaroughcomparisonofthespeedadvantageofanFFToveraDFTbycomputingthe numberofmultiplications foreachsincetheseare usuallymoretimeconsumingthanadditions. For instance, for N =8theDFT,usingtheabove formula,would need82=64complexmultiplications, but the radix-2 FFT requires only $12\ (= \frac{8}{2}\log_2 8 = 4 \times 3)$.



Number of multiplications: DFT vs. FFT

No. of points	No. of complex multiplications		No. of real multiplications	
N	DFT	FFT	DFT	FFT
32	1024	80	4096	320
128	16384	448	65536	1792
1024	1048576	5120	4194304	20480

We consider first the case where the length N of the sequence is an integral power of 2, that is, $N=2^v$where v is an integer. These are called **radix-2 algorithms** of which the **decimation-in-time (DIT)** version is also known as the **Cooley-Tukey algorithm** and the **decimation-in-frequency (DIF)** version is also known as the **Sande-Tukey algorithm**. We show first how the algorithms work; their derivation is given later. For a radix of (r = 2), the **elementary computation** (*EC*) known as the **butterfly** consists of a single complex multiplication and two complex additions.

Ifthe number ofpoints, N, can be expressed as $N = r^m$, and ifthe computation algorithm is carried out by means of a succession of r-point transforms, the resultant FFT is called a **radix*r*algorithm**. In a radix-r FFT, an elementary computation consists of an r-point DFT followed by the multiplication of the r results bythe appropriate twiddle factor. The number of *EC*s required is

$$C_r = \frac{N}{r}\log_r N$$

whichdecreasesas*r*increases.Ofcourse,thecomplexityofan*EC*increaseswithincreasing*r*.For r=4,the *EC*requiresthreecomplexmultiplicationsand severalcomplexadditions.

Supposethatwedesirean$N$-point DFTwhere$N$isacompositenumberthat canbe factored into the product of integers

$N= N1N2\ldots Nm$

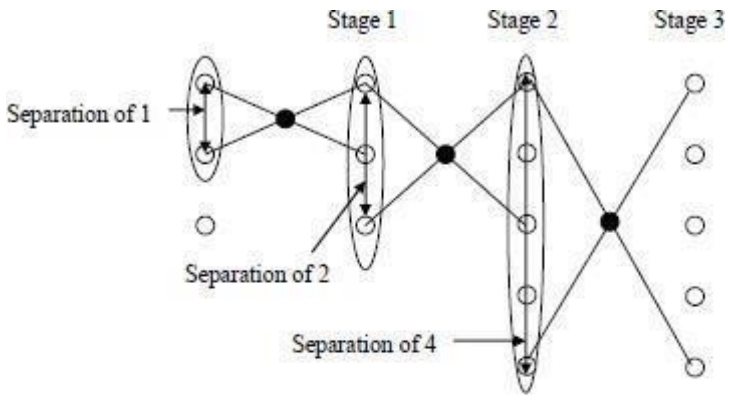
If, for instance, N = 64 and m = 3, we might factor N into the product 64 = 4 x 4 x 4, and the 64-point transform can be viewed as a three-dimensional 4 x 4 x 4 transform. If $N$is a prime numberso that factorization of N is not possible, the original signal can be *zero-padded* and the resulting new composite number of points can be factored.

2. Radix-2decimation-in-timeFFT(Cooley-Tukey)

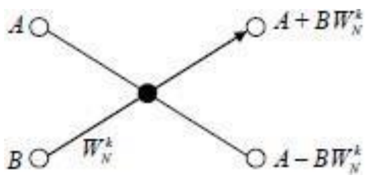
Procedureandimportantpoints

1. Thenumber ofinputsamples is $N= 2^v$where$v$isan integer.
2. The input sequence is shuffled through bit-reversal. The index n of the sequence $x(n)$ isexpressed in binary and then reversed.
3. Thenumberofstagesintheflow graph isgivenby$v$=$\log_2 N$.
4. Eachstageconsistsof$N/2$butterflies.
5. Inputs/outputsforeachbutterflyareseparatedasfollows: Separation=2^{m-1}sampleswhere$m$=stageindex,stagesbeingnumberedfromleftto

right(thatis,*m*=1forstage1,*m*=2forstage2etc.).Thisamountstoseparation increasing from left to right in the order 1, 2, 4… *N/2*.



6. Thenumberofcomplexadditions=$N\log_2 N$and thenumber ofcomplex multiplications $\frac{N}{2}\log_2 N$.
7. The elementarycomputation block in the flow graph, called the butterfly, is shown here. This is an **in-place calculation** in that the outputs$(A + B\,W_N^k)$ and$(A - B\,W_N^k)$can be computed and stored in the same locations as *A* and *B*.



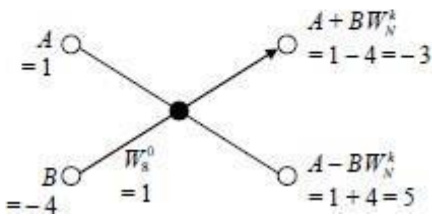
Example1 Radix-2,8-point,decimation-in-time FFTforthesequence

n→01234567*x(n)*= {1,234–4–3–2–1}
Solution The twiddle factors are

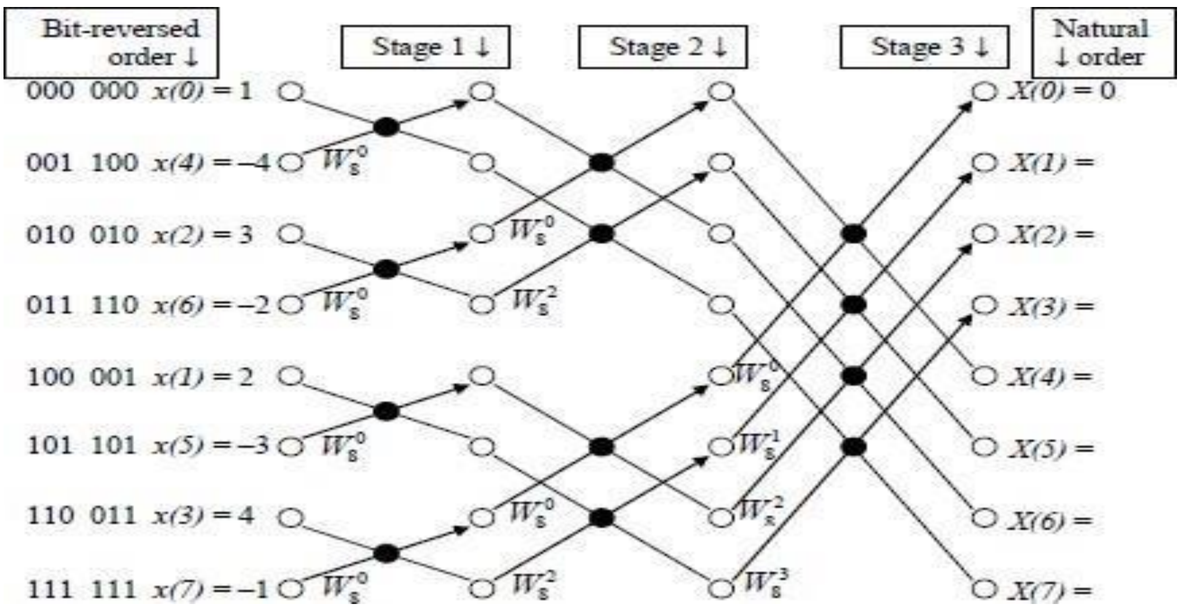
$$W_8^0 = 1 \qquad W_8^1 = e^{-j2\pi/8} = e^{-j\pi/4} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j \qquad W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Oneofthe elementarycomputationsisshown below:



Thesignalflowgraph follows:



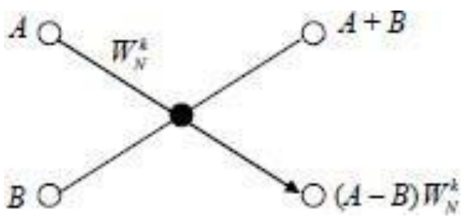
TheDFT is

$X(k)$={0,(5– j12.07),(–4 +j4),(5– j2.07),–4,(5 +j2.07),(–4–j4),(5 +j12.07)}

3. Radix-2decimation-in-frequencyFFT(Sande-Tukey)

Procedureandimportantpoints

1. Thenumber ofinputsamples is $N= 2^v$where$v$isan integer.
2. The inputsequenceisinnaturalorder;theoutputisinbit-reversed order.
3. Thenumberofstagesintheflow graph isgivenby$v=\log_2 N$.
4. EachstageconsistsofN/2butterflies.
5. Inputs/outputs for each butterfly are separated in the reverse order from that of the DIT. The separation decreases *from left to right* in the order $N/2$, … , 4, 2, 1.
6. Thenumberofcomplexadditions= $N\log_2 N$andthenumberofcomplexmultiplications is $\frac{N}{2}\log_2 N$.
7. ThebasiccomputationblockintheflowgraphoftheDIFFFTisthebutterflyshown here. Thisisan**in-placecalculation**in thatthe twooutputs ($A + B$)and($A – B$)W_N^kcan be computed and stored in the same locations as A and B.



Example2: Radix-2,8-point,decimation-in-frequencyFFTforthesequence

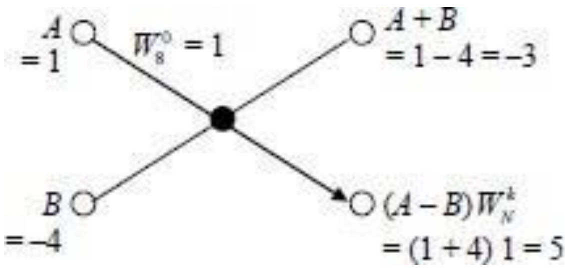
n→01234567

x(n)= {1,234–4–3–2–1}

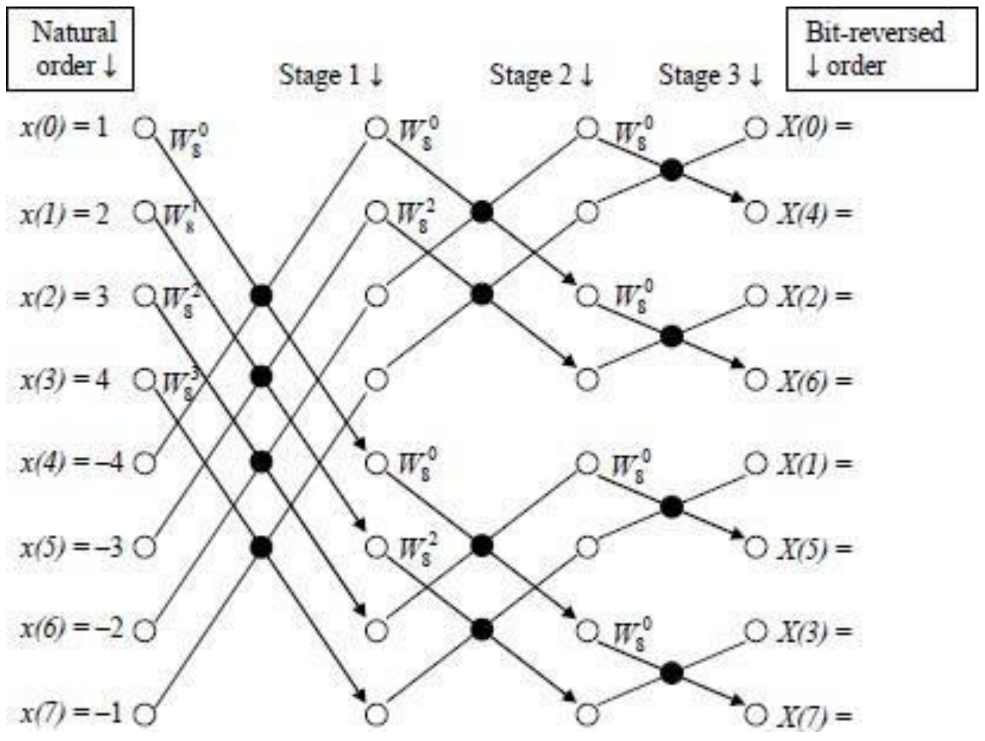
Solution:

Thetwiddlefactorsarethesameasinthe DITFFTdoneearlier (bothbeing8-pointDFTs):

Oneofthe elementarycomputationsisshown below:



Thesignalflowgraph follows:

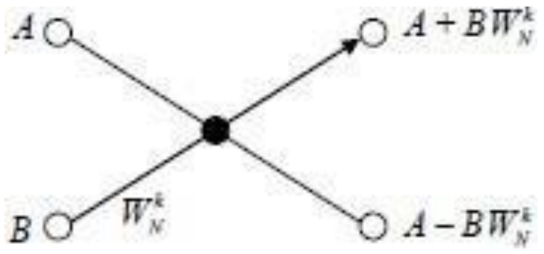


TheDFT is

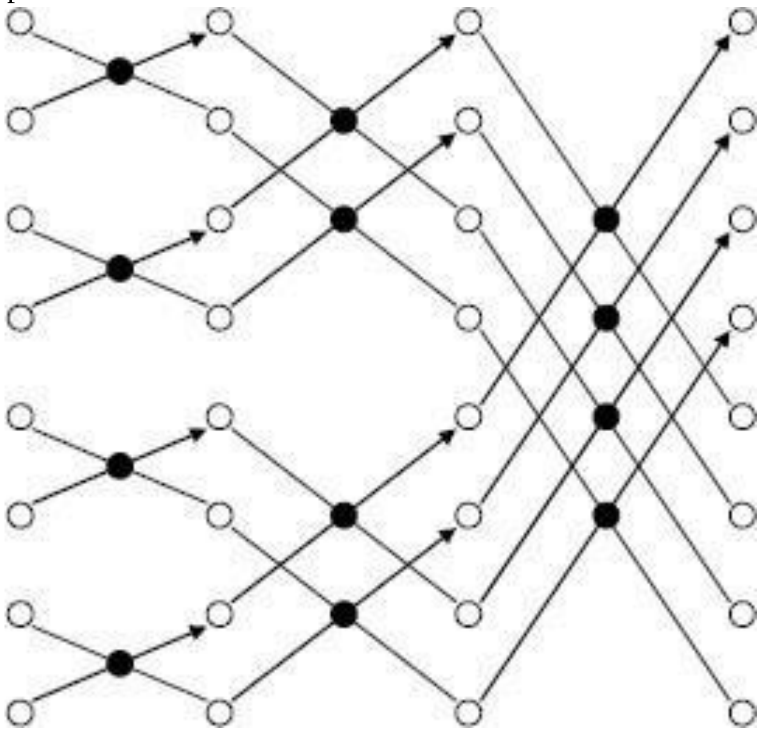
X(k)={0,(5– j12.07),(–4 +j4),(5– j2.07),–4,(5 +j2.07),(–4–j4),(5 +j12.07)}

(DITTemplate)

Theelementarycomputation(Butterfly):

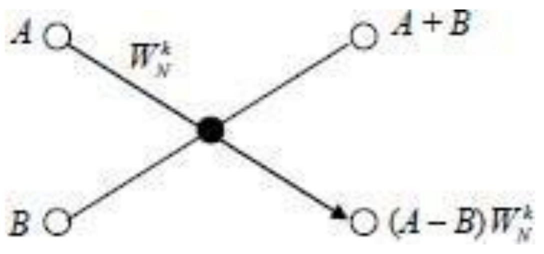


Thesignalflow graph:

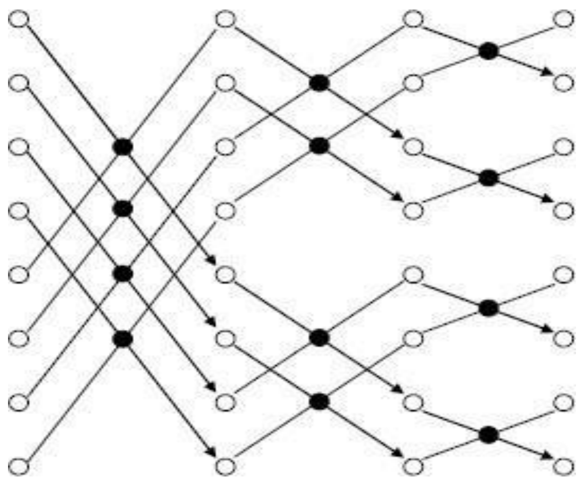


(DIFTemplate)

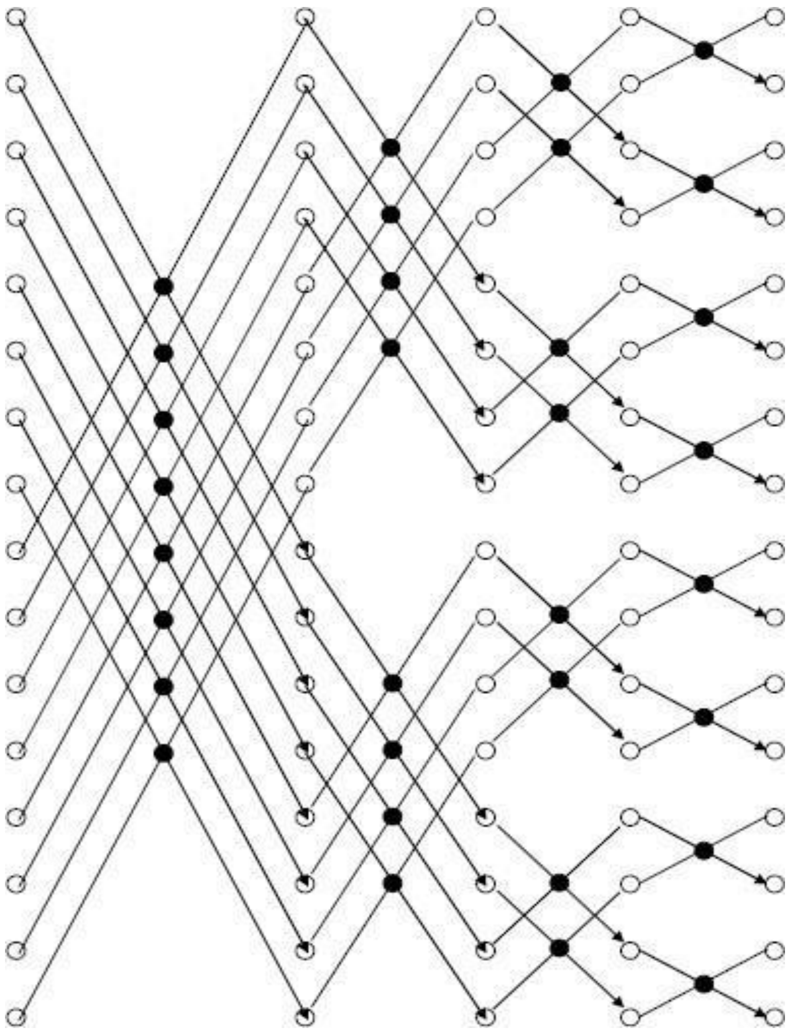
Theelementarycomputation(Butterfly):



Thesignalflow graph:



16-pointDIFFFT



4. InverseDFTusingtheFFTalgorithm

TheinverseDFTofan*N*-pointsequence{*X(k)*,*k*=1,2,…,(*N*–1)}isdefined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n=0,1,\ldots,N-1$$

Where$W_N = e^{-j2\pi/N}$.Takethecomplexconjugateof*x(n)*andmultiplyby*N*toget

$$Nx^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{kn}$$

The right hand side of the above equation is simply the DFT of the sequence $X^*(k)$and can be computed by using any FFT algorithm. The desired output sequence is then found by taking the conjugate of the result and dividing by *N*

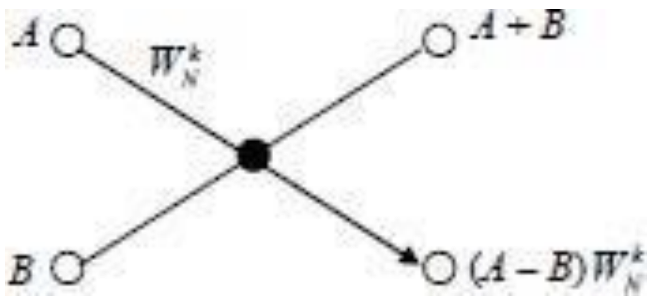
$$x(n) = \frac{1}{N}\left(\sum_{k=0}^{N-1} X^*(k) W_N^{kn}\right)^*$$

Example3: GiventheDFTsequence*X(k)*={0,(–1–j),j,(2+j),0,(2–j),–j,(–1+j)}obtainthe IDFT *x(n)* using the DIF FFT algorithm.

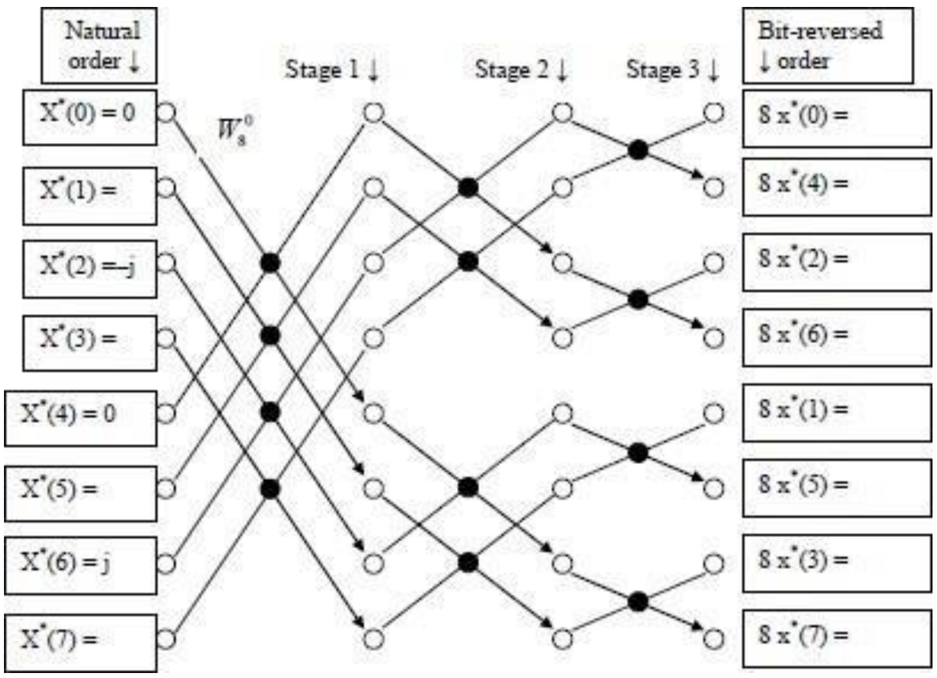
Solution:

Thisisan8-pointIDFT.The8-pointtwiddle factorsare,ascalculatedearlier,

Theelementarycomputation(Butterfly)isshownbelow:



Thesignalflowgraph follows:



Theoutputatstage3givesusthevalues$\{8x^*(n)\}$inbit-reversedorder:

$\{8x^*(n)\}_{bitrevorder}$={2,–2,4,–4,–6.24,2.24,6.24,–2.24}

TheIDFTisgivenbyarrangingthedatainnormalorder,takingthecomplexconjugateofthe sequence and dividing by 8:

$\{8x^*(n)\}_{normalorder}$={2,–6.24, 4,6.24, –2, 2.24,–4, –2.24}

$$x(n) = \{\frac{1}{4},\frac{-6.24}{8},\frac{1}{2},\frac{6.24}{8},\frac{1}{4},\frac{2.24}{8},-\frac{1}{2},\frac{-2.24}{8}\}$$

$$x\ n\ =\ 0.25,-0.78,0.5,0.78,-0.25,0.28,-0.5,-0.28( )\{\ \}$$

Example 4: GiventheDFTsequence *X(k)* ={0,(1–j), j, (2+j), 0, (2–j), (–1+j),–j},obtaintheIDFT *x(n)* usingtheDIFFFTalgorithm.

Solution:
Thereisnoconjugatesymmetryin{*X(k)*}.UsingMATLAB X =
[0, 1-1j, 1j, 2+1j, 0, 2-1j, -1+1j, -1j]
x=ifft(X)

TheIDFT is

x(n)={0.5,(-0.44+0.037i),(0.375-0.125i),(0.088 +0.14i),(-0.75+0.5i),(0.44+0.21i),(-0.125 -0.375i),(-0.088-0.39i)}

5. APPLICATIONSOFFFTALGORITHMS:

1. EfficientComputationoftheDFTofTwoRealSequences

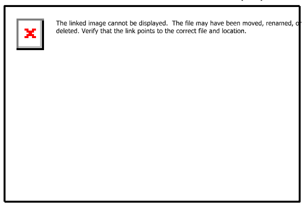
The FFT algorithm is designed to perform complex multiplications and additions, even though the input data may be real valued. The basic reason for this situation is that the phase factors are complexand hence, afterthefirst stageofthealgorithm, all variablesarebasicallycomplex-valued. In view of the fact that the algorithm can handle complex -valued input sequences, we can exploit thiscapabilityinthe computationofthe DFToftwo real-valuedsequences.Supposethat$x_1(n)$and $x_2(n)$are two real-valued sequences of length N, and let x(n) be a complex-valued sequence defined as

$$(n) = x_1(n) + jx_2(n) \qquad 0 \le n \le N-1$$

TheDFToperationis linearandhencetheDFTofx(n)canbeexpressedas

$$(k) = X_1(k) + jX_2(k)$$

Thesequences$x_1(n)$and$x_2(n)$canbeexpressedin termsofx(n) asfollows: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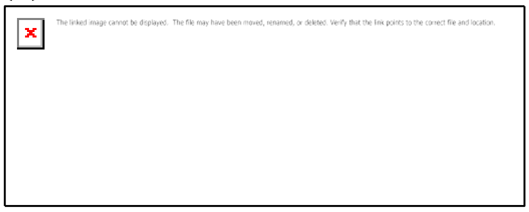
HencetheDFTsof$x_1(n)$and$x_2(n)$are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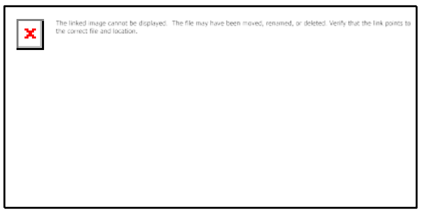
RecallthattheDFTof$x^*(n)$ is $X^*(N-k)$.Therefore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Thus, by performing a single DFT on the complex-valued sequence x(n), we have obtained theDFT ofthe two realsequences with only a smallamount ofadditional computation that is involved in computing $X_1(k)$ and $X_2(k)$from *X(k).*

2. <u>EfficientComputationoftheDFTofa2N-PointRealSequence</u>

Suppose that g(n) is a real-valued sequence of 2N points. We now demonstrate how to obtain the 2N-point DFTofg(n) fromcomputationofone N-point DFT involving complex-valued data. First, we define

$$x_1(n) = g(2n)$$

$$x_2(n)=g(2n+1)$$

Thus we have subdivided the 2N-point realsequence into two N-point realsequences. Now we can apply the method described in the preceding section.
Let x(n)betheN-point complex-valuedsequence

$$(n) = x_1(n)+jx_2(n)$$

Fromtheresultsofthepreceding section,wehave

$$X_1(k) = \frac{1}{2}[X(k) + X^*(N-k)]$$

$$X_2(k) = \frac{1}{2j}[X(k) - X^*(N-k)]$$

Finally,wemust expressthe2N-point DFTintermsofthetwoN-point DFTs,$X_1(k)$and$X_2(k)$.To accomplish this, we proceed as in the decimation-in-time FFT algorithm, namely,

$$(k)=\sum_{n=0}^{N-1} g(2n)W_{2N}^{2nk}+\sum_{n=0}^{N-1} g(2n+1)W_{2N}^{(2n+1)k}$$

$$=\sum_{n=0}^{N-1} x_1(n)W_N^{nk}+W_{2N}^{k}\sum_{n=0}^{N-1} x_2(n)W_N^{nk}$$

Consequently,

$$(k) = X_1(k) + W_2^{k}NX_2(k) \qquad k=0,1,\ldots,N-1(k+$$

$$N)=X_1(k) - W_2^{k}NX_2(k) \qquad k=0,1,\ldots,N-1$$

ThuswehavecomputedtheDFTofa2N-pointrealsequencefromoneN-pointDFTandsome additional computation.

6. TheChirp-zTransformAlgorithm:

TheDFTofanN-point datasequence*x(n)* hasbeenviewedasthez-transformof$x_1(n)$evaluatedat N equally spaced points on the unit circle. It has also been viewed as N equally spaced samples of the Fourier transform of the data sequence*x(n).* In this section we consider the evaluation of *X(z)*on other contours in the z-plane, including the unit circle.

Supposethatwewishtocomputethevaluesofthe z-transformof*x(n)*ata setofpoints$\{z_k\}$.Then,

$$X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} \qquad k=0,1,\ldots,L-1$$

For example,ifthecontourisacircleofradiusr and the$z_k$areNequallyspaced points,then

$$z_k = re^{j2\pi kn/N} \qquad k=0,1,2,\ldots,N-1$$

$$(z_k) = \sum_{n=0}^{N-1} [x(n) r^{-n}] e^{-j2\pi kn/N} \qquad k=0,1,2,\ldots,N-1$$

InthiscasetheFFTalgorithmcanbeappliedonthemodifiedsequence$(n)r^{-n}$.

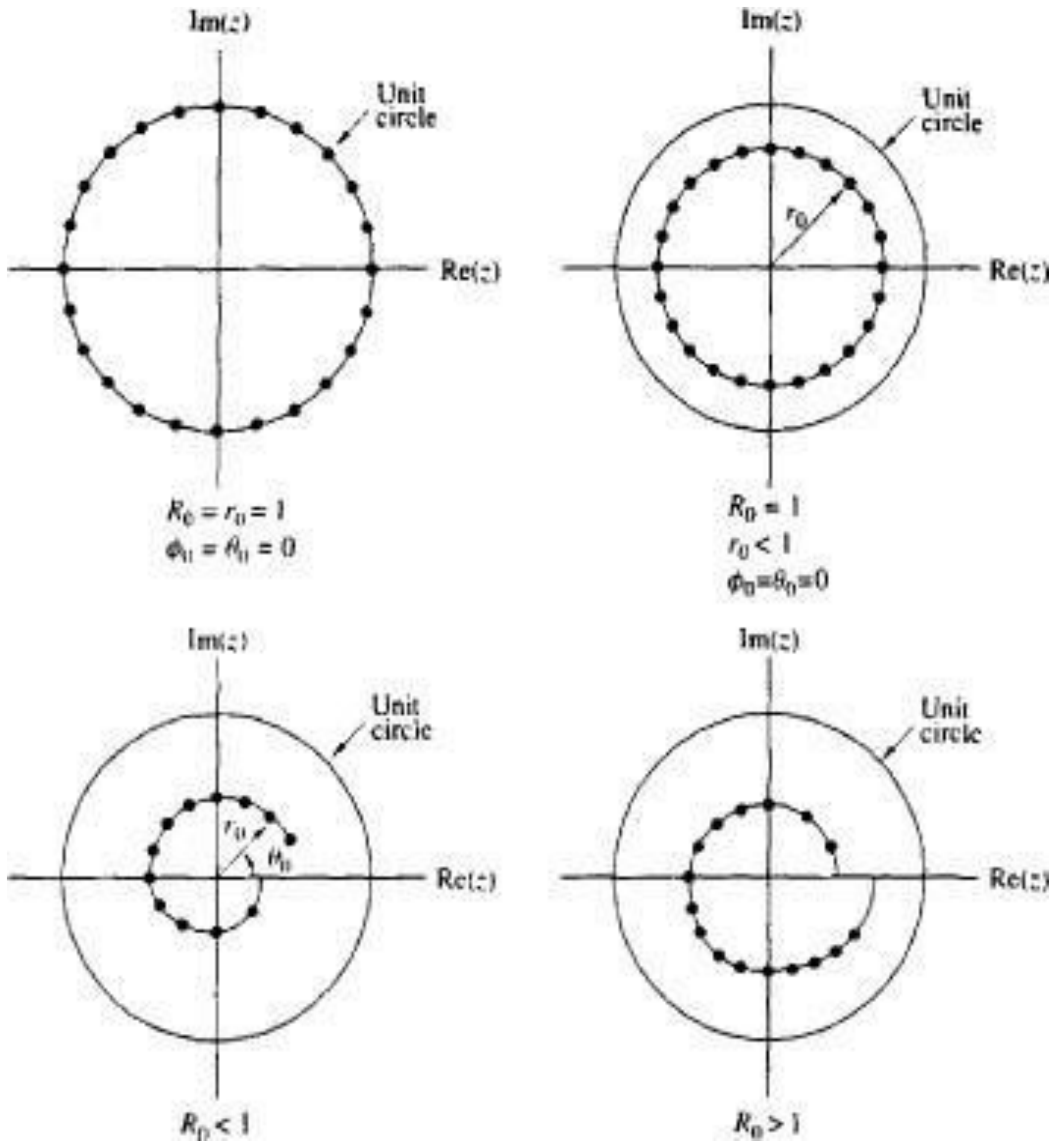
Moregenerally, supposethatthepoints$z_k$inthez-planefallonanarcwhichbeginsat somepoint

$$z_0 = r_0 e_{j\theta_0}$$

and spirals either intowardtheoriginor out awayfromthe origin suchthatthe points $z_k$are defined as

$$_k z_k = r_0 e^{j\theta_0} (R_0 e^{j\phi_0}) \qquad k=0,1,\ldots,L-1$$

Note that if $R_0 < 1$, the points fall on a contour that spirals toward the origin and if $R_0 > 1$, the contour spirals away from the origin.If Ro= 1, the contour is a circular arc of radius r_0. If $r_0 = 1$ and R_0= l, the contouris anarc ofthe unit circle. The lattercontour would allow us to computethe frequency content of the sequence x(n) at a dense set of L frequencies in the range covered by the arc without having to compute a large DFT, that is, a DFT ofthe sequence x(n) padded with many zeros to obtain the desired resolution in frequency. Finally, if $r_0 = R_0 = 1$, = 0, $\Theta_0 = 0$, $\phi_0 = 2n / N$, and L= N, the contour is the entire unit circle and the frequencies are those ofthe DFT.



Whenpoints{z_k} aresubstituted intotheexpression fortheztransform,we obtain

$$(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n}$$

$$= \sum^{N-1} (n)(r_0 e_{j\theta_0})^{-n} \quad V_{-nkn=0}$$

where, bydefinition,$V=R_0e^{j\phi_0}$

Wecanexpresstheaboveequationintheformofaconvolution,bynotingthat

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Letusdefineanewsequenceg(n) as

$$(n)=x(n)(r_0e^{j\theta})^{0-n}V^{-n^2/2}$$

Then,

$$(z_k)=V^{-k^2/2}\sum_{n=0}^{N-1}g(n)V^{(k-n)^2/2}$$

The summation in the above expression can be interpreted as the convolution of the sequence g(n) with the impulse response h(n) of a filter, where

$$h(n)=V^{n^2/2}$$

Hence,

$$X(z_k)=V^{-k^2/2}y(k)=\frac{y(k)}{h(k)}\qquad k=0,1,\ldots,L-1$$

Wherey(k)istheoutputofthefilter

$$(k)=\sum_{n=0}^{N-1}g(n)h(k-n)\qquad k=0,1,\ldots,L-1$$

We observe that both h(n) and g(n) are complex-valued sequences. The sequence h(n) with $R_0 = 1$ hastheform of acomplex exponentialwith argument$wn=n^2\phi_0/2=(n\phi_0/2)n$.Thequantity $n\phi_0/2$represents the frequency of the complex exponential signal, which increases linearly with time. Such signals are used in radar systems and are called chirp signals. Hence the z-transform evaluated is called the chirp-z transform.

MODULE 4:

StructuresforFIRandIIRSystems:

StructureforFIRSystems:

IngeneralaFIRsystemisdescribedbythedifferenceequation

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

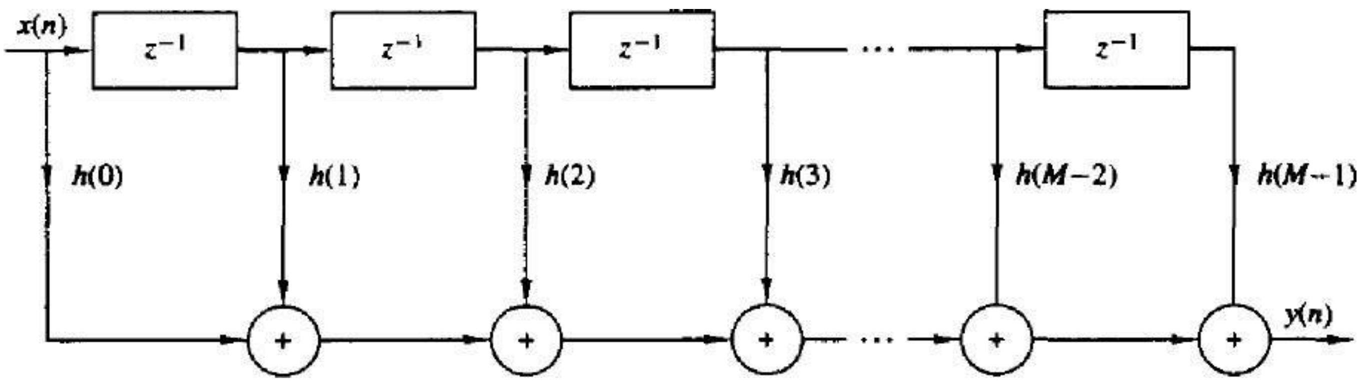
Orequivalently,bythesystemfunction

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

1. Direct-FormStructure:

Thedirect-formrealizationfollowstheconvolutionsummation

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$



DirectformrealisationofFIRsystem

We observe that this structure requires M-1 memory locations for storing the M-1 previous inputs, and has a complexity of M multiplications and M-1 additions per output point. Since the output consists of a weighted linear combination of M-1 past values ofthe input and the weighted current value of the input, the structure inabove figure, resembles a tapped delay line or a transversalsystemconsequently, the direct-form realization is often called a transversal or tapped-delay-line filter.

2. Cascade-FormStructures:

Thecascaderealizationfollowsnaturallyfromthesystemfunctiongivenby

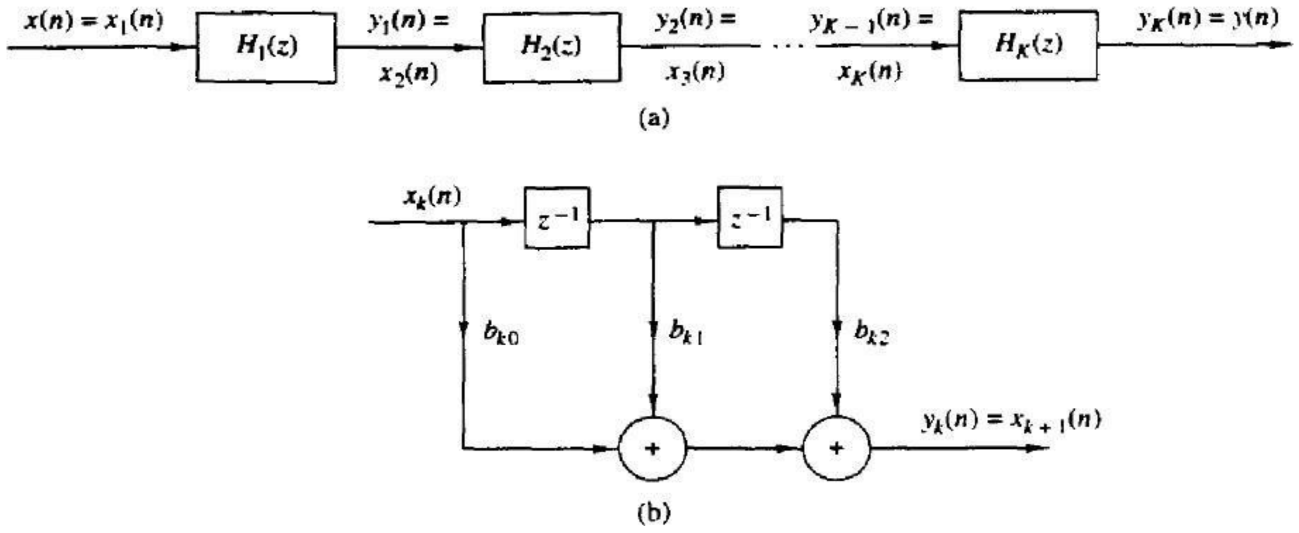
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

Itissimple mattertofactorH(z)intosecondorderFIRsystemsothat

$$H(z) = \prod_{k=1}^{M} H_k(z)$$

Where$H_k(z)=b_{k0}+b_{k1}z^{-1}+b_{k2}z^{-2}$,k=1,2,3 .... k

And K is the integer part of (M + l) /2. The filter parameter b_0 may be equally distributed among the K filter sections, such that $b_0 = b_{10}b_{20}\cdots b_{K0}$or it may be assigned to asingle filter section. The zeros ofH(z) aregrouped inpairs to produce the second-order FIR systems. It is always desirable to form pairs of complex- conjugate roots so that the coefficients $\{b_{ki}\}$ are real valued. On the other hand, real- valued roots can be paired in any arbitrary manner. The cascade-form realization along with the basic second-order section is shown below.



CascadeRealisationofaFIRsystem

DesignofDigitalFilters:

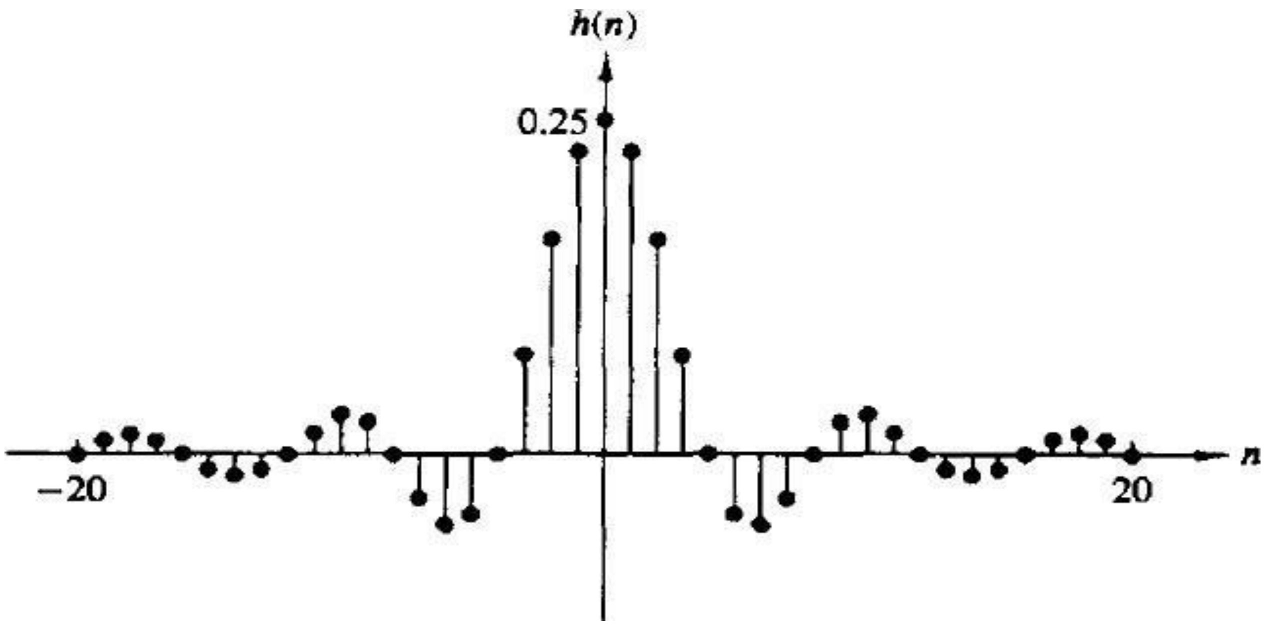
CausalityandItsImplications:

Letusconsidertheissueofcausalityinmoredetailbyexaminingtheimpulseres ponseh(n)ofanideallowpassfilterwithfrequencyresponsecharacteristic

$$H(w)=\begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

Theimpulseresponseofthe filteris

$$h(n)=\begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}, & n \ne 0 \end{cases}$$



Unitsampleresponse ofanideallowpassfilter

Aplot of h{n) for $w_c = \pi/4$ is illustrated in the above figure. It is clear that the ideal low pass filter is noncausal and hence it cannot be realized in practice.

One possible solution is to introduce a large delay n_0 in h(n) and arbitrarily to set h(n)=0 for $n < n_0$. However, the resulting system no longer has an ideal frequency responsecharacteristic.Indeed,ifweseth(n)=0forn<n_0,theFourierseries expansionofH(w)results intheGibbsphenomenon.

Paley-WienerTheorem:

Ifh(n)hasfiniteenergyandh(n)=0for n<0,then

$$\int_{-\pi}^{\pi} |\ln|H(\omega)||d\omega < \infty$$

Conversely, if |H(ω)| is square integrable and if the integral in the above equation is finite, then we can associate with |H(ω)| a phase response$\Theta(\omega)$, so that the resulting filter with frequency response $H(\omega)=|H(\omega)|e^{j\theta(\omega)}$ is causal.

One important conclusion that we draw from the Paley-Wiener theorem is that the magnitude function |H(ω)| can be zero at some frequencies, but it can't be zero over anyfinite bandoffrequencies, since the integralthen becomes infinite. Consequently any ideal filter is noncausal.

Apparently causalty imposes some tight constraints on a linear time invariant system. In addition to the Paley-Wiener condition causalty also implies a strong relation between $H_R(\omega)$ and $H_I(\omega)$, the real and imaginary components of the frequency response H(ω).To illustrate this dependence we decompose h(n).That iseven and an odd sequence, that is

$$H(n)=h_e(n)+h_o(n)$$

Where$h_e(n)=\frac{1}{2}[h(n)+h(-n)]$and $\frac{1}{2}[h(n)-h(-n)]$

Now, if h(n) is causal ,it is possible to recover h(n) from its even part $h_e(n)$ for

$0\le n\le\infty$or fromitsoddcomponent$h_o(n)$ for$1\le n\le\infty$.Indeed, itcanbeeasilyseenthat

$$h(n)=2h_e(n)u(n)-h_e(0)\delta(n) \qquad n\ge 0$$

and

$$h(n)=2h_o(n)u(n)-h_o(0)\delta(n) \qquad n\ge 1$$

Since$h_0(n)$=0forn=0, wecannotrecoverh(0)from$h_0(n)$andhencewealsomust

know h(0). Inanycase, it is apparent that h_0 (n) = h_e(n) for n> 1, so there is a strong relationship between h_0 (n) and h_e(n).
If h (n) is absolutely summable (i.e., BIBO stable), the frequency response H(w) exists, and

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

In addition, if h(n) is real valued and causal, the symmetry properties of the Fourier transform imply that

$$h_e(n) \overset{F}{\longleftrightarrow} H_R(\omega)$$

$$h_o(n) \overset{F}{\longleftrightarrow} H_I(\omega)$$

Since h(n) is completely specified by h_e(n), it follows that H(ω) is completely determined if we know H_R(ω).alternatively H(ω) is completely determined from H_I(ω) and h(0).In short H_R(ω) and H_I(ω) are independent and cannot be specified independently if the system is causal. Equivalently the magnitude and phase responses of a causal filter are interdependent and hence cannot be specified independently.

DesignofLinearPhaseFIRfiltersusingdifferentwindows:

In many cases a linear phase characteristics is required through the passband of the filter. It can be shown that causal IIR filter cannot produce a linear phase characteristicsandonlyspecialformsofcausalFIRfilterscangivelinearphase.If
{h[n]} represents the impulse response of a discrete time linear system a necessary and sufficient condition for linear phase is that {h[n]} have finite duration N, that it be symmetric about its midpoint, i.e.

$$h[n] = h[N-1-n], \quad n = 0, 1, 2, \ldots(N-1)$$

$$\begin{aligned} H(e^{jw}) &= \sum_{n=0}^{N-1} h[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} h[n]^{-j\omega n} + \sum_{n=N/2}^{N-1} h[n] e^{-j\omega n} \\ &= \sum_{n=0}^{N/2-1} h[n] e^{-j\omega n} + \sum_{m=0}^{N/2-1} h[m] e^{-j\omega}(N-1-m) \end{aligned}$$

For N even, we get

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \sum_{n=0}^{N/2-1} 2h[N] \cos(\omega(n-(N-1)/2))$$

For N odd

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2} \left\{ h[\frac{N-1}{2}] + \sum_{n=0}^{\frac{N-3}{2}} 2h[n] \cos[\omega(n - \frac{N-1}{2})] \right\}$$

For N evenwe get a non-integer delay, whichwillcause the value ofthe sequence to change.

OneapproachtodesignFIR filterslinearphaseisto usewindows.Theeasiestwayto obtain FIR filter is to simply truncate the impulse response of an IIR filter. If $\{h_d[n]\}$istheimpulseresponseofthedesignedFIR filterthenthe firfilterwith impulse response $\{h[n]\}$ can be obtained as follows.

$$H[n] = \begin{cases} h_d[n], & N_1 \le n \le N_2 \\ 0, & otherwise \end{cases}$$

This canbethoughtofas beingformedbyaproductof$\{h_d[n]\}$andawindow function $\{w[n]\}$ $\{h[n]\} = \{h_d[n]\}$ $\{w[n]\}$ where $\{w[n]\}$ is the window function.

Usingmodulationpropertyoffouriertransform

$$H(e^{j\omega}) = \frac{1}{2\pi}[H_d(e^{j\omega}) \otimes W(e^{j\omega})]$$

In general for smaller N values spreading of main lobe more, and for larger N narrowerthr mainlobeand$|H(e^{j\omega})|$comescloserto$|H_d(e^{j\omega})|$.Muchwork has been done on adjusting $\{w[n]\}$ to satisfy certain main lobe and side lobe req uirements.Someofthecommonlyusedwindowsaregivenbelow-

(a) RectangularWindow

W$_R$(n)={1,0 □*n* □*N* □1

0,*otherwise*

(b) Bartlett(Triangular)

2*n*

____,0 □*n* □ (*N* □1)/2

N □1

W$_B$(n)={2 □□2*n*,(*N* □1)/2 □*n* □*N* □1 *N*

□1

0,*elsewhere*

(c) HanningWindow

1□ cos[2□*n*/(*N* □1)]

W$_{Han}$(n)={2,0 □*n* □*N* □1 0,*otherwise*

(d) BlackmanWindow

W(n)={.42 □.5cos □2□*n* *N* □1 □□ .08cos □4□*n* *N* □1 □,0 □*n* □*N* □1

$_{Bl}$0,*otherwise*

(e)KaiserWindow

□□*N* □1□^{2}□□ *N* □1□^{2}□$^{1/2}$ –

□ W$_K$(n)={*I*$_0$□ □□□□□2 □□□□□*n*□□ ____2 □□□□□□ ,0 □*n* □*N* □1

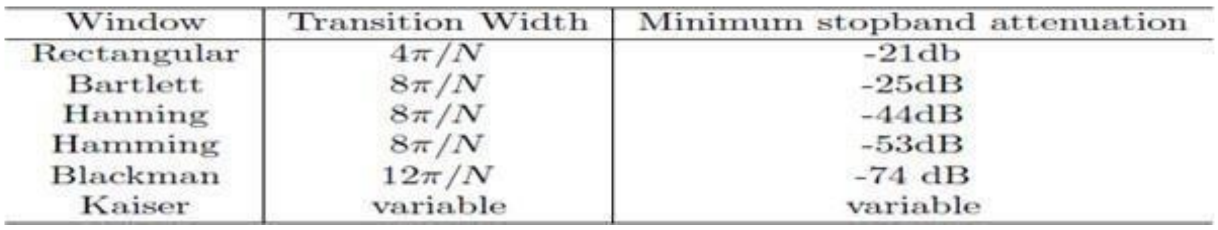
I$_0$ □□*w$_a$*(*N* □1)□□□□

□ 2 □□

0,*otherwise*

Where I$_0$(x) is the modified Zero Order Bessel Function of the first kind.

TheTransitionwidthandthe minimumstoppedattenuationfordifferentwindows arelistedbelow-



Window	Transition Width	Minimum stopband attenuation
Rectangular	$4\pi/N$	-21db
Bartlett	$8\pi/N$	-25dB
Hanning	$8\pi/N$	-44dB
Hamming	$8\pi/N$	-53dB
Blackman	$12\pi/N$	-74 dB
Kaiser	variable	variable

We first choose a window that satisfies the minimum attenuation and the bandwidth that allows us to choose the appropriate value of N.Actual frequency responsecharacteristicsarethencalculatedandwechecktherequirments aremet or not

<u>DesignofIIR Filters:</u>

Therearetwomethods fordesigntheIIRfilter.

1. ImpulseInvariantMethod

2. BilinearTransformationMethod

1. <u>Filterdesignbyimpulseinvariance:</u>

Here the impulse response h[n] of the desire discrete time system is proportional to equallyspacessamplesofthecontinuoustimefilteri.e,

$H[n]=T_d h_a(nT_d)$

Where$T_d$representsasampleinterval.Sincethespecificationofthefilterare given in discrete time domain it turns out that T_d has no role to play in design of the filter. From the sampling theorem the frequency response of the discrete time filter is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right)$$

Since anypracticalcontinuous time filter is not strictlyband limited there is some aliasing. However if the continuous time filter approaches zero at high frequency the aliasing may be negligible. Then the frequency response of the discrete time filter is

$$H(e^{j\omega}) \approx H_a\left(j\frac{\omega}{T_d}\right), |\omega| \le \pi$$ Typeequation here.

We first convert digital filter specifications to continuous time filter specifications.NeglectingaliasingwegetH$_a$(jΩ)specificationbyapplyingthe relation Ω= ω/T$_{d.}$ Where H$_a$(jΩ) is transferred to the designed filter H(z).

Letusassumethatthepolesofthecontinuoustimefilterarearesimple,

thenH(s)= □$_N$□ A_ka

k□s □s k

ThecorrespondingImpulseresponseish(t)={ □$_N$$Ae_{s_k t}$,$t$ □ 0

a □k1k

0,t □ 0

Thenh[n]=T$_d$h$_a$(nT$_d$)= □$_N$$T_d A e_{s_k n T_d} u[n]$

k

k1□ □

N $T_d A^k$

Thesystemfunctionfunctionforthisis H(z)= □□ $s_k T_d$ □1

k□1 □e z

Weseethatapoleats=s$_k$inthes-plane istransferredtoapoleatz= $e^{s_k T_d}$inthez- plane. If the continuous time filter is stable i.e Re{s$_k$}<0, then the magnitude of $e^{s_k T_d}$will be less than 1.So the pole will be inside the unit circle. Thus the causal discrete filter is stable. The mapping of zero is not so straight forward.

<u>BilinearTransformation:</u>

This technique avoids the problem of aliasing by mapping jΩ axis in the s-planeto one revolution of unit circle in the z-plane. If H$_a$(s) is the continuous time transfer functionthe discrete time transfer functionis detained byreplacings with 2

□1□z□ □□ S= □□ □1□□

□ T_d □ 1□z □□

Fromwhichwegetz=1 □□T_d /2 □s

1□ (T_d/2)s

□ 1□□$T_d$$j$□ □$T_d$

Substitutings= □+jΩ,wegetz □1□□T2$_d$$j$□ □$T2_d$

2 2

If<□0, it is then magnitude ofthe realpart inthe denominator is more thanthat of the numerator and so │z│<1. Similarly if>□0 then│z│>1 for all Ω.Thus pole in thelefthalfofthes-planewillgetmappedtothepolesinsidetheunitcircleinz-plane.If=□0then

□ 1□j□T□

□ z □______2

□ 1j□ □2T̲dso│z│=1,writingz=e^j □□ weget

1j□ □Tde

j□1j□ □2Td

2

Rearrangingwegetj2T□ _□□e̲ejj□□□□1̲1 □□e̲ejj□□// 22((e̲e □□jj□□// 22 □□e̲ejj□□ □□// 22)) .□□ j _______cossin□□// 22

Or □□ 2tan □/2or □□ 2tan^1□ ____□Td.
Td 2