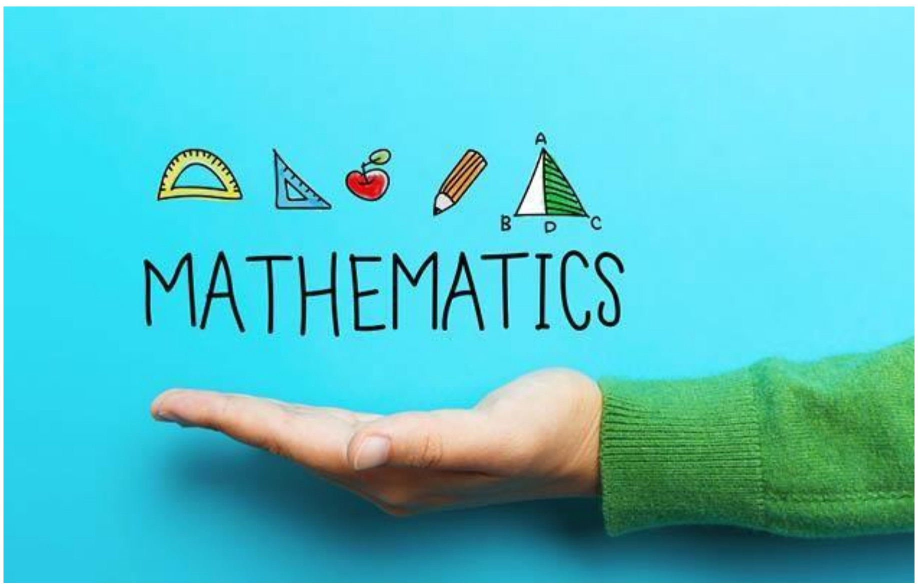
E-Learning Material

on

Engineering Mathematics-II

of

2nd semester of all Engineering Branches



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SYLLABUS
Th.3. Engineering Mathematics – II
(2nd Semester Common)

Theory: 5 Periods per Week | **I.A: 20 Marks**
Total Periods: 75 Periods | **Term End Exam: 80 Marks**
Examination: 3 Hours | **TOTAL MARKS: 100 Marks**

Objective:

Principles and application in Engineering are firmly ground on abstract mathematical structures. Students passing from secondary level need familiarization with such structure with a view to develop their knowledge, skill and perceptions about the applied science. Calculus is the most important mathematical tool in forming engineering application into mathematical models. Wide application of calculus makes it imperative to develop methods of solving differential equations. The knowledge of limit, derivative and derivative needs to be exhaustively practiced. To help a systematic growth of skill in solving equation by calculus method will be the endeavor of this course content. Understanding the concept of co-ordinate system in 3D in case of lines, planes and sphere and it's use to solve Engineering problems. After completion of the course the student will be equipped with basic knowledge to form equations and solve them competently.

SL NO	TOPICS	PERIODS	MARKS
1	Vector Algebra	15	12
2	Limits and Continuity	12	12
3	Derivatives	21	20
4	Integration	15	24
5	Differential Equation	12	12
	TOTAL	75	80

1) VECTOR ALGEBRA

a) Introduction**b)** Types of vectors (null vector, parallel vector , collinear vectors)(in component form)**c)** Representation of vector**d)** Magnitude and direction of vectors **e)**Addition and subtraction of vectors **f)** Position vector **g)** Scalar product of two vectors **h)** Geometrical meaning of dot product**i)** Angle between two vectors**j)** Scalar and vector projection of two vectors**k)** Vector product and geometrical meaning (Area of triangle and parallelogram)

2) LIMITS AND CONTINUITY

a) Definition of function, based on set theory **b)** Types of functions i) Constant function ii) Identity function iii) Absolute value function iv)The Greatest integer function v) Trigonometric function vi) Exponential function vii) Logarithmic function **c)** Introduction of limit **d)** Existence of limit **e)** Methods of evaluation of limit (1) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where a>0 and n€R

(2) $\lim_{x\to 0}\frac{a^x-1}{x}$ = $\log_e a$ In particular $\lim_{x\to 0}\frac{e^x-1}{x}$ = 1 (3)$\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ = e
(4) $\lim_{x\to\infty}(1+\frac{1}{x})^x$ = e (5) $\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$ In particular $\lim_{x\to 0}\frac{\log_e(1+x)}{x} = \log_e e$ =1
(6) $\lim_{x\to 0}\cos x$ = 1 (7)$\lim_{x\to 0}\sin x$ =0 (8)$\lim_{x\to 0}\frac{\sin x}{x}$ = 1 (9) $\lim_{x\to 0}\frac{\tan x}{x}$ = 1
f) Definition of continuity of a function at a point and problems based on it

3) DERIVATIVES

a) Derivative of a function at a point **b)** Algebra of derivative **c)** Derivative of standard functions **d)** Derivative of composite function (Chain Rule) **e)** Methods of differentiation of i) Parametric function ii) Implicit function iii) Logarithmic function iv) a function with respect to another function **f)** Applications of Derivative i) Successive Differentiation (up to second order) ii) Partial Differentiation (function of two variables up to second order) **g)** Problems based on above

4) INTEGRATION

a) Definition of integration as inverse of differentiation
b) Integrals of some standard function
c) Methods of Integration
i) Substitution
Integration of forms $\int\frac{dx}{x^2+a^2}, \int\frac{dx}{x^2+a^2}$ etc using substitution method
ii) Integration By parts
d) Integration of forms $\int\sqrt{a^2-x^2}dx, \int\sqrt{a^2+x^2}dx$ using by parts
e) Definition of Definite Integral
Fundamental theorem of Integral Calculus
Properties of Definite Integral
f)Application of Integration
i) Area enclosed by a curve and X- axis.
ii) Area of a circle with centre at origin.

5) DIFFERENTIAL EQUATION

Order and degree of a differential equation
Solution of Differential equation 1st order and 1st degree
i) variable separable method
ii) Linear equation $\frac{dy}{dx}+Py=Q$, where P,Q are functions of x.

VECTORS

INTRODUCTION:-

In our real life situation we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively.

We have also come across physical quantities such as displacement, velocity, acceleration, momentum etc, which are of different type in comparison to above.

Consider the figure-1, where A, B, C are at a distance 4k.m. from P. If we start from P, then covering 4k.m. distance is not sufficient to describe the destination where we reach after the travel, So here the end point plays an important role giving rise the need of direction. So we need to study about direction of a quantity, along with magnitude.

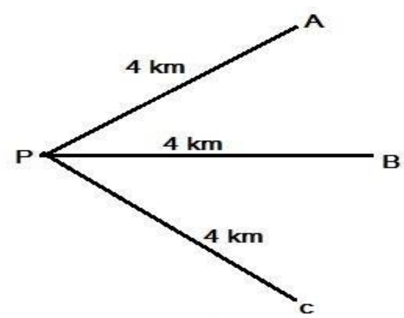


Fig - 1

OBJECTIVE

After completion of the topic you are able to :-

i) Define and distinguish between scalars and vectors.
ii) Represent a vector as directed line segment.
iii) Classify vectors in to different types.
iv) Resolve vector along two or three mutually perpendicular axes.
v) Define dot product of two vectors and explain its geometrical meaning.
vi) Define cross product of two vectors and apply it to find area of triangle and parallelogram.

Expected background knowledge

i) Knowledge of plane and co-ordinate geometry
ii) Trigonometry.

Scalars and vectors

All the physical quantities can be divided into two types.
i) Scalar quantity or Scalar.
ii) Vector quantity or Vector.

Scalar quantity: - The physical quantities which requires only magnitude for its complete specification is called as scalar quantities.
Examples: - Speed, mass, distance, velocity, volume etc.

Vector: - A directed line segment is called as vector.

Vector quantities:- A physical quantity which requires both magnitude & direction for its complete specification and satisfies the law of vector addition is called as vector quantities.
Examples: - Displacement, force, acceleration, velocity, momentum etc.

Representation of vector:- A vector is a directed line segment $\overrightarrow{AB}$ where A is the initial point and B is the terminal point and direction is from A to B. (see fig-2).

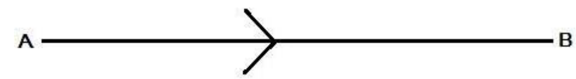


Fig - 2

Similarly $\overrightarrow{BA}$ is a directed line which represents a vector having initial point B and terminal point A.

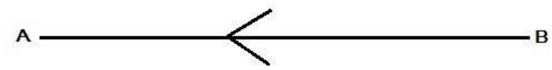


Fig - 3

Notation: - A vector quantity is always represented by an arrow (→) mark over it or by bar (–) over it. For example $\overrightarrow{AB}$. It is also represented by a single small letter with an arrow or bar mark over it. For example $\vec{a}$.

Magnitude of a vector: - Magnitude or modulus of a vector is the length of the vector. It is a scalar quantity.

Magnitude of $\overrightarrow{AB}$ = $|\overrightarrow{AB}|$ = Length AB. = AB

Types of Vector: - Vectors are of following types.

1) **Null vector or zero vector or void vector: -** A vector having zero magnitude and arbitrary direction is called as a null vector and is denoted by $\vec{0}$.

Clearly, a null vector has no definite direction. If $\vec{a} = \overrightarrow{AB}$, then $\vec{a}$ is a null (or zero) vector iff $|\vec{a}| = 0$ i.e. if $|\overrightarrow{AB}| = 0$

For a null vector initial and terminal points are same.

2) Proper vector: - Any non zero vector is called as a proper vector. If $|\vec{a}| \neq 0$ then $\vec{a}$ is a proper vector.

3) Unit vector : - A vector whose magnitude is unity is called a unit vector. Unit vectors are denoted by a small letter with ^ over it. For example $\hat{a}$. $|\hat{a}| = 1$

Note: - The unit vector along the direction of a vector $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

4) Co-initial vectors:- Vectors having the same initial point are called co-initial vector.

In figure-4, $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$ and $\vec{OE}$ are Co-initial vectors.

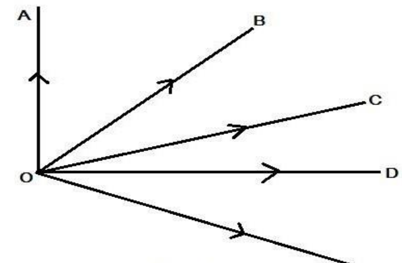


Fig - 4

5) Like and unlike vectors: - Vectors are said to be like if they have same direction and unlike if they have opposite direction.

6) Co-Linear vectors:- Vectors are said to be co-linear or parallal if they have the same line of action. In f figure-5 $\vec{AB}$ and $\vec{BC}$ are collinear.

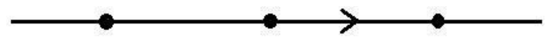


Fig - 5

7) Parallel vectors: - Vectors are said to be parallel if they have same line of action or have line of action parallel to one another. In fig-6 the vectors are parallel to each other.

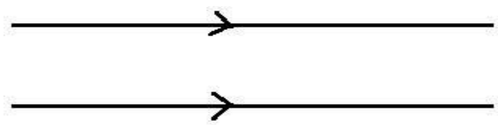


Fig - 6

8) Co-planner Vectors: - Vectors are said to be co-planner if they lies on the same plane. In fig-7 vector $\vec{a}, \vec{b}$ *and* $\vec{c}$ are coplanner.

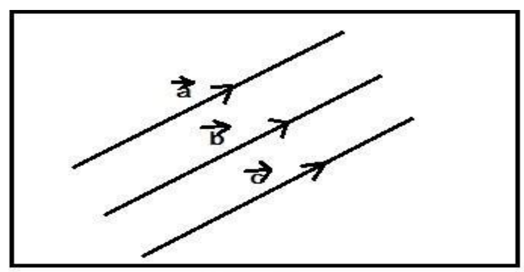


Fig - 7

9) Negative of a vector: - A vector having same magnitude but opposite in direction to that of a given vector is called negative of that vector. If $\vec{a}$ is any vector then negative vector of it is written as $-\vec{a}$ and $|\vec{a}| = |-\vec{a}|$ but both have direction opposite to each other as shown in fig-8.

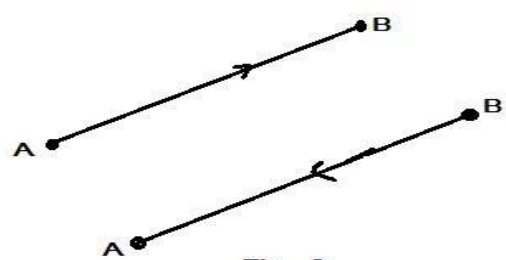


Fig - 8

10) Equal Vectors: - Two vectors are said to be equal if they have same magnitude as well as same direction.

Thus $\vec{a} = \vec{b}$

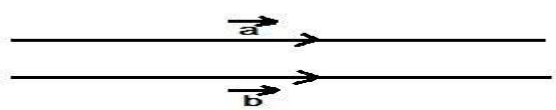


Fig - 9

Remarks:- Two vectors can not be equal

i) If they have different magnitude .
ii) If they have inclined supports.
iii) If they have different sense.

Vector operations

Addition of vectors: -

Triangle law of vector addition: - The law states that If two vectors are represented by the two sides of a triangle taken in same order their sum or resultant is represented by the 3rd side of the triangle with direction in reverse order.

As shown in figure-10 $\vec{a}$ and $\vec{b}$ are two vectors represented by two sides OA and AB of a triangle ABC in same order. Then the sum $\vec{a} + \vec{b}$ is represented by the third side OB taken in reverse order i.e. the vector $\vec{a}$ is represented by the directed segment $\vec{OA}$ and the vector $\vec{b}$ be the directed segment $\vec{AB}$, so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then $\vec{OB}$ represents the sum (or resultant) $(\vec{a} + \vec{b})$. Thus $\vec{OB} = \vec{a} + \vec{b}$

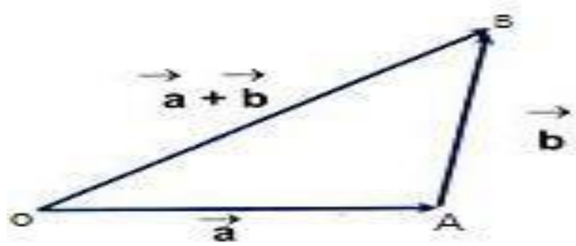


Fig – 10

Note-1 – The method of drawing a triangle in order to define the vector sum $(\vec{a} + \vec{b})$ is called triangle law of addition of the vectors.

Note-2 – Since any side of a triangle is less than the sum of the other two sides
$|\vec{OB}| \neq |\vec{OA}| + |\vec{AB}|$

Parallelogram law of vector addition: - If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogram in magnitude and direction, then their sum (resultant) is represented in magnitude and direction by the diagonal which is passing through the common initial point of the two vectors.

As shown in fig-II if OA is $\vec{a}$ and AB is $\vec{b}$ then OB diagonal represent $\vec{a} + \vec{b}$.

i.e. $\vec{a} + \vec{b} = \vec{OA} + \vec{AB}$

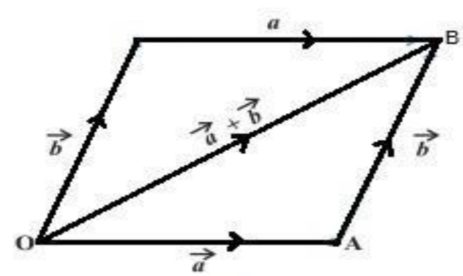


Fig - 11

Polygon law of vector addition: - If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the four sides of a polygon in same order then their sum is represented by the last side of the polygon taken in opposite order as shown in figure-12.

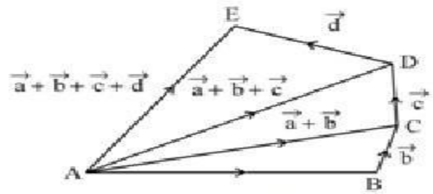


Fig - 12

Subtraction of two vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the subtraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a} - \vec{b}$ is defined as addition of $-\vec{b}$ with $\vec{a}$. i.e. $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Properties of vector addition:- i) Vector addition is commutative i.e. if $\vec{a}$ & $\vec{b}$ are any two vectors then:-

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

ii) Vector addition is associative i.e. if $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

iii) Existence of additive identity i.e. for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e. $\vec{0} + \vec{a} = \vec{a} + \vec{0} = \vec{a}$ where $\vec{0}$ is a null vector.

iv) Existence of additive Inverse :- If $\vec{a}$ is any non zero vector then $-\vec{a}$ is the additive inverse of $\vec{a}$, so that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Multiplication of a vector by a scalar : -

If $\vec{a}$ is a vector and k is a nonzero scalar then the multiplication of the vector $\vec{a}$ by the scalar k is a vector denoted by k$\vec{a}$ or $\vec{a}$k whose magnitude $|k|$ times that of $\vec{a}$.

i.e k$\vec{a}$ = $|k|$ x $|\vec{a}|$

= k x $|\vec{a}|$ if $k \geq 0$.

= (-k) x $|\vec{a}|$ if k<0 .

The direction of k$\vec{a}$ is same as that of $\vec{a}$ if k is positive and opposite as that of $\vec{a}$ if k is negative.

k$\vec{a}$ and $\vec{a}$ are always parallel to each other.

Properties of scalar multiplication of vectors :-

If h and k are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

i) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

ii) $(h+k)\vec{a} = h\vec{a} + k\vec{a}$, (Distributive law)

iii) $(hk)\vec{a} = h(k\vec{a})$, (Associative law)

iv) $1.\vec{a} = \vec{a}$

v) $0.\vec{a} = \vec{0}$

Position Vector of a point

Let O be a fixed point called origin, let P be any other point, then the vector $\overrightarrow{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{p}$.

As shown in figure-13, let AB be any vector , then applying triangle law of addition we have

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ where $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

=> $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

= (Position vector of B)–(Position vector of A)

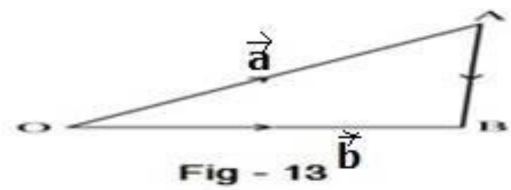


Fig - 13

Section Formula:- Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB , dividing it in the ration m:n. internally. Then the position vector of P i.e.$\vec{r}$ is given by the formula: $\vec{r} = \frac{m\vec{b} + n\,\vec{a}}{m+n}$

If P divides AB externally in the ratio m:n then $\vec{r} = \frac{m\vec{b} - n\,\vec{a}}{m-n}$

If P is the midpoint of AB then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

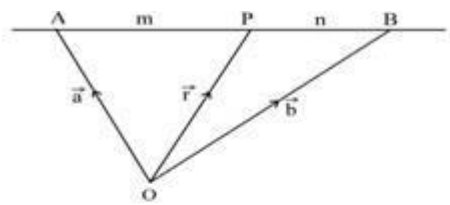


Fig - 14

Example-1 :- Prove that by vector method the medians of a triangle are concurrent.

Solution:- Let ABC be a triangle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A,B and C respectively. We have to show that the medians of this triangle are concurrent.

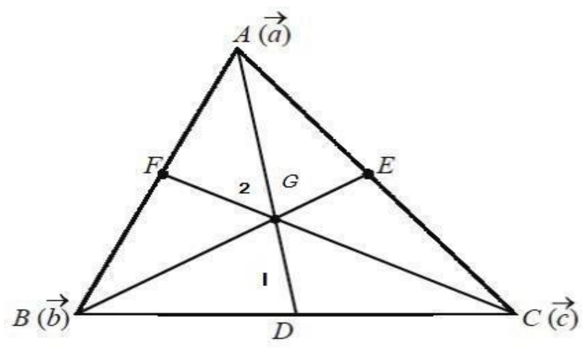


Fig - 15

Let AD , BE and CF are the three medians of the triangle.

Now as D be the midpoint of BC, so position vector of D i.e. $\vec{d} = \frac{\vec{b} + \vec{c}}{2}$.

Let G be any point of the median AD which divides AD in the ratio 2:1 . Then position vector of G is given

by $\vec{g} = \frac{2\vec{d} + \vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right) + 1\vec{a}}{3}$ (by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

Let G′ be a point which divides BE in the ratio 2:1 ,

Position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$.

Then position vector of G′ is given by $\vec{g}' = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{2\left(\frac{\vec{a}+\vec{c}}{2}\right)+1\vec{b}}{3}$ (by applying section formula)

$$\Rightarrow \vec{g}' = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

As position vector of a point is unique , so G = G′ .

Similarly if we take G″ be a point on CF dividing it in 2:1 ratio then the position vector of G″ will be same as that of G.

Hence G is the one point where three median meet.

∴ The three medians of a triangle are concurrent. (proved)

Example2: - Prove that i) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (It is known as Triangle Inequality).

ii) $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$

iii) $|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Proof:- Let O,A and B be three points, which are not collinear and then draw a triangle OAB.

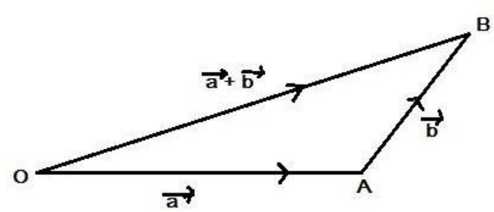


Fig - 16

Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$, then by triangle law of addition we have $\vec{OB} = \vec{a}+\vec{b}$

From properties of triangle we know that the sum of any two sides of a triangle is greater than the third side.

- ⇨ OB < OA + AB
- ⇨ $|\vec{OB}| < |\vec{OA}| + |\vec{AB}|$
- ⇨ $|\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ ----------------- (1)

When O,A, B are collinear then

From figure-17 it is clear that

OB = OA + AB

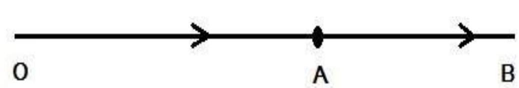


Fig-17

- ⇨ $|\vec{OB}| = |\vec{OA}| + |\vec{AB}|$
- ⇨ $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ ----------------- (2)

From (1) and (2) we have,

$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$ ---------- (1)

But $|(\vec{a}-\vec{b})+\vec{b}| \leq |\vec{a}-\vec{b}| + |\vec{b}|$ (From triangle inequality) ----------- (2)

From (1) and (2) we get $|\vec{a}| \leq |\vec{a}-\vec{b}| + |\vec{b}|$

⇨ $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$(proved)

iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \leq |\vec{a}| + |-\vec{b}|$ (From triangle inequality)

$= |\vec{a}| + |\vec{b}|$(as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

Components of vector in 2D

Let XOY be the co-ordinate plane and P(x,y) be any point in this plane.

The unit vector along direction of X axis i.e. $\vec{OX}$ is denoted by $\hat{\imath}$.

The unit vector along direction of Y axis i.e. $\vec{OY}$ is denoted by $\hat{\jmath}$.

Then from figure-18 it is clear that $\vec{OM} = x\,\hat{\imath}$ and $\vec{ON} = y\hat{\jmath}$.

So, the position vector of P is given by

$$\vec{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath}$$

And OP $= |\vec{OP}| = r = \sqrt{x^2+y^2}$

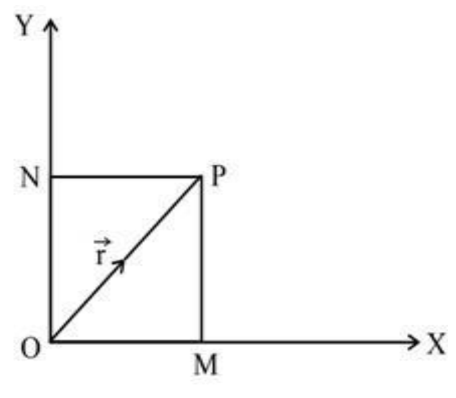


Fig-18

Representation of vector in component form in 2D

If $\vec{AB}$ is any vector having end points A(x_1, y_1) and B (x_2, y_2) , then it can be represented by

$\vec{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$

Components of vector in 3D

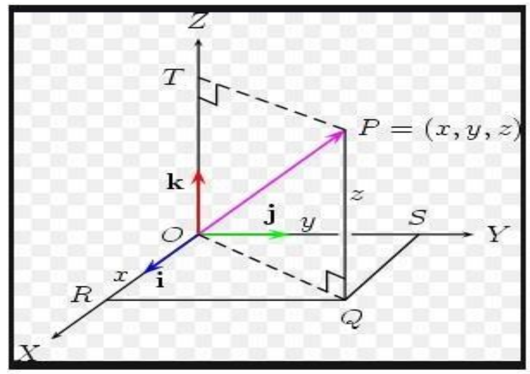


Fig-19

Let P(x,y,z) be a point in space and $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along X axis , Y axis and Z axis respectively. (as shown in fig-19)

Then the position vector of P is given by

$\vec{OP} = x\,\hat{i} + y\,\hat{j} + \hat{k}$, The vectors $x\hat{i}$, $y\,\hat{j}$, $z\hat{k}$ are called the components of $\vec{OP}$ along x–axis, y-axis and z-axis respectively.

And OP $= |\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$

Addition and scalar multiplication in terms of component form of vectors: -

For any vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

i) $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

ii) $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

iii) k $\vec{a} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$, where K is a scalar.

iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\Leftrightarrow a_1=b_1, a_2=b_2, a_3=b_3$

Representation of vector in component form in 3-D & Distance between two points:

If $\vec{AB}$ is any vector having end points A(x_1, y_1, z_1) and B (x_2, y_2, z_2) , then it can be represented by

$\vec{AB}$ **= Position vector of B – Position vector of A**

$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 3:-

Show that the points A(2,6,3) , B(1,2,7) and c(3,10,-1) are collinear.

Solution:- From given data Position vector of A, $\vec{OA} = 2\,\hat{i} + 6\hat{j} + 3\,\hat{k}$.

Position vector of B, $\vec{OB} = \hat{i} + 2\hat{j} + 7\hat{k}$

Position vector of C, $\vec{OC} = 3\,\hat{i} + 10\hat{j} - \hat{k}$

Now $\vec{AB} = \vec{OB} - \vec{OA} = (1-2)\,\hat{i} + (2-6)\,\hat{j} + (7-3)\,\hat{k} = -\hat{i} - 4\hat{j} + 4\hat{k}$.

$\overline{AC} = \overline{OC} - \overline{OA} = (3-2)\hat{\imath} + (10-6)\hat{\jmath} + (-1-3)\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$.

$= -(-\hat{\imath} - 4\hat{\jmath} + 4\hat{k}) = -\overline{AB}$

⇨ $\overline{AB} \parallel \overline{AC}$ or collinear.

∴ They have same support and common point A.

As 'A' is common to both vector , that proves A,B and C are collinear.

Example-4: - Prove that the points having position vector given by $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ form a right angled triangle. [2009(w)]

S0lution :- Let A,B and C be the vertices of a triangle with position vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ respectively

Then . $\overline{AB}$ = Position vector of B – Position vector of A.

$= (1-2)\hat{\imath} + (-3-(-1))\hat{\jmath} + (-5-1)\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$.

$\overline{BC}$ = Position vector of C – Position vector of B.

$= (3-1)\hat{\imath} + (-4-(-3))\hat{\jmath} + (-4-(-5))\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$.

$\overline{AC}$ = Position vector of C – Position vector of A.

$= (3-2)\hat{\imath} + (-4-(-1))\hat{\jmath} + (-4-1)\hat{k} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$.

Now AB = $|\overline{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

BC = $|\overline{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

AC = $|\overline{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

From above $BC^2 + AC^2 = 6+35 = 41 = AB^2$.

Hence ABC is a right angled triangle.

Example-5 :- Find the unit vector in the direction of the vector $\vec{a} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k}$. (2017-W)

Ans:- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{\imath} - 4\hat{y} + \hat{k}}{\sqrt{3^2+(-4)^2+1^2}} = \frac{3\hat{\imath} - 4\hat{y} + \hat{k}}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}\hat{\imath} - \frac{4}{\sqrt{26}}\hat{\jmath} + \frac{1}{\sqrt{26}}\hat{k}.$$

Example-6 :- Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = = \hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans:- Let $\vec{r} = \vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + (\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 2\hat{\imath} + 2\hat{k}$.

Unit vector along direction of $\vec{a} + \vec{b}$ is given by $= \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{\imath} + 2\hat{k}}{\sqrt{2^2+2^2}} = \frac{2\hat{\imath} + 2\hat{k}}{\sqrt{8}} = \frac{2}{\sqrt{8}}\hat{\imath} + \frac{2}{\sqrt{8}}\hat{k}$

$$= \frac{2}{2\sqrt{2}}\hat{k}\hat{\imath} + \frac{2}{2\sqrt{2}}\hat{\jmath} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}.$$

Angle between the vectors:-

As shown in figure-20 angle between two vectors $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ can be determined as follows.

Let $\overrightarrow{OB}$ be a vector parallel to $\overrightarrow{RS}$ and $\overrightarrow{OA}$ is a vector parallel to $\overrightarrow{PQ}$ such that $\overrightarrow{OB}$ and $\overrightarrow{OA}$ intersect each other.

Then θ = ∟AOB = angle between $\overrightarrow{RS}$ and $\overrightarrow{PQ}$.

If θ = 0 then vectors are said to be parallel.

If θ $= \frac{\pi}{2}$ then vectors are said to be orthogonal or perpendicular.

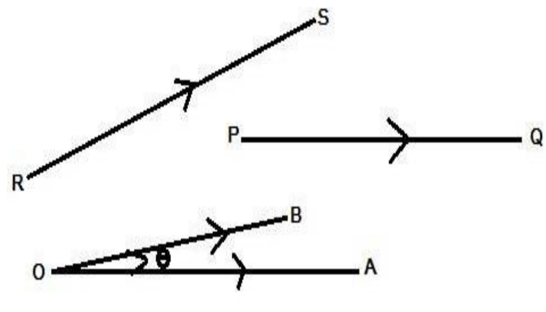


Fig-20

Dot Product or Scalar product of vectors

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are, a and b respectively denoted by $\vec{a}.\vec{b}$ is defined as the scalar abcosθ , where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$.

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\,b \cos\theta$$

Geometrical meaning of dot product

In figure21(a), $\vec{a}$ and $\vec{b}$ are two vectors having θ angle between them. Let M be the foot of the perpendicular drawn from B to OA.

Then OM is the Projection of $\vec{b}$ on $\vec{a}$ and from figure-21(a) it is clear that ,

$|OM| = |OB| \cos\theta = |\vec{b}| \cos\theta$.

Now $\vec{a}.\vec{b} = |\vec{a}| (|\vec{b}| \cos\theta) = |\vec{a}|$ x projection of $\vec{b}$ on $\vec{a}$

which gives projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Similarly we can write $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$= |\vec{b}| (|\vec{a}| \cos\theta) = |\vec{b}|$ projection of $\vec{a}$ on $\vec{b}$.

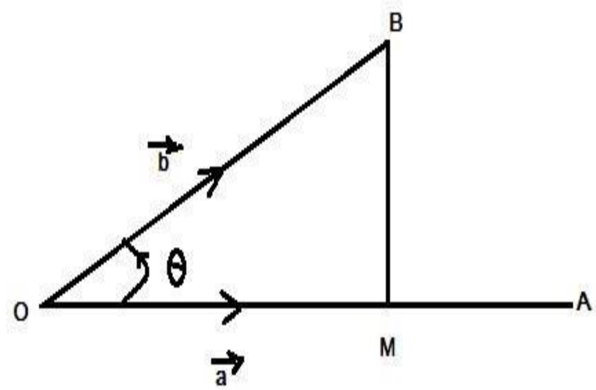


Fig-21

(a)

Similarly, let us draw a perpendicular from A on OB and let N be the foot of the perpendicular in fig-21(b).

Then ON = Projection of $\vec{a}$ on $\vec{b}$

and ON = OAcosθ = $|\vec{a}|\cos\theta$.

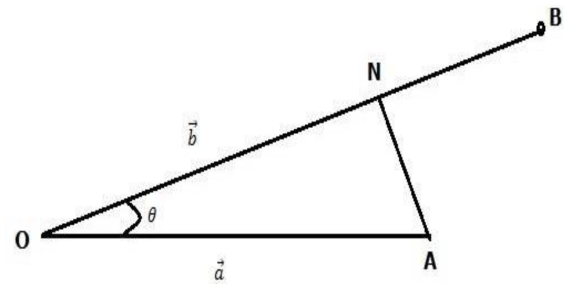


Fig-21(b)

Properties of Dot product

i) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

ii) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributive)

iii) If $\vec{a} \parallel \vec{b}$, then $\vec{a}.\vec{b}$ = ab { as $\theta = 0$ in this case $\cos 0 = 1$ }

In particular $(\vec{a})^2 = \vec{a}.\vec{a} = |\vec{a}|^2$

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

iv) If $\vec{a} \perp \vec{b}$, then $\vec{a}.\vec{b} = 0$. { as $\theta = 90°$ in this case $\cos 90° = 0$ }

In particular $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0 = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k}$

v) $\vec{a}.\vec{0} = \vec{0}.\vec{a} = 0$

vi) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = a^2 - b^2$ {Where $|\vec{a}| = a$ and $|\vec{b}| = b$ }

viii) Work done by a Force:- The work done by a force $\vec{F}$ acting on a body causing displacement $\vec{d}$ is given by W = $\vec{F}.\vec{d}$

Dot product in terms of rectangular components

For any vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ we have,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (by applying distributive(ii) , (iii) and (iv) successively)

Angle between two non zero vectors

For any two non zero vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, having θ is the angle between them we have,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \hat{a}.\hat{b} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$ **(In terms of components.)**

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Condition of Perpendicularity: -

Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are perpendicular to each other $\Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

Condition of Parallelism :-

Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel to each other $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$)

<u>Scalar & vector projections of two vectors (Important formulae)</u>

Scalar Projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\hat{a} = \left[\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}\right]\vec{a}$

Scalar Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$

Examples: -

Q.- 7. Find the value of p for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$, $\hat{i} + p\hat{j} + 3\hat{k}$ are perpendicular to each other.

Solution:- Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$.

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 9$

$b_1 = 1$, $b_2 = p$ & $b_3 = 3$

Given $\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

$\Rightarrow 3.1+2.p+9.3 = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow 2p = -30$

$\Rightarrow p = -15$ (Ans)

Q-8 Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel to each other. (2014-W)

Solution:- Given $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Leftrightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$ { Taking 1st two terms}

$\Leftrightarrow 3 = \frac{2}{p} \Leftrightarrow p = \frac{2}{3}$ (Ans) {Note:- any two expression may be taken for finding p.}

Q-9 Find the scalar product of $3\hat{i} - 4\hat{j}$ and $-2\hat{i} + \hat{j}$. (2015-S)

Solution:- $(3\hat{i} - 4\hat{j}) . (-2\hat{i} + \hat{j}) = (3\times(-2)) + ((-4)\times1) = (-6)+(-4) = -10$

Q-10 Find the angle between the vectors $5\hat{i} + 3\hat{j} + 4\hat{k}$ and $6\hat{i} - 8\hat{j} - \hat{k}$. (2015-W)

Solution:- Let $\vec{a} = 5\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 6\hat{i} - 8\hat{j} - \hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\theta = cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$

$= cos^{-1}\left(\frac{5.6+3.(-8)+4.(-1)}{\sqrt{5^2+3^2+4^2}\sqrt{6^2+(-8)^2+(-1)^2}}\right) = cos^{-1}\left(\frac{30-24-4}{\sqrt{50}\sqrt{101}}\right) = cos^{-1}\left(\frac{2}{\sqrt{50}\sqrt{101}}\right)$

Q-11 Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where,

$\vec{a} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$. { 2013-W ,2017-W, 2017-S}

Solution:- Scalar Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{1.3+(-1).1+(-1).3}{\sqrt{3^2+1^2+3^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$

Vector Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{1.3+(-1).1+(-1).3}{(\sqrt{3^2+1^2+3^2})^2}(3\hat{i} + \hat{j} + 3\hat{k})$

$= \frac{3-1-3}{19}(3\hat{i} + \hat{j} + 3\hat{k}) = \frac{-1}{19}(3\hat{i} + \hat{j} + 3\hat{k})$

Q-12 Find the scalar and vector projection of $\vec{b}$ on $\vec{a}$ where,

$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$. { 2015-S}

Solution: - Scalar Projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|} = \frac{3.2+1.3+(-2).(-4)}{\sqrt{3^2+1^2+(-2)^2}} = \frac{6+3+8}{\sqrt{14}} = \frac{17}{\sqrt{14}}$

Vector Projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{3.2+1.3+(-2).(-4)}{(\sqrt{3^2+1^2+(-2)^2})^2}(3\hat{i} + \hat{j} - 2\hat{k})$

$= \frac{17}{14}(3\hat{i} + \hat{j} - 2\hat{k})$.

Q-13 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Proof:- Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

⇨ $(\vec{a}.\vec{b}) - (\vec{a}.\vec{c}) = \vec{0}$ => $\vec{a}.(\vec{b} - \vec{c}) = \vec{0}$ { applying distributive property}

Dot product of above two vector is zero indicates the following conditions

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{o}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

⇨ $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ (proved)

Example:-14 Find the work done by the force $\vec{F} = \hat{i} + \hat{j} - \hat{k}$. acting on a particle if the particle is displace A from A(3,3,3) to B(4,4,4).

Ans:- Let O be the origin, then

Position vector of A $\vec{OA} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

Position vector of B $\vec{OB} = 4\hat{i} + 4\hat{j} + 4\hat{k}$

Then displacement is given by, $\vec{d} = \vec{AB} = (\vec{OB} - \vec{OA}) = (4\hat{i}+4\hat{j}+4\hat{k}) - (3\hat{i}+3\hat{j}+3\hat{k}) = \hat{i}+\hat{j}+\hat{k}$.

So work done by the force W = $\vec{F}.\vec{d} = \vec{F}.\vec{AB} = (\hat{i}+\hat{j}-\hat{k}).(\hat{i}+\hat{j}+\hat{k})$

$= 1.1+1.1+(-1).1 = 1$ units

Example:-15 If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$$

Proof: - $(|\hat{a} - \hat{b}|)^2 = (\hat{a} - \hat{b}).(\hat{a} - \hat{b}) = (\hat{a}.\hat{a}) - (\hat{a}.\hat{b}) - (\hat{b}.\hat{a}) + (\hat{b}.\hat{b})$ { Distributive property}

$= (|\hat{a}|)^2 - (\hat{a}.\hat{b}) - (\hat{a}.\hat{b}) + (|\hat{b}|)^2$ {commutative property}

$= 1^2 - 2\,\hat{a}.\hat{b} + 1^2$ { as $\hat{a}$ and $\hat{b}$ are unit vectors so their magnitudes are 1}

$= 2 - 2\,\hat{a}.\hat{b} = 2(1 - \hat{a}.\hat{b})$

$= 2(1 - |\hat{a}|.|\hat{b}|\cos\theta)$ { as θ is the angle between $\hat{a}$ and $\hat{b}$ }

$= 2(1 - 1.1.\cos\theta)$

$= 2(1 - \cos\theta) = 2.2\,sin^2\frac{\theta}{2}$

Taking square root of both sides we have $|a - b| = 2\sin\frac{\theta}{2}$

⇨ $\sin\frac{\theta}{2} = \frac{1}{2}|a - b|$ (proved)

Example:-16 If the sum of two unit vectors is a unit vector. Then show that the magnitude of their difference is $\sqrt{3}$.

Proof:-$\hat{a}$,$\hat{b}$ and $\hat{c}$ are three unit vectors such that $\hat{a} + \hat{b} = \hat{c}$

Squaring both sides we have,

=> $(|\hat{a} + \hat{b}|)^2 = (|\hat{c}|)^2$

=> $(|\hat{a}|)^2 + (|\hat{b}|)^2 + 2\,\hat{a}.\hat{b} = 1^2$

=> $1^2 + 1^2 + 2|\hat{a}||\hat{b}|\cos\theta = 1$ { where θ is the angle between $\hat{a}$ and $\hat{b}$ }

=> $1 + 1 + 2\cos\theta = 1$

=> $2\cos\theta = -1$

=> $\cos\theta = \frac{-1}{2}$

Now we have to find the magnitude of their difference i.e $\hat{a} - \hat{b}$.

So $(|\hat{a} - \hat{b}|)^2 = (|\hat{a}|)^2 + (|\hat{b}|)^2 - 2\,\hat{a}.\hat{b} = 1^2 + 1^2 - 2|\hat{a}||\hat{b}|\cos\theta$

$= 2 - 2\cos\theta = 2 - 2(\frac{-1}{2}) = 2 - (-1) = 3$

$\therefore |\hat{a} - \hat{b}| = \sqrt{3}$ (Proved)

Vector Product or Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them , then the vector product of these two vectors denoted by $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}| \sin\theta \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

As shown in figure-21 the direction of $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$.

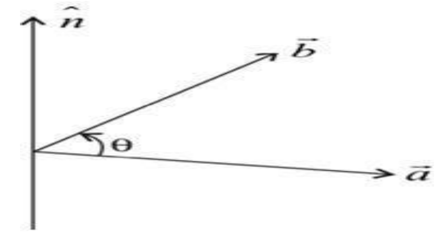


Fig-22

Properties of cross product

i) Vector product is not commutative $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

ii) For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

iii) For any scalar m, $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

iii) Distributive $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

iv) Vector product of two parallel or collinear vectors is zero.

$\vec{a} \times \vec{a} = \vec{0}$ and if $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$ { as $\theta = 0$ or 180° => $\sin\theta = 0$ }

Using this property we have,

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

v) Vector product of orthonormal unit vectors form a right handed system.

As shown in figure- 23 the three mutually perpendicular unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ form a right handed system , i.e. $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$ (as $\theta = 90$, then $\sin\theta = 1$)

$\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$

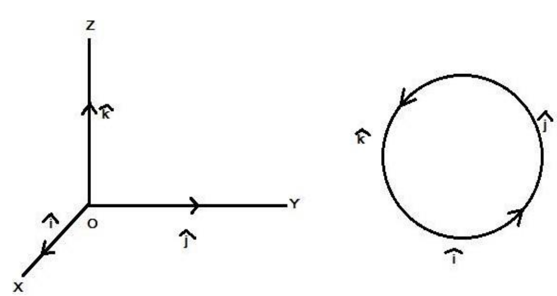


Fig-23

Unit vector perpendicular to two vectors:- Unit vector perpendicular to two given vectors $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Angle between two vectors

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then $\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}| \sin\theta)\,\hat{n}$.

Taking modulus of both sides we have,

$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}| \sin\theta$

⇨ $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

Hence $\theta = sin^{-1}\left\{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right\}$

Geometrical Interpretation of vector product or cross product

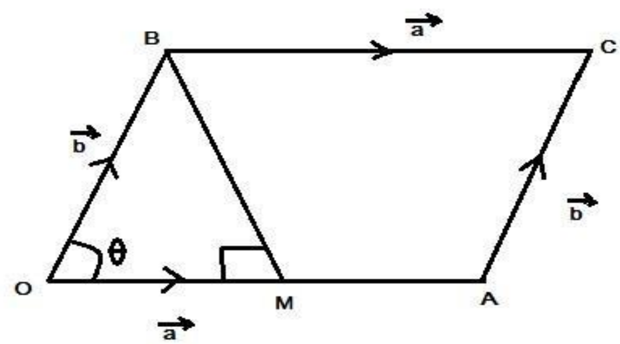


Fig-24

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Then $\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}| \sin\theta)\,\hat{n}$

$= (|\vec{a}|).(|\vec{b}| \sin\theta)\hat{n}$

From fig-24 below it is clear that

BM = OB sin θ = $|\vec{b}| \sin\theta$ = $|\vec{a}|\,|BM|\,\hat{n}$

{ as sin θ = BM/OB & $\vec{OB} = \vec{b}$}

Now $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|\,|\hat{n}|$ = OA.

BM = Area of the parallelogram with side $\vec{a}$ and $\vec{b}$.

Therefore the magnitude of cross product of two vectors is equal to area of the parallelogram formed by these vectors as two adjacent sides.

From this it can be concluded that area of △ ABC = ½ $|\vec{AB} . \vec{AC}|$

Application of cross product

1. Moment of a force about a point ($\vec{M}$) :-Let O be any point and Let $\vec{r}$ be the position vector w.r.t. O of any point 'P' on the line of action of the force $\vec{F}$, then the moment or torque of the force F about origin 'O' is given by

$\vec{M} = \vec{r} \times \vec{F}$

2. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a triangle then the area of the triangle is given by
$\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

3. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram then area of the parallelogram is given by
$\Delta = |\vec{a} \times \vec{b}|$ Sq. unit

4. If $\vec{a}$ and $\vec{b}$ represent two diagonals of a parallelogram then area of the parallelogram is given by
$\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

Vector product in component form : -

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k})$
$+ a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k})$

{ using properties $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{i} \times \hat{j} = \hat{k} = -(\hat{j} \times \hat{i})$, $\hat{j} \times \hat{k} = \hat{i} = -(\hat{k} \times \hat{j})$ and
$\hat{k} \times \hat{i} = \hat{j} = -(\hat{i} \times \hat{k})$ }

$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{i.e. } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Condition of Co-planarity

If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ lies on the same plane then the perpendicular to $\vec{a}$ and $\vec{b}$ must be perpendicular to $\vec{c}$.

In particular $(\vec{a} \times \vec{b}) \perp \vec{c}$ => $(\vec{a} \times \vec{b}) . \vec{c} = 0$

In component form if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $(\vec{a} \times \vec{b}) . \vec{c} = 0$

⇨ $(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 = 0$

⇨ $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$ (interchanging rows two times R_1 and R_2, then R_2 and R_3)

⇨ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Example:- 17

If $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{b} = -\hat{\imath} + 3\hat{k}$ then find $|\vec{a} \times \vec{b}|$

Ans: - We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \{(3\times3)-(0\times(-2))\}\hat{\imath} - \{(1\times3)-((-1)\times(-2))\}\hat{\jmath} + \{(1\times0)-((-1)\times3)\}\hat{k}$

$= 9\hat{\imath} - \hat{\jmath} + 3\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81+1+9} = \sqrt{91}$ (Ans)

Example:-18 Determine the area of the parallelogram whose adjacent sides are the vectors

$\vec{a} = 2\hat{\imath}$ and $\vec{b} = 3\hat{\jmath}$. (2013-W)

Ans:- Area of the parallelogram with adjacent sides given by $\vec{a}$ and $\vec{b}$ is given by

area = $|\vec{a} \times \vec{b}| = |2\hat{\imath} \times 3\hat{\jmath}| = |6\hat{k}| = 6$ sq units (Ans)

Example:-19 Find a unit vector perpendicular to both the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans: - (2015-W and 2017-S)

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ ....(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix}$

$= (3-1)\hat{\imath} - (6+3)\hat{\jmath} + (-2-3)\hat{k}$

$= 2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}$ -----------------(2)

From (1) and (2) we have,

$\hat{n} = \frac{2\hat{\imath} - 9\hat{y} - 5\hat{k}}{|2\hat{\imath} - 9\hat{y} - 5\hat{k}|} = \frac{2\hat{\imath} - 9\hat{y} - 5\hat{k}}{\sqrt{2^2+(-9)^2+(-5)^2}} = \frac{2\hat{\imath} - 9\hat{y} - 5\hat{k}}{\sqrt{110}}$

$= \frac{2}{\sqrt{110}}\hat{\imath} - \frac{9}{\sqrt{110}}\hat{\jmath} - \frac{5}{\sqrt{110}}\hat{k}$ (ans)

Example:-20 If $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$, then find the sine of the angle between these vectors. (2016-w)

Ans :- We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$ ....(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\hat{\imath} - (-2-3)\hat{\jmath} + (8+3)\hat{k} = -3\hat{\imath} + 5\hat{\jmath} + 11\hat{k}$

Hence $|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9 + 25 + 121} = \sqrt{155}$ ....(2)

Again $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$....................................(3)

and $|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$....................................(4)

From equation (1),(2),(3) and (4) we have,

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$ (Ans)

Q-21 Calculate the area of the triangle ABC (by vector method) where A(1,1,2), B(2,2,3) and C(3,-1,-1). (2013-W)

Solution: - Let the position vector of the vertices A,B and C is given by $\vec{a}$,$\vec{b}$ and $\vec{c}$ respectively.

Then $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{c} = 3\hat{i} - \hat{j} - \hat{k}$

Now $\vec{AB} = Position\ vector\ of\ B - Position\ vector\ of\ A$

$= 2\hat{i} + 2\hat{j} + 3\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$= (2-1)\hat{i} + (2-1)\hat{j} + (3-2)\hat{k}$

$= \hat{i} + \hat{j} + \hat{k}$

Similarly $\vec{AC} = Position\ vector\ of\ C - Position\ vector\ of\ A$

$= 3\hat{i} - \hat{j} - \hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$= (3-1)\hat{i} + (-1-1)\hat{j} + (-1-2)\hat{k}$

$= 2\hat{i} - 2\hat{j} - 3\hat{k}$

Now $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= (-3 + 2)\hat{i} - (-3 - 2)\hat{j} + (-2 - 2)\hat{k} = -\hat{i} + 5\hat{j} - 4\hat{k}$

Hence area of the triangle is given by

$\Delta = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{(-1)^2 + 5^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{1 + 25 + 16} = \frac{1}{2}\sqrt{42}$ sq units. (Ans)

Example:-22 Find the area of a parallelogram whose diagonals are determined by the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$. (2014-W, 2017-W)

Ans: - Area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ are given by

$$\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= (4 - 6)\,\hat{\imath} - (12 + 2)\,\hat{\jmath} + (-9 - 1)\,\hat{k} = -2\,\hat{\imath} - 14\,\hat{\jmath} - 10\,\hat{k}$

Now area $\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$

$= \frac{1}{2}\sqrt{4 + 196 + 100} = \frac{\sqrt{300}}{2} = \frac{10\sqrt{3}}{2} = 5\sqrt{3}$ sq unit. (ans)

Example:-23 For any vector $\vec{a}$ and $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}.\vec{b})^2$ where a and b are magnitude of $\vec{a}$ and $\vec{b}$ respectively.

Proof: - $(\vec{a} \times \vec{b})^2 = (|\vec{a}|.|\vec{b}|\sin\theta\,\hat{n})^2$

$= (ab\sin\theta\,\hat{n})^2 = a^2b^2\sin^2\theta$ (As $(\hat{n})^2 = (|\hat{n}|)^2 = 1^2 = 1$)

$= a^2b^2(1 - \cos^2\theta) = a^2b^2 - a^2b^2\cos^2\theta$

$= a^2b^2 - (ab\cos\theta)^2 = a^2b^2 - (\vec{a}.\vec{b})^2$ (Proved)

Example:-24 In a ΔABC , prove by vector method that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$,

where BC = a, CA = b and AB =c. (2017-S)

Proof:- As shown in figure- 25 ABC is a triangle having, $\vec{a} = \overrightarrow{BC}, \vec{b} = \overrightarrow{CA}$ and $\vec{c} = \overrightarrow{AB}$.

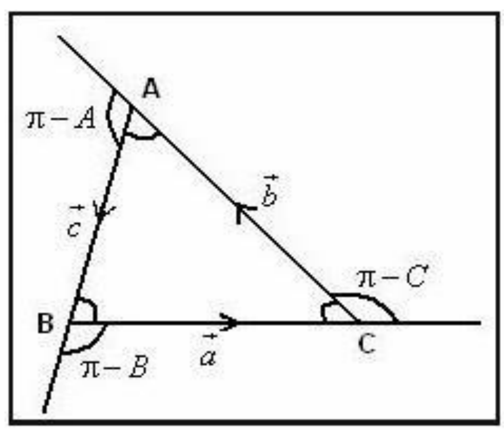


Fig-25

From triangle law of vector we know that ,

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

⇨ $\vec{a} + \vec{b} = -\vec{c}$

⇨ $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ....(1)

(taking cross product of both sides with $\vec{a}$ we get)

⇨ $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

⇨ $(\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $\vec{0} + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $(\vec{a} \times \vec{b}) = -(\vec{a} \times \vec{c})$

⇨ $(\vec{a}\times\vec{b}) = (\vec{c}\times\vec{a})$ -------------------- (2)

Similarly taking cross product with $\vec{b}$ both sides of (1) we have,

⇨ $(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c})$----------------- (3)

From (2) and (3) , $(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a})$

⇨ $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

⇨ $ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

As from fig-25 it is clear that angle between $\vec{a}$ and $\vec{b}$ is $\pi - C$, $\vec{b}$ and $\vec{c}$ is $\pi - A$ and $\vec{c}$ and $\vec{a}$ is $\pi - B$.

Dividing above equation by abc we have,

⇨ $\frac{ab\sin(\pi-C)}{abc} = \frac{bc\sin(\pi-A)}{abc} = \frac{ca\sin(\pi-B)}{abc}$

⇨ $\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Hence $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ (Proved).

Example:-25 What inference can you draw when $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

Ans: - Given $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

⇨ { Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ II $\vec{b}$} and { $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$ }

⇨ As $\vec{a}$ II $\vec{b}$ and $\vec{a} \perp \vec{b}$ cannot be hold simultaneously so $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ Hence either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

Example:-26 If $|\vec{a}| = 2$ and $|\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, then find $\vec{a}.\vec{b}$.

Ans: - Given $|\vec{a}\times\vec{b}| = 8$

⇨ $|\vec{a}||\vec{b}|\sin\theta = 8$

⇨ $2\times 5\sin\theta = 8$

⇨ $\sin\theta = \frac{8}{10} = \frac{4}{5}$

∴ $\cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1-(\frac{4}{5})^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Hence $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 2\times 5\times\frac{3}{5} = 6$ (Ans)

Example:-27 Show that the vectors $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$, and $4\hat{\imath} - 7\hat{\jmath} + 10\hat{k}$ are co-planar.(2017-S)

Ans: - Now let us find the following determinant ,

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) - (-3)(20-8) + 4(-14+4) = 4 + 36 - 40 = 0$$

Hence the three given vectors are co-planar.

Exercise

1. Show that the points (3,4) ,(1,7) and (-5,16) are collinear. (2 Marks)
2. If $\vec{a} = 3\hat{i} - 5\hat{j}$ and $\vec{b} = 2\hat{i} + 3\hat{j}$, then find the unit vector parallel to $\vec{a} + 2\vec{b}$. (2 Marks)
3. Show that the vectors $\vec{a} = 3\sqrt{3}\hat{i} - 3\hat{j}$, $\vec{b} = 6\hat{j}$ and $\vec{c} = 3\sqrt{3}\hat{i} + 3\hat{j}$ form the sides of an equilateral triangle. (5 Marks)
4. Find the unit vector parallel to the sum $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. (2014-W,2017-W).(2 Marks)
5. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$,where $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{j} + \hat{k} - 2\hat{i}$.(2015-W) (5 Marks)
6. The position vector of A,B and C are $2\hat{i} + \hat{j} - \hat{k}, 3\hat{i} - 2\hat{j} + \hat{k}, \hat{i} + 4\hat{j} - 3\hat{k}$ respectively . Show that A, B and C are collinear. (2 Marks)
7. Find the value of 'a' such that the vectors $\hat{i} - \hat{j} + \hat{k},\ 2\hat{i} + \hat{j} + \hat{k}$ and $a\hat{i} - \hat{j} + a\hat{k}$ are coplanar. (2 Marks)
8. Find the value of 'k' so that the vectors $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = k\hat{i} + \hat{j} + 5\hat{k}$ are perpendicular to each other. (2015-W) (2 Marks)
9. Find the unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$. (5 Marks)
10. Find the angle between the vectors $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$. (5 Marks)
11. Calculate the area of the triangle ABC by vector method where A(1,2,4), B(3,1,-2) and C(4,3,1). (5 Marks)
12. Obtain the area of the parallelogram whose adjacent sides are given by vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $-3\hat{i} - 2\hat{j} + \hat{k}$. (5 Marks)
13. Determine the sine of the angle between $\vec{a} = \hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$. (5 Marks)
14. Find the unit vector along the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}$. (2015-S) (2 Marks)
15. Find the area of the parallelogram having adjacent sides $\hat{i} - 2\hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j}$. (5 Marks)
16. Find the unit vector perpendicular to both $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} - \hat{k}$. (5 Marks)
17. Find the area of the parallelogram having vertices A(5,-1,1), B(-1,-3,4), C(1,-6,10) and D(7,-4,7). (5 Marks)
18. Find the vector joining the points (2,-3) and (-1,1).Find its magnitude and the unit vector along the same direction. Also determine the component vectors along the co-ordinate axes. (5 Marks)
19. Prove by vector method , that in a triangle ABC,

$cosA = \frac{b^2+c^2-a^2}{2bc}$ where BC = a, CA = b and AB =c. (5 Marks)

20. Find the work done by the force $4\hat{i} - 3\hat{k}$ on a particle to displace it from (1,2,0) to (0,2,3) (2 Marks)
21. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then show that $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$. (2 Marks)

22. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. (10 Marks)

23. Find the area of the parallelogram whose diagonals are $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -3\hat{\imath} + 4\hat{\jmath} - \hat{k}$. (5 Marks)

Answers

2) $\frac{7}{5\sqrt{2}}\hat{\imath} + \frac{1}{5\sqrt{2}}\hat{\jmath}$

4) $\frac{3\hat{\imath}+6\hat{y}-2\hat{k}}{7}$

5) $\frac{-1}{\sqrt{6}}$, $\frac{-1}{6}(\hat{\jmath} + \hat{k} - 2\hat{\imath})$

7) 1

8) 3

9) $\frac{11}{\sqrt{122}}\hat{\imath} - \frac{1}{\sqrt{122}}\hat{k}$

10) $\frac{\pi}{2}$

11) $\frac{5\sqrt{10}}{2}$ sq units

12) $6\sqrt{5}$ sq units

13) $\frac{4\sqrt{2}}{\sqrt{33}}$

14) $\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$

15) $\sqrt{45}$ sq units

16) $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{k}$

17) $\sqrt{2257}$ sq units

18) $-3\hat{\imath} + 4\hat{\jmath}$, 5 , $\frac{-3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$, $\frac{-3}{5}\hat{\imath}$ and $\frac{4}{5}\hat{\jmath}$.

20) -13 units

23) $\frac{3\sqrt{30}}{2}$ sq units.

LIMITS AND CONTINUITY

INTRODUCTION:-

The concept of limit and continuity is fundamental in the study of calculus. The fragments of this concept are evident in the method of exhaustion formulated by ancient Greeks and used by Archimedes (287-212 BC) in obtaining a formula for the area of the circular region conceived as successive approximation of areas of inscribed polygons with increased number of sides.

The concept of calculus is used in many engineering fields like Newton's Law of cooling derivation of basic Fluid mechanics equation etc.

In general the study of the theory of calculus mainly depends upon functions. Thus it is desirable to discuss the idea of functions before study of calculus.

OBJECTIVES:-

After studying this topic, you will be able to

(i) Define function and cited examples there of

(ii) State types of functions.

(iii) Define limit of a function.

(iv) Evaluate limit of a function using different methods.

(v) Define continuity of a function at a point.

(vi) Test continuity of a function at a point.

EXPECTED BACKGROUND KNOWELDGE :-

(1) Set Theory

(2) Concept of order pairs.

RELATION:-

Definition : - If A and B are two nonempty sets, then any subset of AXB is called a relation 'R' from A to B.

Mathematically $R \subset AXB$

Since $\emptyset \subset AXB$ and AXB is a subset of itself, therefore Ø and AXB are relations from A to B.

EXAMPLE:-1

$A = \{1,2\}$ $B = \{a, b, c\}$

Then $A\times B = \{ (1,a), (1,b),(1,c),(2,a),(2,b),(2,c)\}$

Now the following subsets of $A\times B$ give some examples of relation from A to B.

$R_1 = \{(1, a), (2, b)\}$, $R_2 = \{(1, c)\}$, $R_3 = \{(2, b)\}$ and $R_4 = \{(1, a), (1, b), (2, c)\}$ are examples of some relations. The above relations can be represented by figure1-4 as follows.

$R_1 = \{(1,a),(2,b)\}$

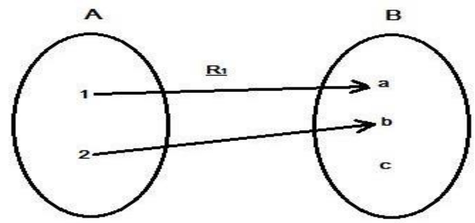


Fig-1

$R_2 = \{(1,c)\}$

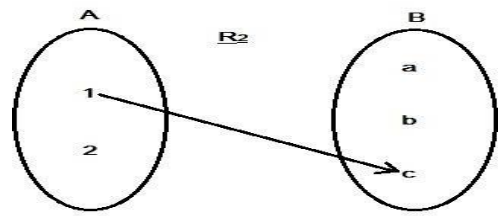


Fig-2

$R_3 = \{(2,b)\}$

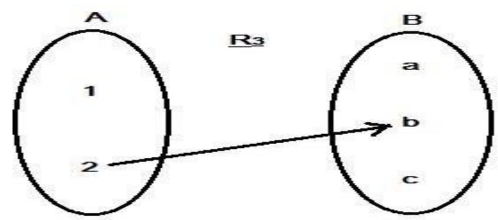


Fig-3

$R_4 = \{(1,a),(1,b),(2,c)\}$

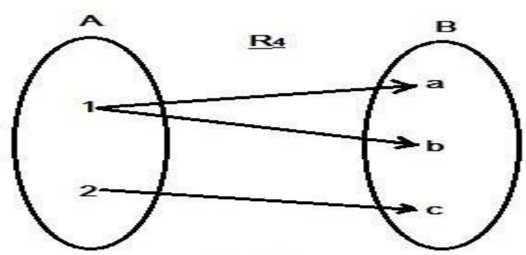


Fig-4

Example – 2

A = Players = $\{sachin, dhoni, pele, saina\}$

B = Game = $\{cricket, badminton, football\}$

Then R = player related to their games.

The pictorial representation is given in Fig-5

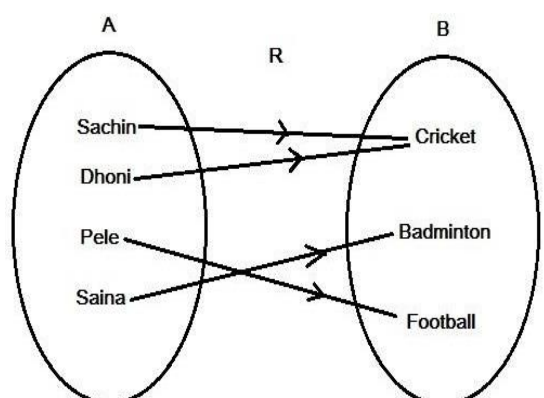


Fig-5

FUNCTION: -

A relation 'f' from X to Y is called a function if it satisfies the following two conditions

(i) All elements of X are related to the elements of Y.

(ii) Each element of X related to only one element of Y.

In above example -1 only relation R_1 is a function, because all elements of A is related. Each element (i.e. 1-a and 2-b) is related to only one element of B.

R_2 is not a function as '2' is not related.

R_3 is not a function as '1' is not related.

R_4 is not a function as 1 related a and b.

EXAMPLE – 2 represent a function.

EXAMPLE -4

Let us consider a function F from A = $\{1,2,3\}$ to $B = \{a, b, c, d\}$ as follows.

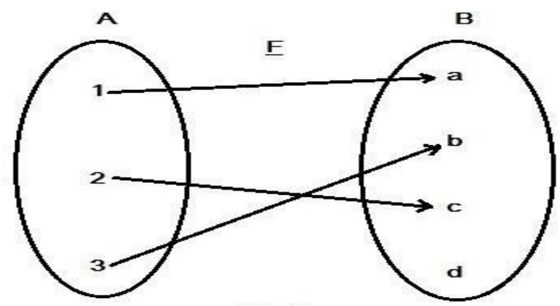


Fig-6

Here we can write F(1) = a

F (2) = c

F(3) = b

DOMAIN (D_F)

Let F:X→Y is a function then the First set 'X' is called domain of F.

X= Dom F = D_F

In example-4 $\{1,2,3\}$ is the domain.

Co-domain –

If F: X→Y is a function, then the 2nd set Y is called co-domain of F. In example -4 $\{a, b, c, d\}$ is the co-domain.

IMAGE:- If f: X→Y is a function and for any $x \in X$, we have $f(x) \in Y$.

This f (x) =y $\in$ Y is called theimage of x.

In example -4 'a' is the image of 1

'b' is the image of 3

'c' is the image of 2

Range (R_F): -

The image set of 'X' i.e. domain is called range of F.

F(X) = Range of F

In example -4 {a, b, c} represent the range of F.

In above discussion we have taken examples of finite sets. But when we consider infinite sets it is not possible to represent a function either in tabular form or in figure form. So, we define function in another way as follows.

CONSTANT- A quantity which never changes its value. Constants are denoted by A, B, C etc.

VARIABLE:-A quantity which changes its value continuously x, y, z etc are used for variables.

TWO TYPES OF VARIABLE:- i) Independent variable ii) dependent variable

Independent variable → Variable which changes its value independently .Generally we take 'x' as independent variable.

Dependent Variable → Variable which changes its value depending upon independent variable. We take 'y' as dependent Variable.

DEFINITION OF FUNCTION:-

Let X and Y be two non empty sets. Then a function or mapping 'f' assigned from set X to the set Y is a sort of correspondence which associated to each element $x \in X$ a unique element $y \in Y$ and is written as

f : X → Y (read as " f maps X into Y).

The element 'y' is called the image of x under f and is denoted by f(x) i.e. y = f (x) and x is called pre-image of y.

Example - Let y = F(x) = x^2 where X ={ 1,1.5,2}

Pictorial representation is given in Fig-7

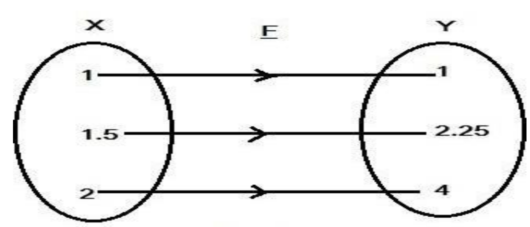


Fig-7

Here values of x form domain and values of y form range.

FUNCTIONAL VALUE f (a)→ The value of f (x) obtained by replacing x by a is called functional value of f(x) at x=a, denoted by f(a)

Example :- Let y = f (x) = x^2

Then functional value of f(x) at x=2 is f(2)= 2^2 =4

The functional value of f(x) at x=1.5 is f(1.5) = 2.25

If f(x) =$\frac{1}{x}$ then f(1)= 1/1=1 , f(2) = ½. But f(0) =$\frac{1}{0}$ which is undefined

So the function value of function is either finite or undefined.

Classification of function

Functions are classified into following categories

Into function

A function F : A → B is said to be into if there exist at least one element in B which has no pre-image in A. In this case Range set is a proper subset of co-domain Y.

Let A = {1,2,3} and B = { a,b,c,d}

Then the function F given by fig-8 represent one into function from A to B.

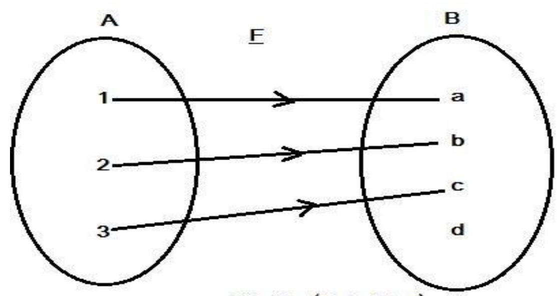


Fig-8 (Into Map)

In above figure d has no pre-image in A.

Onto function

A function F : A → B is said to be onto if Range of F i.e. F(A) = B. In other words every element of B has a pre-image in A.

Let A = {1,2,3,4} and B = { a,b,c,}

Then the function F given by fig-9 represent one onto function from X to Y.

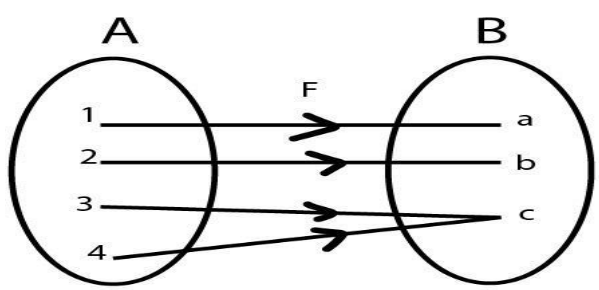


fig-9
(Onto Map)

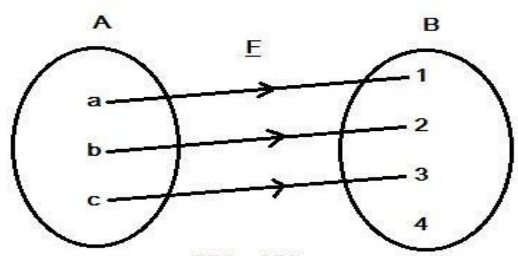
From above figure it is clear that F(A) = B.

One-one function

A function F: A → B is said to be one-one if each distinct elements in A have distinct images in B i.e. if $x_1 \neq x_2$ in A => F(x_1) ≠ F(x_2) in B.

Let A = {a,b,c} and B = { 1,2,3,4}

Then the function F given by fig-10 represent an one-one function from A to B .



one - one

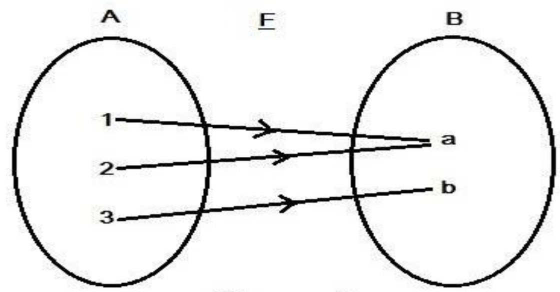
Fig-10

Many-one function

A function F : A → B is said to be many-one if there exists at least one element in B, which has more than one pre-image in A.

Let A = {1,2,3} and B = { a,b}

Then the function F given by fig-11 represent a many-one function from A to B .



(Many one)

Fig-11

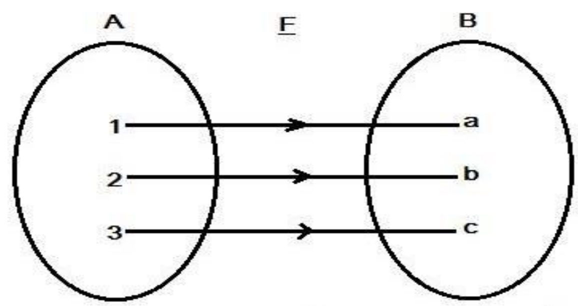
From above figure it is clear that a has two pre-images 1 and 2 in A .

One-one and onto function or bijective function

A function F : A → B is said to be one-one and onto if it is one-one and onto i.e each distinct element in A has distinct image in B and every element of B has a pre-image in A.

Let A = {1,2,3} and B = { a,b,c}

Then the function F given by fig-12 represent an one-one and onto function from A to B .



(One-one and onto)

Fig-12

Inverse function:-

If F : X → Y is a bIjective function then it's inverse function defined from Y to X denoted by F^{-1} . If x ∈ X and y ∈ Y such that y= F(x) then x= F^{-1}(y). One example of inverse function is given in Fig-13 .

Let X = { a,b,c} and Y = {1,2,3} .

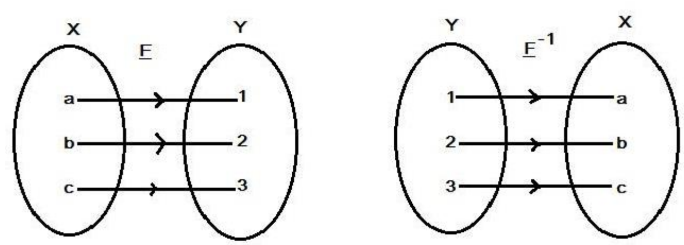


Fig-13

Composition of two function

If f : X → Y and g : Y → Z be any two functions having Range f = Domain g, then composition of f and g denoted by $g_o f$ is defined by $g_o f(x) = g(f(x))$, x∈ X.

Domain of $g_o f$ = X and Range of $g_o f$ = Range of g.

The composition of two function f and g is shown in fig-14.

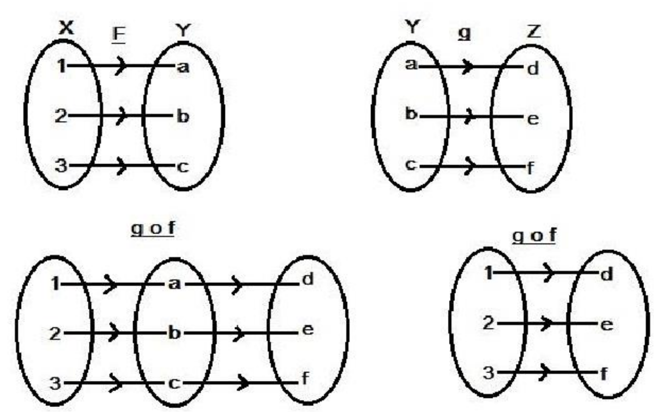


Fig-14

Examples of some composite functions is given below y = f(x) = sinx^2 formed by composition of $x^2 and\ sinx$.

y = f(x) = $\sqrt{cotx}$ formed by composition of cotx and $\sqrt{x}$.

y = f(x) = $\log_a sin\sqrt{x}$ formed by composition of $\sqrt{x}$, sinx and $\log_a x$

Real Valued Function:-

F : X→Y is called a real valued Function.

If dom F = X ⊂ R and Y ⊂ R.

Generally we discuss our topics on this type of functions.

DIFFERENT TYPES OF FUNCTIONS:-

(1) **CONSTANT FUNCTION** :-

The function F (x) = K for all x ∈ R, where K is some real number is called a constant function

For Constant Function D_f = R

R_f= {K}

(2) **IDENTITY FUNCTION** :-

F(x)=x ∀x ∈ R, is called Identity Function. (∀ x means for all x)

It is also denoted by I_x or I.

Dom_I= D_I= R

(3) **TRIGNOMETRIC FUNCTIONS** :-

sinx, cosx, tanx, cotx, secx, cosecx are called trigonometric functions.

we know the defination of these functions.

We know that -1≤sinx≤1 and -1≤cosx≤1 ∀x ∈ R. Here x is radian measure of an angle.

Function	Domain	Range
Sinx	R	[-1,1]
Cosx	R	[-1,1]
Tanx	R- {(2n+1)$\frac{\pi}{2}$: $n€z$}	R
Cotx	R- {nπ: n€z}	R
Secx	R- {(2n+1)$\frac{\pi}{2}$:n€z}	R-(-1,1)
Cosecx	R- {nπ: n€z}	R-(-1,1)

(4) INVERSE TRIGNOMETRIC FUNCTIONS :-

$sin^{-1}x$, $cos^{-1}x$, $tan^{-1}x$, $cot^{-1}x$, $sec^{-1}x$, $cosec^{-1}x$ are called inverse trigonometric functions. These are real functions.

Function	Domain	Range
$sin^{-1}x$	[-1,1]	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$cos^{-1}x$	[-1,1]	$[0,\pi]$
$tan^{-1}x$	R	$(-\frac{\pi}{2},\frac{\pi}{2})$
$cot^{-1}x$	R	$(0,\pi)$
$sec^{-1}x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$
$cosec^{-1}x$	$x \leq -1$ or $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$

(5) EXPONENTIAL FUNCTION (a^x):-

An exponential Function is defined by $F(x) = a^x$ ($a>0$, $a\neq1$), for all x€R

$D_F=R$

$R_F=R_+$

Properties

(1) $a^{x+y} = a^x.a^y$

(2) $(a^x)^y = a^{xy}$

(3) $a^x=1$,<=> $x=0$

(4) If $a>1$, $a^x<a^y$ if $x<y$

If $a<1$, $a^x>a^y$ if $x<y$

(5) LOGARITHMIC FUNCTION

The inverse of a^x is called logarithmic function.

$f(x) = \log_a x$ (log x base a) is called the logarithmic Function.

Dom f = R^+ , $R_f=R$

PROPERTIES

(1) $\log_a(xy) = \log_a x + \log_a y$

(2) $\log_a(\frac{x}{y}) = \log_a x - \log_a y$

(3) $\log_a x = 0 \iff x=1$

(4) $\log_a a = 1$

(5) $\log_a x = \frac{1}{Log_x a}$ $(x \neq 1)$

(6) $\log_a x = \log_b x . \log_a b$

(7) $\log_a x^n = n \log_a x$

(8) $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$

(6) ABSOLUTE VALUE FUNCTION OR MODULUS FUNCTION (IxI) :-

The function f defined by f(x) = $|x| = \begin{cases} -x & when\ x < 0 \\ x & when\ x \geq 0 \end{cases}$

Is called Absolute Value Function.

D_f= R

$R_F = R^+ \cup \{0\}$

E.g. |5| = 5 (as 5>0)

|-2| = -(-2) = 2 (as-2<0)

|0| = 0

|3.7| = 3.7

|-5.2 | = 5.2

(8) GREATEST INTEGER FUNCTION OR BRACKET x ([x]) : -

The greatest integer Function [x] is defined as

$[x] = \begin{cases} x & if\ x \in Z \\ n & if\ x \not\in Z\ and\ n < x < n+1\ where\ n \in Z \end{cases}$

Example

[2] = 2 (as $2 \in Z$)

[0] = 0 (as $0 \in Z$)

[2.5] = 2 $\{as\ 2.5 \not\in Z\ and\ 2.5\ lies\ between\ 2\ to\ 3\ i.e.\ 2 < 2.5 < 3\}$

[-1.5] = -2 $\{as\ -2 < -1.5 < -1\}$

[$\sqrt{3}$] = 1 { as 1 < $\sqrt{3}$ < 2 }

[-e] =-3 {as 2< e<3 => -3<-e<-2 }

Dom[x] = R and Range [x] = Z

Functions are categories under two types as follows

(1) ALGEBRAIC FUNCTION (Three types)

i) **Polynomial P(x)** = a_0+a_1x----------$+a_n x^n$

E.g→F(x) = x^2+2x+3, F(x)=3x+5 e

(ii) **Rational Function** $\left(\frac{P(x)}{Q(x)}\right) = \frac{a_0+a_1x+\text{------------}+a_n x^n}{b_0+b_1x+\text{------------}+b_m x^m}$

E.g. $\frac{x}{x^2+1}$, $\frac{x^2+2x+5}{3x+1}$ are rational functions.

iii) **Irrational Function {P(x)} $^{p/q}$** e.g→$\sqrt{x}$,($x^2+2x+1)^{2/3}$ etc.

2) TRANSCENDENTAL FUNCTION

Trigonometric, logarithmic, exponential functions are called transcendental functions.

Again there are following types of functions as follows.

EXPLICT FUNCTION

y=f(x) i.e. if y is expressed directly in terms of independent variable x, then it is called explicit function .

Example $y = x^2$

$y = 2x + 1$ etc.

IMPLICT FUNCTION

Function in which x and y cannot be separated from each other (i.e.) F(x,y) = 0 is called implicit Function. E.g. → $x^2+y^2 = 1$

$x^3+2xy+3x^2y^2 = 7$ are examples implicit functions.

EVEN FUNCTION:-

If f(-x) = f(x), thenf(x) is called even function.

Example f(x) = cosX

f(-x) =cos (-x) =cosx =f(x)

Hence f(x) = cosX is an even function

Similarly f(x) = x^2, x^4 ------------------- are even functions.

ODD FUNCTION:-

If f(-x) = -f(x), then f(x) is called odd function.

f(x) = sinx, x, x^3,------------------------- are example of odd functions

INTRODUCTION TO LIMIT:-

The concept of limits plays an important role in calculus. Before defining the limit of a function near a point let us consider the following example

$$\text{Let } F(x) = \frac{x^2-1}{x-1}$$

$$\text{Now } F(1) = \frac{1^2-1}{1-1} = \frac{0}{0} \text{ undefined}$$

But if we take x close to 1, we obtain different values for F(x) as follows

TABLE -1

X	0.91	0.93	0.99	0.9999	0.99999
F(X)	1.91	1.93	1.99	1.9999	1.99999

TABLE – 2

X	1.1	1.01	1.001	1.00001	1.000001
F(X)	2.1	2.01	2.001	2.00001	2.000001

In above we can see that when x gets closer to 1,F(x) gets closer to 2.however,in this case F(x) is not defined at x=1,but as x approaches to 1 F (x) approaches to 2.

This generates a new concept in setting the value of a function by approach method. The above value is called limiting value of a Function.

SOME DEFINATIONS ASSOCIATED WITH LIMIT:-

NEIGHBOURHOOD:-

For every a€R, the open interval (a-δ, a+ δ) is called a neighborhood of a where δ>0 is a very very small quantity.

Example (1.9,2.1) is a neighborhood of 2. (δ =0.1)

DELETED NEIGHBOURHOOD of 'a':-

(a- δ, a+ δ)- {a} is called deleted neighborhood of a.

Left neighborhood of a is given by (a- δ, a) .

Right neighborhood of a given by (a, a+ δ).

Example

(1.9.2.1)- {2} is a deleted neighborhood of 2.

(1.9, 2) is left neighborhood of 2.

(2, 2.1) is a right neighborhood of 2.

DEFINATION OF LIMIT:-

Given €>0, there exist δ>0 depending upon € only such that , |x-a|< δ => |f(x)-l|<€

Then $\lim_{x\to a} f(x)$ **= l**

EXPLANATION

If for every ∈>0,we can able to find δ ,which depends upon ∈ only such that x∈ (a- δ,a+ δ),=> f(x) ∈ (l-∈,l+∈) . In other words when x gets closer to a then f(x) gets closer to l.

We read x→a as x tends to 'a' i.e. x is nearer to a but x≠a

$\lim_{x\to a} f(x)$→ limit x tends a f(x). 'l' is called limiting value of f(x)at x = a.

In 1st example $\lim_{x\to 1} \frac{x^2-1}{x-1}$= 2

i.e limiting value of f(x) at x=1 is 2.

Note:-

Functional value always gives the exact value of a function at a point where as limiting value gives an approximated value of function.

Functional value is either defined or undefined. Similarly limiting value is either exist or does not exist.

EXISTENCY OF LIMITING VALUE:-

In our first example if we observe table -1 then we see we approach 2 from left in that table. In table -2 we approach 2 from right.

So in table -1 x€ (2- δ,2)

And in table -2 x€ (2,2+ δ)

These two approaches give rise to two definitions.

LEFT HAND LIMIT

When x approaches a from left then the value to which f(x) approaches is called left hand limit of f(x) at x=a written as L.H.L.= $\lim_{x\to a^-} f(x)$

X→a^- means x€$(a-\delta, a)$.

RIGHT HAND LIMIT: -

When x approaches a form right then the value to which f(x) approaches is called right hand limit.

Mathematically

R.H.L. = $\lim_{x\to a^+} f(x)$

x→a^+ means x∈ (a,a+ δ)

EXISTENCY OF LIMIT

If L.H.L = R.H.L i.e.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$$

Then the limit of the function exists and $\lim_{x\to a} f(x) = l$

Otherwise limit does not exist.

ALGEBRA OF LIMIT: -

IF $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$

Then

i) $\lim_{x\to a}\{ f(x) + g(x) \} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m$

ii)) $\lim_{x\to a}\{ f(x) - g(x) \} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m$

iii) $\lim_{x\to a}\{ f(x) . g(x) \} = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = l\, m$

iv)) $\lim_{x\to a}\{ \frac{f(x)}{g(x)} \} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$ (provided m≠0)

v) $\lim_{x\to a} K = K$ (K is constant)

vi) $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x) = K\, l$

(Vii) $\lim_{x\to a} \log_b f(x) = \log_b \lim_{x\to a} f(x) = \log_b l$

Viii) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$

(ix) $\lim_{x\to a} f(x)^n = \{\lim_{x\to a} f(x)\}^n = l^n$

(x) $\lim_{x\to a} f(x) = \lim_{y\to a} f(y) = l$

(xi) $\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)| = |l|$

(xii) If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

EVALUATION OF LIMIT:-

When we evaluate limits it is not necessary to test the existency of limit always. So in this section we will discuss various methods of evaluating limits.

(1) EVALUATION OF ALGEBRAIC LIMITS :-

(2) Method -> (i) Direct substitution (ii) Factorisation Iii) Rationalisation

i) Direct Substitution :-

If f(x) is an algebraic function and f(a) is finite. Then $\lim_{x\to a} f(x)$ is equal to f(a) i.e. we can substitute x by a.

Let us consider following examples.

Example -1

Evaluate $\lim_{x\to 0}(x^2 + 2x + 1)$

ANS →

$\lim_{x\to 0}(x^2 + 2x + 1)$= 0^2+2x0+1 = 1

Example -2

Evaluate $\lim_{x\to -1} \frac{x-1}{x^2+2x-1}$

ANS→

$$\lim_{x\to -1} \frac{x-1}{x^2+2x-1} = \frac{(-1)-1}{(-1)^2+2\times(-1)-1} = \frac{-2}{1-2-1} = \frac{-2}{-2} = 1$$

Example – 3

$\lim_{x\to 1} \frac{\sqrt{x}}{\sqrt{x+2}}$ = ?

Ans :-

$$\lim_{x\to 1} \frac{\sqrt{x}}{\sqrt{x+2}} = \frac{\sqrt{1}}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$$

Example -4

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$,Which cannot be determined.

NOTE: -

So here direct substitution method fails to find the limiting value. In this case we apply following method.

ii) FACTORISATION METHOD :-

If the given Function is a rational function $\frac{f(x)}{g(x)}$, and $\frac{f(a)}{g(a)}$ is in $\frac{0}{0}$ form then we apply factorisation method i.e we factorise f(x) and g (x) and cancel the common factor. After cancellation we again apply direct substitution, if result is a finite number otherwise we repeat the process .

This method is clearly explained in following example.

Example -4

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Ans : - $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)}$

= $\lim_{x\to 1}$(x+1) {x→ 1 means x≠1 => (x-1) ≠ 0}

= 1+1 = 2 {after cancellation we can apply the direct substitution}

Example -5

Evaluate $\lim_{x\to -3}\frac{x^2+7x+12}{x^2+5x+6}$

ANS :-

$\lim_{x\to -3}\frac{x^2+7x+12}{x^2+5x+6}$ {by putting x=-3 we can easily check that the question is in $\frac{0}{0}$form }

$= \lim_{x\to -3}\frac{x^2+4x+3x+12}{x^2+2x+3x+6}$

$= \lim_{x\to -3}\frac{x(x+4)+3(x+4)}{x(x+2)+3(x+2)}$

$= \lim_{x\to -3}\frac{(x+4)(x+3)}{(x+2)(x+3)}$ {x→ -3 then x+3 ≠ 0}

$= \lim_{x\to -3}\frac{(x+4)}{(x+2)} = \frac{-3+4}{-3+2} = \frac{1}{-1} = -1$

Example – 6

Evaluate $\lim_{x\to 4}\frac{x^3-3x^2-3x-4}{x^2-4x}$

ANS ->

$\lim_{x\to 4}\frac{x^3-3x^2-3x-4}{x^2-4x}$ ($\frac{0}{0}$form)

As x=4 gives $\frac{0}{0}$form

⇨ X-4 is a factor of both polynomials.

x-4 | x^3-3$x^2 - 3x - 4$| x^2+x+1

| x^3 -4x^2 |

\- +

| $x^2 - 3x - 4$|

| $x^2 - 4x$ |

\- +

| x - 4 |

| x - 4 |

0

Hence x^3 -3x^2-3x-4 = (x-4)(x^2+x+1)

Now $\lim_{x\to 4}\frac{x^3-3x^2-3x-4}{x^2-4x} = \lim_{x\to 4}\frac{(x-4)(x^2+x+1)}{x(x-4)} = \lim_{x\to 4}\frac{x^2+x+1}{x} = \frac{4^2+4+1}{4} = \frac{21}{4}$.

iii) Rationalisation method :-

When either the numerator or the denominator contain some irrational functions and direct substitution gives $\frac{0}{0}$ form, then we apply rationalisation method. In this method we rationalize the irrational function to eliminate the $\frac{0}{0}$ form.This can be better explained in following examples.

Example -7

Evaluate $\lim_{x\to0}\frac{x}{\sqrt{x+1}-1}$

ANS :-

$\lim_{x\to0}\frac{x}{\sqrt{x+1}-1}$ { In order to rationalize $\sqrt{x+1}-1$ *we have to apply* a^2-b^2 *formula*

$a^2-b^2 = (a+b)(a-b)$ *so here* $a-b$ *is present, so we have to multiply* $a+b$ *i.e.* $\sqrt{x+1}+1$}

$$= \lim_{x\to0}\frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$$

$$= \lim_{x\to0}\frac{x(\sqrt{x+1}+1)}{\sqrt{x+1}^2-1^2}$$

$$= \lim_{x\to0}\frac{x(\sqrt{x+1}+1)}{x+1-1} = \lim_{x\to0}\frac{x(\sqrt{x+1}+1)}{x}$$

$$= \lim_{x\to0}(\sqrt{x+1}+1) = \sqrt{0+1}+1 = 1+1 = 2$$

Example – 8

Evaluate $\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

Ans :-

$$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x} \quad (\tfrac{0}{0}\text{ form})$$

$$= \lim_{x\to0}\left(\frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x}+\sqrt{1-x})}\right)$$

$$= \lim_{x\to0}\left(\frac{(\sqrt{1+x})^2-(\sqrt{1-x})^2}{2x(\sqrt{1+x}+\sqrt{1-x})}\right) = \lim_{x\to0}\frac{(1+x)-(1-x)}{2x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to0}\frac{2x}{2x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to0}\frac{1}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{1}{\sqrt{1+0}+\sqrt{1-0}} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

(3) Evaluating limit when x-> ∞

In order to evaluate infinite limits we use some formulas and techniques.

Formulas (i) $\lim_{x\to\infty} x^n = \infty$, n>0

(ii) $\lim_{x\to\infty} \frac{1}{x^n} = 0$, n>0

When we evaluate functions in $\frac{f(x)}{g(x)}$ form , the we use the following technique

Divide both f(x) and g(x) by x^k where $x^k$is the highest order term in g(x).

It can be better understood by following examples.

Example – 9

Evaluate $\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-7x+5}$

ANS:- $\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-7x+5}$

{Dividing numerator and denominator by highest order term in denominator i.e. x^2}

$$= \lim_{x\to\infty} \frac{\frac{3x^2+x-1}{x^2}}{\frac{2x^2-7x+5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{7x}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}-\frac{1}{x^2}}{2-\frac{7}{x}+\frac{5}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3+\lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{7}{x} +\lim_{x\to\infty}\frac{5}{x^2}}$$ (applying algebra of limits)

$$= \frac{3+0-0}{2-0+0} = \frac{3}{2}$$

Example – 10

Evaluate $\lim_{x\to\infty} \frac{x^3+2x^2+3}{x^4-3x+1}$

ANS :-

$$\lim_{x\to\infty} \frac{x^3+2x^2+3}{x^4-3x+1}$$

{ Dividing numerator and denominator by highest order term x^4}

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x^4}+\frac{2x^2}{x^4}+\frac{3}{x^4}}{\frac{x^4}{x^4}-\frac{3x}{x^4}+\frac{1}{x^4}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^4}}{1-\frac{3}{x^3}+\frac{1}{x^4}} = \frac{0+0+0}{1-0+0} = \frac{0}{1} = 0$$

Example – 11

Evaluate $\lim_{x\to\infty} \frac{x^4+5x+2}{x^3+2}$

ANS :-

$$\lim_{x\to\infty} \frac{x^4+5x+2}{x^3+2}$$

{ Dividing numerator and denominator by highest order term of denominator i.e. x^3}

$$= \lim_{x\to\infty} \frac{\frac{x^4}{x^3}-\frac{5x}{x^3}+\frac{2}{x^3}}{\frac{x^3}{x^3}+\frac{2}{x^3}} = \lim_{x\to\infty} \frac{x-\frac{5}{x^2}+\frac{2}{x^3}}{1+\frac{2}{x^3}} = \frac{\lim_{x\to\infty} x-\lim_{x\to\infty}\frac{5}{x^2}+\lim_{x\to\infty}\frac{2}{x^3}}{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{2}{x^3}}$$

$$= \frac{\lim_{x\to\infty} x-0+0}{1+0} = \lim_{x\to\infty} x = \infty$$

Example – 12

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$$

ANS :- $\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3} = \lim_{x\to\infty} \frac{\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{x}}{\frac{4x+3}{x}}$

{ Dividing numerator and denominator by highest order term in denominator i.e. x.}

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2-1}{x^2}}-\sqrt{\frac{2x^2-1}{x^2}}}{\frac{4x}{x}+\frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2}{x^2}-\frac{1}{x^2}}-\sqrt{\frac{2x^2}{x^2}-\frac{1}{x^2}}}{4+\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3-\frac{1}{x^2}}-\sqrt{2-\frac{1}{x^2}}}{4+\frac{3}{x}} = \frac{\{\lim_{x\to\infty}(3-\frac{1}{x^2})\}^{1/2}-\{\lim_{x\to 0}(2-\frac{1}{x^2})\}^{1/2}}{\lim_{x\to\infty} 4+\lim_{x\to\infty}\frac{3}{x}}$$

$$= \frac{(3-0)^{1/2}-(2-0)^{1/2}}{4+0}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4} \quad \text{(ans)}$$

Important note in ∞ limit evaluation:-

$$\lim_{x\to\infty} \frac{a_0+a_1x+------+a_mx^m}{b_0+b_1x+b_2x^2+------+b_nx^n} = \begin{cases} \frac{a_m}{b_n} & if\ m=n \\ 0 & if\ m<n \\ \infty & if\ m>n \end{cases}$$

Example-13

If $\lim_{x\to\infty} \left(\frac{x^2-1}{x+1} - ax - b\right) = 2, find\ the\ values\ of\ a\ and\ b.$

Solution -> Given $\lim_{x\to\infty} \left(\frac{x^2-1}{x+1} - ax - b\right) = 2$

⇨ $\lim_{x\to\infty}\left(\frac{x^2-1-ax^2-ax-bx-b}{x+1}\right) = 2$

⇨ $\lim_{x\to\infty} \frac{x^2(1-a)-x(a+b)-(b+1)}{x+1} = 2$

As result is finite non zero quantity

⇨ Degree of numerator polynomial = degree of denominator polynomial

⇨ Degree of polynomial in numerator = 1

{As x+1 has degree = 1}

⇨ 1-a = 0 => a-=1

Now putting a = 1 in above evaluation

$\lim_{x\to\infty} \frac{-x(1+b)-(b+1)}{x+1} = 2$

⇨ $\frac{-(1+b)}{1} = 2$ {by important note}

⇨ $-1-b=2$ $\{\lim_{x\to\infty} \frac{a_0+a_1x---+a_mx^m}{b_0+b_1x+---+b_nx^n} = \frac{a_m}{b_n}$ $where\ m = n\}$

=> b=-1-2 =-3

Therefore a=1 and b=-3

(4) <u>Important Formulas in limit</u>

(1) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$ where a>0 and n€R

(2) $\lim_{x\to 0} \frac{a^x-1}{x} = \log_e a$

In particular $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

(3) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(4) $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

(5) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$

In particular $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = \log_e e = 1$

(6) $\lim_{x\to 0} \cos x = 1$

(7) $\lim_{x\to 0} \sin x = 0$

(8) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(9) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

<u>SUBSTITUTION METHOD:-</u>

In order to apply known formula sometimes we apply substitution method. In this method x is replaced by another variable u,and then we apply formula on 'u'.

Let us consider the following example.

Example – 14:-

Evaluate $\lim_{x\to 0}\frac{\sin 2x}{x}$

ANS:- Let 2x=u => when x→0

U →0 (as u = 2x)

Now $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{u\to 0}\frac{\sin u}{\frac{u}{2}} = 2\lim_{u-0}\frac{\sin u}{u}$

= 2×1 = 2

In general

Putting λx=u

$$\lim_{x\to 0} f(\lambda x) = \lim_{u\to 0} f(u) = \lim_{x\to 0} f(x)$$

Hence some of the formulas may be stated as fllows

1) $\lim_{x\to 0}\frac{a^{\,x}-1}{\lambda x} = \log_e a$

In particular $\lim_{x\to 0}\frac{e^{\,x}-1}{\lambda x} = 1$

2) $\lim_{x\to 0}(1+\lambda x)^{\frac{1}{x}} = e$

3) $\lim_{x\to\infty}(1+\frac{1}{\lambda x})^{\lambda x} = e$

4) $\lim_{x\to 0}\frac{\log_a(1+\lambda x)}{\lambda x} = \log_a e$

In particular $\lim_{x\to 0}\frac{\log_e(1+\lambda x)}{\lambda x} = 1$

5) $\lim_{x\to 0}\frac{\sin \lambda x}{\lambda x} = 1$

6) $\lim_{x\to 0}\frac{\tan \lambda x}{\lambda x} = 1$

Some examples based on the formulas

(1) Evaluate $\lim_{x\to 0}\frac{\sin 3x}{\tan 5x}$

Ans :-

$\lim_{x\to 0}\frac{\sin 3x}{\tan 5x} = \lim_{x\to 0}\frac{\frac{3\sin 3x}{3x}}{\frac{5\tan 5x}{5x}}$

$= \frac{3}{5}\lim_{x\to 0}\frac{(\frac{\sin 3x}{3x})}{(\frac{\tan 5x}{5x})} = \frac{3}{5}\times\frac{1}{1} = \frac{3}{5}$

(2) Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (2014 S)

Ans :- $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2sin^{2}\frac{x}{2}}{x^2}$

$= \lim_{x\to 0} 2.\frac{\sin\frac{x}{2}}{\frac{x}{2}2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}2} = \frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$

(3) Evaluate $\lim_{x\to0}\frac{e^{3x}-e^{x}}{x}$

Ans :- $\lim_{x\to0}\frac{e^{3x}-e^{x}}{x} = \lim_{x\to0}\frac{e^{3x}-1+1-e^{x}}{x}$

$= \lim_{x\to0}\frac{(e^{3x}-1)}{x} - \lim_{x\to0}\frac{(e^{x}-1)}{x}$

$= \lim_{x\to0} 3(\frac{e^{3x}-1}{3x}) - \lim_{x\to0}\frac{e^{x}-1}{x}$ = 3-1 = 2

(4) Evaluate $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)$

Ans :-

$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim_{x\to\frac{\pi}{2}}(\frac{1}{\cos x}-\frac{\sin x}{\cos x})$

$= \lim_{x\to\frac{\pi}{2}}(\frac{1-\sin x}{\cos x}) = \lim_{x\to\frac{\pi}{2}}(\frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)})$

$= \lim_{x\to\frac{\pi}{2}}(\frac{1-sin^2x}{\cos x(1+\sin x)}) = \lim_{x\to\frac{\pi}{2}}\frac{cos^2x}{\cos x(1+\sin x)}$

$= \lim_{x\to\frac{\pi}{2}}(\frac{\cos x}{1+\sin x}) = \frac{\cos\frac{\pi}{2}}{1+sin\frac{\pi}{2}} = \frac{0}{1+1} = \frac{0}{2} = 0$

(5) Evaluate $\lim_{x\to0}(\frac{\tan x-\sin x}{sin^3x})$

$\lim_{x\to0}(\frac{\tan x-\sin x}{sin^3x}) = \lim_{x\to0}(\frac{\frac{\sin x}{\cos x}-\sin x}{sin^3x})$

$= \lim_{x\to0}(\frac{\sin x-\sin x cosx}{sin^3x\cos x}) = \lim_{x\to0}(\frac{\sin x(1-cosx)}{sin^3x\cos x}) = \lim_{x\to0}(\frac{1-\cos x}{sin^2x\cos x})$

$= \lim_{x\to0}(\frac{1-cosx}{(1-cos^2x)cosx}) = \lim_{x\to0}(\frac{(1-cosx)}{(1-cosx)(1+cosx)cosx})$

$= \lim_{x\to0}(\frac{1}{(1+\cos x)\cos x}) = \frac{1}{(1+cos0)cos0} = \frac{1}{(1+1).1} = \frac{1}{2}$

(6) Evaluate $\lim_{x\to0}\frac{sin^{-1}x}{x}$.

Ans :-

$\lim_{x\to0}\frac{sin^{-1}x}{x} = \lim_{u\to0}\frac{u}{\sin u}$

{put $\sin^{-1}x$ =u => x=sinu when x->0 u-> $\sin^{-1}x$ -> 0 {as $\sin^{-1}0 = 0$}}

$= \lim_{u\to0}\frac{1}{\frac{sinu}{u}} = \frac{1}{1} = 1$

(7) Evaluate $\lim_{x\to0}\frac{tan^{-1}x}{x}$

Ans :-

$\lim_{x\to0}\frac{tan^{-1}x}{x} = \lim_{u\to0}\frac{u}{\tan u}$ {put $\tan^{-1}x$ = u => x= tanu when x->0 u->0}

$= \lim_{u\to0}\frac{1}{\frac{tanu}{u}} = {}^{1}/_{1} = 1$

(8) Evaluate $\lim_{x\to 2}\frac{x^3-8}{x^5-32}$

Ans :-

$$\lim_{x\to 2}\frac{x^3-8}{x^5-32} = \lim_{x\to 2}\frac{x^3-2^3}{x^5-2^5} \quad \{ \text{ as } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\}$$

$$= \lim_{x\to 2}\frac{\frac{x^3-2^3}{x-2}}{\frac{x^5-2^5}{x-2}} = \frac{3\ 2^{3-1}}{5\ 2^{5-1}} = \frac{3\times 2^2}{5\times 2^4} = \frac{3}{5\times 2^2} = \frac{3}{20}\ .$$

(9) Evaluate $\lim_{x\to 0}\frac{(3+x)^3-27}{x}$

Ans :- $\lim_{x\to 0}\frac{(3+x)^3-27}{x}$ { put x+3 = u when x--> 0 then u-->3}

$= \lim_{u\to 0}\frac{u^3-3^3}{u-3}$ $\{\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\}$

$= 3\ 3^{3-1} = 3\times 3^2 = 3\times 9 = 27$

(10) Evaluate $\lim_{x\to 0}\frac{tan^{-1}\,3x}{\sin 7x}$

Ans :- $\lim_{x\to 0}\frac{tan^{-1}\,3x}{\sin 7x} = \lim_{x\to 0}\frac{\frac{tan^{-1}3x}{3x}.3}{\frac{sin7x}{7x}.7}$

$$= \frac{3}{7}\lim_{x\to 0}\frac{(\frac{tan^{-1}3x}{3x})}{(\frac{sin7x}{7x})}$$

$$= \frac{3}{7}\times\frac{1}{1} = \frac{3}{7}$$

(11) Evaluate $\lim_{x\to 1}\frac{\log_e 2x-1}{x-1}$

Ans :- $\lim_{x\to 1}\frac{\log_e 2x-1}{x-1}$

{ For applying log formula x→ 0 ,but here x→ 1, so we have to substitute a new variable u as, u = x-1}

$= \lim_{u\to 0}\frac{\log_e 2(u+1)-1}{u}$

$= \lim_{u\to 0}\frac{\log_e 2u+1}{u}$ when x-->1 then u=x-1 -->0}

$= \lim_{u-0}\frac{\log_e(1+2u)}{2u}.2 = 1\times 2 = 2$

(12) Evaluate $\lim_{x\to 0}\frac{4^x-5^x}{3^x-4^x}$.

Ans :- $\lim_{x\to 0}\frac{4^x-5^x}{3^x-4^x} = \lim_{x\to 0}\frac{\frac{4^x-1+1-5^x}{x}}{\frac{3^x-1+1-4^x}{x}}$

$= \lim_{x\to 0}\frac{\frac{(4^x-1)}{x}-\frac{(5^x-1)}{x}}{(\frac{3^x-1}{x})-(\frac{4^x-1}{x})} = \frac{\log_e 4-\log_e 5}{\log_e 3-\log_e 4}$

$= \frac{ln4-ln5}{ln3-ln4} = \frac{ln\frac{4}{5}}{ln\frac{3}{4}}$ $\{\log e\ is written as \ln\ i.e.\ natural logarithm\}$

(13) Evaluate $\lim_{x\to 0}(1+3x)^{\frac{1}{x}}$

Ans :-

$$\lim_{x\to 0}(1+3x)^{\frac{1}{x}} = \lim_{x\to 0}\{(1+3x)^{\frac{1}{3x}}\}^3$$

$$= \{\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\}^3 = e^3$$

(14) Evaluate $\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{2x}}$

Ans:-

$$\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{2x}} = \lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1/3}{(\frac{2}{3}x)}}$$

$$= \{\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{(\frac{2x}{3})}}\}^{1/3} = e^{1/3}$$

Use of L.H.L and R.H.L to find limit of a function

L.H.L and R.H.L used to find limit of a function where the definition of a function changes. For example |x| at 0 or [x] at any integral point etc.

Also the same concept is used when we come across following terms.

$\lim_{x\to 0^+}\frac{1}{x} = \infty$

$\lim_{x\to 0^-}\frac{1}{x} = -\infty$

$\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

$\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

Examples :-

(1) Evaluate $\lim_{x\to 0}\frac{|x|}{x}$

Ans :- L.H.L = $\lim_{x\to 0^-}\frac{|x|}{x}$ {x → 0^- => x€($-\delta$,0)i.e. x<0 => |x| = -x}

$= \lim_{x\to 0^-}\frac{-x}{x}$

$= \lim_{x\to 0^-}(-1) = -1$

R.H.L = $\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x}$ {x→0^+ => x € (0,δ) i.e.x>0 => |x| = x}

$= \lim_{x\to 0^+} 1 = 1$

From above

L.H.L ≠ R.H.L =>$\lim_{x\to 0}\frac{|x|}{x}$ does not exist

(2) Evaluate $\lim_{x\to-1}\frac{x+1}{|x+1|}$

Ans :- $\lim_{x\to-1}\frac{x+1}{|x+1|}$ = $\lim_{u-0}\frac{u}{|u|}$ { Let x+1=u . when x→-1 then u→0}

L.H.L. = $\lim_{u\to0^-}\frac{u}{|u|}$= $\lim_{u\to0^-}\frac{u}{-u}$ { u→0^- => u<0 => IuI = -u }

= $\lim_{u\to0^-}(-1)$ = - 1

R.H.L = $\lim_{u\to0^+}\frac{u}{|u|}$ = $\lim_{u\to0^+}\frac{u}{u}$= $\lim_{u\to0^+}1$ = 1 { u→0^+ => u>0 => IuI =u }

Hence $\lim_{u\to0}\frac{u}{|u|}$ does not exist

Therefore $\lim_{x\to1}\frac{x+1}{|x+1|}$ does not exist.

(3) Find $\lim_{x\to0^+}\{[x]+10\}$

Ans $\lim_{x\to0^+}\{[x]+10\}$

=$\lim_{x\to0^+}(0+10)$ {As x-->0^+=> x € (0,δ) i.e 0 <x <1 } => [x]=0 }

=$\lim_{x\to0^+}10 = 10$

(4) Find $\lim_{x\to3.7}[x]$

Ans :-

$\lim_{x\to3.7}[x]$ =[3.7] = 3

(5) Find $\lim_{x\to-1}[x]$

Ans :-

[x] changes its definition at each integral point. So, we have to go through L.H.L and R.H.L.

L.H.L. = $\lim_{x\to-1^-}[x]$ = $\lim_{x\to-1^-}(-2)$ = -2

{ As x→ -1^- => x€(-1-δ,-1) i.e.-2<x<-1 =>[x]=-2}

R.H.L. = $\lim_{x\to-1^+}[x]$ = $\lim_{x\to-1^+}-1$ = -1 {as x-->-1^+ => -1<x<0 =>[x]=-1}

As from above L.H.L ≠ R.H.L.

⇨ $\lim_{x\to-1}[x]$ does not exist.

(6) Evaluate $\lim_{x\to\frac{4}{3}}[3x-1]$

Ans :-

$\lim_{x\to\frac{4}{3}}[3x-1]$ = $\lim_{u\to3}[u]$ { Put 3x-1 = u => when x →$\frac{4}{3}$ u→ 3 X $\frac{4}{3}-1$ i.e.u→3}

Now L.H.L = $\lim_{u\to3^-}[u]$ = $\lim_{u\to3^-}2$ = 2 { As $u\to3^-$ => $2<u<3$ => $[u]=2$}

R.H.L = $\lim_{u\to3^+}[u]$ = $\lim_{u\to3^+}3$ =3 { As $u\to3^+$ => $3<u<4$ => $[u]=3$}

Hence L.H.L ≠ R.H.L =>$\lim_{u\to3}[u]$ does not exist.

Therefore $\lim_{x\to\frac{4}{3}}[3x-1]$ does not exist.

(7) Evaluate $\lim_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} -x & x < 1 \\ x+1 & x \geq 1 \end{cases}$$

Ans :- As f(x) does not changes its definition at '2' so,

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+1) = 2+1 = 3$

{As x-->2 => x ∈ (2-δ,2+δ) => x> 1 => f(x)= x+1 }

(8) Evaluate $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} x^2 & x < 1 \\ 2x+1 & 1 \leq x \leq 2 \\ 5 & x > 2 \end{cases}$

Ans :- As function changes its definition at x=1 and 2, so we have to go through L.H.L and R.H.L. step.

$\lim\limits_{x\to 1} f(x)$

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2$ = $1^2 = 1$

R.H.L = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x+1)$ = 2 x 1 + 1 = 3

{ when x-->1^- => x<1 so we use f(x) = x^2 }

{when x-->1^+ => x > 1 i.e. 1 < x < 2 = > f(x) = 2x+1}

From above L.H.L ≠ R.H.L

⇨ $\lim_{x\to 1} f(x)$ *does not exist*

$\lim\limits_{x\to 2} f(x)$

L.H.L = $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+1)$ = 2 x 2 + 1 = 5

{when x-->2^- => x ε (2-δ,2) i.e. 1<x<2 => f(x)=2x+1}

R.H.L = $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5$ = 5

{x-->2^+ => x>2 => f(x) = 5 from definition}

As L.H.L = R.H.L

Therefore

$\lim_{x\to 2} f(x)$ = 5

(9) Evaluate $\lim_{x\to 0}\frac{1}{x}$

Ans :- L.H.L. = $\lim_{x\to 0^-}\frac{1}{x}$ = -∞

R.H.L. = $\lim_{x\to 0^+}\frac{1}{x}$ = ∞

L.H.L ≠ R.H.L.

=>$\lim_{x\to 0^-}\frac{1}{x}$ does not exist.

Note

X-->a^+
X-->a
X-->a^-

So when we use direct substitution method either for x-->a^+ or x-->a^- in both case we have to replace x by a.

Sandwich theorem or squeezing theorem

If $\lim_{x\to a} f(x)$ = $\lim_{x\to a} g(x)$ = l and a function h (x) is such that f(x)$\leq h(x) \leq g(x)$ for all xε(a-$\delta, a+\delta$) ,then

$$\lim_{x\to a} h(x) = l$$

Example

Find $\lim_{x\to 0} x \sin\frac{1}{x}$

Solution: - We know $|\sin\frac{1}{x}| \leq 1$

⇨ $|x \sin\frac{1}{x}| \leq |x|$

Again $|x \sin\frac{1}{x}| \geq 0$

Hence $0 \leq |x \sin\frac{1}{x}| \leq |x|$

Now $\lim_{x\to 0} 0 = 0$

And $\lim_{x\to 0} |x| = 0$

{As $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-}(-x)$ =- 0 = 0 and $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ }

Hence by sandwich theorm

$$\lim_{x\to 0} |x \sin\frac{1}{x}| = 0$$

When x--> 0 , x sin $\frac{1}{x}$ = (+)ve. So $|x \sin\frac{1}{x}| = x\sin\frac{1}{x}$

{when x-->0^- then x∈$(-\delta, 0)$, x = (-)ve, sin $\frac{1}{x}$ = -ve => x sin $\frac{1}{x}$ = +ve}

{When x-->0^+ then $x \in (0,\delta)$, x = +ve, sin $\frac{1}{x}$ = +ve => x sin $\frac{1}{x}$ = +ve}

Hence,

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0$$

ILLUSTRATIVE EXAMPLES

1. Evaluate $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ a,b $\neq 0$ (2015-S) (2019-w)

Ans.

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\frac{\sin ax}{ax}.a}{\frac{\sin bx}{bx}.b}$$

$$= \frac{a}{b}\lim_{x\to 0}\frac{(\frac{\sin ax}{ax})}{(\frac{\sin bx}{bx})}$$

$$= \frac{a}{b} \text{ X } \frac{1}{1} = \frac{a}{b}$$

2. Evaluate $\lim_{x\to 0}\frac{x - x\cos 2x}{sin^3 2x}$ (2015-S)

Ans.

$$\lim_{x\to 0}\frac{x - xcos2x}{sin^3 2x} = \lim_{x\to 0}\frac{x(1-cos2x)}{sin^3 2x}$$

$$= \lim_{x\to 0}\frac{x\, 2\, sin^2 x}{sin^3 2x} = 2\lim_{x\to 0}\frac{\frac{x\, sin^2 x}{x^3}}{\frac{sin^3 2x}{(2x)^3}.2^3}$$

$$=2\lim_{x\to 0}\frac{\frac{sin^2 x}{x^2}}{\frac{sin^3 2x}{(2x)^3}.8}$$

$$= \frac{2}{8}\lim_{x\to 0}\frac{(\frac{sinx}{x})^2}{(\frac{sin2x}{2x})^3}$$

$$= \frac{1}{4} \text{ X } \frac{1^2}{1^3}$$

$$= \frac{1}{4} \text{ (Ans)}$$

3. Evaluate $\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2}$ (2017-s old)

Ans.

$$\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2} = \lim_{x\to 0}\frac{2\sin(\frac{mx+nx}{2})\sin\frac{nx-mx}{2}}{x^2} \quad \{\text{cosC-cosD} = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\}$$

$$=\lim_{x\to 0} 2\,\frac{\sin(\frac{n+m}{2})x}{x}\,\frac{\sin(\frac{n-m}{2})x}{x}$$

$$= \lim_{x\to 0} 2\left(\frac{m+n}{2}\right)\frac{\sin(\frac{m+n}{2})x}{(\frac{m+n}{2})x}\left(\frac{n-m}{2}\right)\frac{\sin(\frac{n-m}{2})x}{(\frac{n-m}{2})x}$$

$$= 2\left(\frac{m+n}{2}\right)\left(\frac{n-m}{2}\right)\lim_{x\to 0}\frac{\sin(\frac{m+n}{2})x\,\sin(\frac{n-m}{2})x}{(\frac{m+n}{2})x\quad(\frac{n-m}{2})x}$$

$= \quad 2\,\frac{(m+n)}{2}\,\frac{(n-m)}{2}$x1x1

$= \quad \frac{(m+n)(n-m)}{2}$

$= \quad \frac{n^2-m^2}{2}$

4. Evaluate $\lim_{x\to\frac{\pi}{2}}(\frac{\pi}{2}-x)\tan x$

Ans.

$\lim_{x\to\frac{\pi}{2}}(\frac{\pi}{2}-x)\tan x = \lim_{u\to 0} u\tan(\frac{\pi}{2}-u)$ {put $\frac{\pi}{2}-x = u\ when\ x\to\frac{\pi}{2}\ \ u\to 0$}

$= \lim_{u\to 0} u\ \cot u$

$= \lim_{u\to 0}\frac{u}{\tan u}$

$= \lim_{u\to 0}\frac{u/u}{\tan u/u}$

$= \lim_{u\to 0}\frac{1}{(\frac{\tan u}{u})}$

$= \frac{1}{1} = 1$

5. Evaluate $\lim_{x\to 1}\frac{x^2-2x+1}{x^2-x}$ (2017 S)

Ans.

$\lim_{x\to 1}\frac{x^2-2x+1}{x^2-x} = \lim_{x-1}\frac{(x-1)^2}{x(x-1)}$

$= \lim_{x-1}\frac{x-1}{x}$

$= \frac{1-1}{1} = \frac{0}{1} = 0$

6. Evaluate $\lim_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ (a>b) (2017 W)

Ans.

$\lim_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ $(\frac{0}{0}\ form)$

$= \lim_{x\to a}\frac{(\sqrt{x-b}-\sqrt{a-b})(\sqrt{x-b}+\sqrt{a-b})}{(x^2-a^2)(\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a}\frac{(x-b)-(a-b)}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a-b})} = \lim_{x\to a}\frac{x-b-a+b}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a}\frac{(x-a)}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a+b})} = \lim_{x\to a}\frac{1}{(x+a)(\sqrt{x-b}+\sqrt{a-b})} = \frac{1}{(a+a)(\sqrt{a-b}+\sqrt{a-b})}$

$= \frac{1}{2a\ 2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-b}}$

7. Evaluate $\lim_{x \to \frac{1}{2}} \frac{|2x-1|}{2x-1}$

Ans. $\lim_{x \to \frac{1}{2}} \frac{|2x-1|}{2x-1}$

{Put 2x-1 = u => when x -->$\frac{1}{2}$, u --> 2X$\frac{1}{2}$ - 1 = 0 }

$= \lim_{u \to 0} \frac{|u|}{u}$

L.H.L = $\lim_{u \to 0^-} \frac{|u|}{u} = \lim_{u \to 0^-} \frac{-u}{u} = \lim_{u \to 0^-} (-1)$ = -1

R.H.L = $\lim_{u \to 0^+} \frac{|u|}{u} = \lim_{u \to 0^+} \frac{u}{u} = \lim_{u \to 0^+} 1$ = 1

As L.H.L ≠ R.H.L, so $\lim_{u \to 0} \frac{|u|}{u}$ does not exist.

=>$\lim_{x \to \frac{1}{2}} \frac{|2x-1|}{2x-1}$ does not exist.

8. Evaluate $\lim_{x \to \infty} \frac{x}{[x]}$

Ans:- From definition of [x] we know that ,

x – 1 < [x] ≤ x

⇨ $\frac{x}{x-1} > \frac{x}{[x]} \geq \frac{x}{x}$

⇨ $1 \leq \frac{x}{[x]} < \frac{x}{x-1}$

Now, $\lim_{x \to \infty} 1 = 1$

$\lim_{x \to \infty} \frac{x}{x-1} = \lim_{x \to \infty} \frac{\frac{x}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{1-\frac{1}{x}} = \frac{1}{1-0} = 1$

Hence by sandwich theorem $\lim_{x \to \infty} \frac{x}{[x]} = 1$

9. Evaluate$\lim_{n \to \infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

Ans.

$$\lim_{n \to \infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

$= \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$

$= \frac{1}{6} \lim_{n \to \infty} \frac{n}{n} (\frac{n+1}{n})(\frac{2n+1}{n})$

$= \frac{1}{6} \lim_{n \to \infty} 1(1+\frac{1}{n})(2+\frac{1}{n})$

$= \frac{1}{6}$ X 1(1+0)(2+0) = $\frac{2}{6} = \frac{1}{3}$

10. Evaluate $\lim_{x\to\alpha} \frac{x\sin\alpha-\alpha\sin x}{x-\alpha}$

Ans.

$\lim_{x\to\alpha} \frac{x\sin\alpha-\alpha\sin x}{x-\alpha}$ Put x-∝ = u, when x-->∝ , then u-->0

$= \lim_{u\to 0} \frac{(\propto+u)\sin\alpha-\alpha\sin(u+\propto)}{u}$

$= \lim_{u\to 0} \frac{\propto \sin\alpha+u\sin\propto-\alpha\sin u\cos\propto-\propto\cos u\sin\propto)}{u}$

$= \lim_{u\to 0} \frac{\propto \sin\alpha-\alpha\cos u\sin\propto+u\sin\propto-\propto\sin u\cos\propto)}{u}$

$= \lim_{u\to 0} \{\frac{\propto\sin\propto(1-\cos u)}{u} + \sin\propto - \propto\cos\propto\frac{\sin u}{u}\}$

$= \lim_{u\to 0} \{\frac{\propto\sin\propto 2\sin^2\frac{u}{2}}{u} + \sin\propto - \propto\cos\propto\frac{\sin u}{u}\}$

$= \lim_{u\to 0} \{2\propto\sin\propto\frac{\sin\frac{u}{2}}{\frac{u}{2}.2}.\sin\frac{u}{2} + \sin\propto - \propto\cos\propto\frac{\sin u}{u}\}$

$= \lim_{u\to 0} \{\propto\sin\propto\frac{\sin\frac{u}{2}}{\frac{u}{2}}.\sin\frac{u}{2} + \sin\propto - \propto\cos\propto\frac{\sin u}{u}\}$

$= \propto\sin\propto . 1. 0 + \sin\propto - \propto\cos\propto . 1 = \sin\propto - \propto\cos\propto$

11. Evaluate $\lim_{x\to 5} \frac{\log_e x-\log_e 5}{x-5}$

Ans. $\lim_{x\to 5} \frac{\log_e x-\log_e 5}{x-5}$ (Put u = x – 5, when x-->5 then u --> 0)

$= \lim_{u\to 0} \frac{\log_e(u+5)-\log_e 5}{u}$

$= \lim_{u\to 0} \frac{\log_e(\frac{u+5}{5})}{u}$

$= \lim_{u\to 0} \frac{\log_e(\frac{u}{5}+1)}{u}$

$= \lim_{u\to 0} \frac{\log_e(\frac{u}{5}+1)}{\frac{u}{5}.5}$

$= \frac{1}{5}\lim_{u\to 0} \frac{\log_e(1+\frac{u}{5})}{\frac{u}{5}}$

$= \frac{1}{5}.1 = \frac{1}{5}$

12. Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\log_e(1+x)}$

Ans. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\log_e(1+x)}$

$= \lim_{x\to 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{\log_e(1+x)(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1+x-1}{\log_e(1+x)(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1}{\frac{\log_e(1+x)}{x}.(\sqrt{1+x}+1)}$

$= \frac{1}{1X\sqrt{1+0}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

13. Evaluate $\lim_{x\to 2} \frac{\log_7(2x-3)}{(x-2)}$

Ans. $\lim_{x\to 2} \frac{\log_7(2x-3)}{(x-2)}$ { Put x-2 = u when x-->2 then u-->0 }

$= \lim_{u\to 0} \frac{\log_7\{2(u+2)-3\}}{u} = \lim_{u\to 0} \frac{\log_7(2u+4-3)}{u}$

$= \lim_{u\to 0} \frac{2\log_7(1+2u)}{2u}$

$= 2 \lim_{u\to 0} \frac{\log_7(1+2u)}{2u} = 2 . \log_7 e$

14. Find the value of a for which $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a} = 5$

Ans. Given $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a} = 5$------------------------------ (1)

Now, $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a}$

$= \lim_{u\to 0} \frac{5^{u+1}-5}{u \log_e a}$ (Put x-1 = u, when x--> 1, then u-->0)

$= \lim_{u\to 0} \frac{5^u.5-5}{u \log_e a}$

$= \frac{5}{\log_e a} \lim_{u\to 0} \frac{5^u-1}{u}$

$= \frac{5}{\log_e a} \log_e 5$ ---(2)

From (1) and (2) we have,

$\frac{5}{\log_e a} \log_e 5 = 5$

$=> \log_e 5 = \log_e a$

=> a = 5

15. Evaluate $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-6x+5}$

Ans. $\lim_{x\to 1} \frac{x^2-3x-x+3}{x^2-5x-x+5}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 1} \frac{x(x-3)-1(x-3)}{x(x-5)-1(x-5)}$

$= \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-5)(x-1)}$

$= \lim_{x\to 1} \frac{x-3}{x-5}$

$= \frac{1-3}{1-5} = \frac{-2}{-4} = \frac{1}{2}$.

16. Evaluate $\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$

Ans.

$$\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2} = \lim_{x\to 0}\frac{3^x+\frac{1}{3^x}-2}{x^2}$$

$$= \lim_{x\to 0}\frac{3^{2x}+1-2\ 3^x}{3^x x^2} = \lim_{x\to 0}\frac{(3^x)^2-2.3^x.1+1^2}{3^x x^2}$$

$$= \lim_{x\to 0}\frac{(3^x-1)^2}{3^x x^2} = \lim_{x\to 0}\frac{1}{3^x}\left(\frac{3^x-1}{x}\right)^2$$

$$= \frac{1}{3^\circ}(\log_e 3)^2$$

$$= (\ln 3)^2$$

17. Evaluate $\lim_{x\to\infty}\{\sqrt{x^2+1}-\sqrt{x^2-1}\}x$

Ans.

$$\lim_{x\to\infty}\{\sqrt{x^2+1}-\sqrt{x^2-1}\}x$$

$$= \lim_{x\to\infty}\frac{\{(\sqrt{x^2+1}-\sqrt{x^2-1})(\sqrt{x^2+1}+\sqrt{x^2-1})\}x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{\{x^2+1-x^2+1\}x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+1}}{x}+\frac{\sqrt{x^2-1}}{x}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{2} = 1$$

18. Evaluate $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$

Ans.

$$\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$$ {Put u = tanx when x-->0 then u-->0}

$$= \lim_{u\to 0}\frac{e^u-1}{\tan^{-1}u}$$

$$= \lim_{u\to 0}\frac{\frac{e^u-1}{u}}{\frac{\tan^{-1}u}{u}} = \frac{1}{1} = 1$$

19. Evaluate $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2+5x+7}$

Ans.

$$\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2+5x+7} = \lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4x^2}{x^3}+\frac{6x}{x^3}-\frac{1}{x^3}}{\frac{2x^3}{x^3}+\frac{x^2}{x^3}+\frac{5x}{x^3}+\frac{7}{x^3}}$$

$$= \lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{2+\frac{1}{x^2}+\frac{5}{x^2}+\frac{7}{x^3}} = \frac{3-0+0-0}{2+0+0+0} = \frac{3}{2}$$

20. Evaluate $\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

Ans.

$$\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \lim_{x\to 2}\frac{\frac{4-x^2}{4x^2}}{x-2}$$

$$= -\frac{1}{4}\lim_{x\to 2}\frac{x^2-4}{x^2(x-2)}$$

$$= -\frac{1}{4}\lim_{x\to 2}\frac{(x-2)(x+2)}{x^2(x-2)}$$

$$= -\frac{1}{4}\left(\frac{2+2}{2^2}\right) = -\frac{1}{4} \quad \text{(Ans)}$$

Exercise

1. Evaluate the following limits(2 marks)

(i) $\lim_{x\to 0}\frac{\sqrt{x+1}\ -1}{x}$ (2016-S) (2018-S)

(ii) $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ (a;b $\neq$ 0) (2015-S)

(iii) $\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}$ (2019-w)

(iv) $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(v) $\lim_{x\to 0}\frac{1-cosx}{x^2}$ (2014-S)

(vi) $\lim_{x\to \pi}\frac{\sin x}{\pi-x}$ (2016-S)

(vii) $\lim_{x\to 0} ln(1+bx)^{\frac{1}{x}}$ (2016-S)

(viii) $\lim_{x\to 0}\frac{tan5x}{tan7x}$ (2017-w)

2. Evaluate the following limits (5 marks)

(i) $\lim_{x\to 1} \frac{2^{x-1}-1}{\sqrt{x}-1}$

(ii) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

(iii) $\lim_{x\to 1} \frac{x^{\frac{1}{m}}-1}{x^{\frac{1}{n}}-1}$

(iv) $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\cdots+n^3}{n^4}$

(v) $\lim_{x\to\infty} x^2\{\sqrt{x^4+a^2}-\sqrt{x^4-a^2}\}$

(vi) $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}$

(vii) $\lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$

(viii) $\lim_{x\to 1} (\frac{2}{1-x^2}+\frac{1}{x-1})$

(ix) $\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x^2-6x+8}$

(x) $\lim_{x\to 2} \frac{\log_e(x-1)}{x^2-3x+2}$

(xi) $\lim_{x\to 0} \frac{tanx-sinx}{x^3}$ (2018-S)

(xii) $\lim_{x\to 0} \frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

(xiii) $\lim_{x\to 0} \frac{a^x-b^x}{c^x-d^x}$ (2016-S)

(xiv) $\lim_{x\to 0} \frac{cos2x-cos3x}{x^2}$ (2017-S)

(xv) $\lim_{x\to 0} \frac{\sqrt{3-2x}-\sqrt{3}}{x}$

(xvi) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$ (2017-w)

3. Find the value of a on following cases.(5 marks)

(i) $\lim_{x\to\alpha} \frac{tana(x-\alpha)}{x-\alpha} = \frac{1}{2}$

(ii) $\lim_{x\to 0} \frac{e^{ax}-e^x}{x} = 2$

(iii) $\lim_{x\to 2} \frac{\log_e(2x-3)}{a(x-2)} = 1$

Answers

1. i) $\frac{1}{2}$ ii) $\frac{a}{b}$ iii) $\frac{3}{2}$ iv) 3 v) $\frac{1}{2}$ vi) 1 vii) b viii) 5/7

2. i) 2 ln2 ii) ¼ iii)n/m iv)1/4 v) a^2 vi) ½ vii) -11 viii) 1/2
   ix) ½ x) 1 xi) $\frac{1}{2}$ xii) $\frac{2}{3}$ xiii) $\frac{\ln\frac{a}{b}}{\ln\frac{c}{d}}$ xiv) 5/2 xv) -1/√3 xvi) 1

3. (i) ½ ii) 3 iii) 2

Continuity and Discontinuity of Function

In the figure we observe that the 1st graph of a function in Fig-1 can be drawn on a paper without raising pencil i.e. 1st graph is continuously moving where as Fig -2 represents a graph , which cannot be drawn without raising the pencil. Because there are gaps or breaks. So, it is discontinuous.

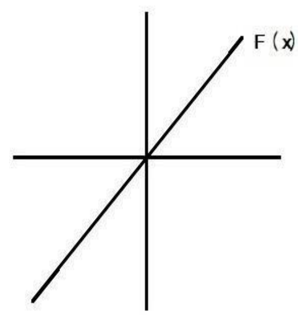


Fig-1

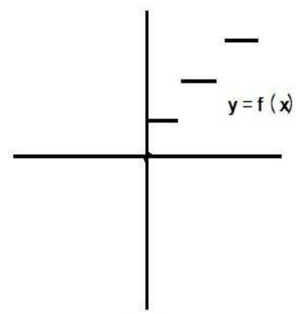


Fig-2

The feature of the graph of a function displays an important property of the function called continuity of a function.

<u>Continuity of a Function at a point</u>

Definition – A function f(x) is said to be continuous at x = a, if it satisfies the following conditions

(i) $\lim_{x \to a} f(x)$ exists.

(ii) f(a) is defined i.e. finite

(iii) $\lim_{x \to a} f(x)$ = *f(a)*

If one or more of the above condition fail, the function f(x) is said to be discontinuous at x= a.

<u>Continuous Function</u>

A function is said to be continuous if it is continuous at each point of its domain.

Working procedure for testing continuity at a point x = a

<u>1st step</u> – First find $\lim_{x \to a} f(x)$ by using concepts from previous chapter.

If $\lim_{x \to a} f(x)$ does not exit then, f(x) is discontinuous at x = a.

If $\lim_{x \to a} f(x)$ = l , then go to 2nd step.

<u>2^{nd} step</u> – Find f(a) from the given data

If f(a) is undefined then f(x) is not continuous at x =a.

If f(a) has finite value then go to 3^{rd} step.

<u>3^{rd} step</u> – Compare $\lim_{x\to a} f(x)$ and f(a)

If $\lim_{x\to a} f(x)$ = f(a), then f(x) is continuous at x= a, otherwise f(x) is discontinuous at x = a.

Examples

Q1. Examine the continuity of the function f(x) at x = 3.

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

Ans:-

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x\to 3}(x+3) = 3+3 = 6 \qquad \{\text{As } x \to 3,\ x \neq 3 \Rightarrow x-3 \neq 0\}$$

From given data f(3) = 6
Now from above $\lim_{x\to 3} f(x)$ = f (3)
Therefore, f(x) is continuous at x = 3.

Q2. Test continuity of f(x) at '0' where,

$$f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & x \neq 0 \\ e^3 & x = 0 \end{cases}$$

Ans:- $\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+3x)^{\frac{1}{x}}$

$$= \lim_{x\to 0} (1+3x)^{\frac{1}{3x}.3}$$

$$= \lim_{x\to 0} \{ (1+3x)^{\frac{1}{3x}} \}^3$$

$$= \{ \lim_{x\to 0}(1+3x)^{\frac{1}{3x}} \}^3$$

$$= e^3$$

{ As $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ = e

$\lim_{x\to 0}(1+\lambda x)^{\frac{1}{x}}$ = e

In particular $\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}$ = e

and we know, $\lim_{x\to a}\{f(x)\}^n = \{\lim_{x\to a} f(x)\}^n$ }

From given data f(0) = e^3

Hence, $\lim_{x\to 0} f(x)$ = f(0)

Therefore, f(x) is continuous at x = 0.

Q3. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ans. $\lim_{x\to0} f(x) = \lim_{x\to0} \frac{|x|}{x}$

As |x| is present and x→0 , so we have to evaluate the above limit by L.H.L and R.H.L method

L.H.L = $\lim_{x\to0^-} \frac{|x|}{x}$ { x → 0^- => x <0 }

$= \lim_{x\to0^-} \frac{-x}{x}$

$= \lim_{x\to0^-}(-1)$ = (-1)

R.H.L = $\lim_{x\to0^+} \frac{|x|}{x}$ { x →0^+ => x> 0 }

$= \lim_{x\to0^+} \frac{x}{x}$ $= \lim_{x\to0^+}(1)$ = 1

Hence, L.H.L ≠ R.H.L

Therefore, f(x) does not exist.

Hence f(x) is not continuous at x = 0.

Q4. Test continuity of $\frac{x^2-4}{x-2}$ at x = 2.

Ans. Here, f(2) = $\frac{2^2-4}{2-2}$ = $\frac{0}{0}$ undefined.

Hence, f(x) is not continuous at x = 2.

Q5. Test continuity of f(x) at '0'.

$$f(x) = \begin{cases} \frac{sin3x}{\tan 5x} & x \neq 0 \\ \frac{5}{3} & x = 0 \end{cases}$$

Ans. $\lim_{x\to0} f(x) = \lim_{x\to0} \frac{\sin 3x}{\tan 5x} = \lim_{x\to0} \frac{\frac{sin3x}{x}}{\frac{tan5x}{x}}$

$= \lim_{x\to0} \frac{\frac{sin3x\ .3}{3x}}{\frac{\tan 5x\ .5}{5x}} = \frac{3}{5}\lim_{x\to0}\{(\frac{sin3x}{3x})/(\frac{tan5x}{5x})\}$

$= \frac{3}{5} \quad (\frac{1}{1}) \quad = \frac{3}{5}$

Given that, f(0) = $\frac{5}{3}$

Thus, $\lim_{x\to0} f(x) \neq$ f(0)

Hence f(x) is not continuous at x = 0.

Q6. Test continuity of f(x) at x = $\frac{1}{2}$

$$f(x) = \begin{cases} 1-x & x \le 1/2 \\ x & x > 1/2 \end{cases}$$

Ans. First understand the function properly

When $x < \frac{1}{2}$, $f(x) = 1 - x$

$x > \frac{1}{2}$, $f(x) = x$

When $x = \frac{1}{2}$, f(x) = 1 − x = 1 - $\frac{1}{2}$ = $\frac{1}{2}$

Now let us find the $\lim_{x\to 1/2} f(x)$

L.H.L = $\lim_{x\to \frac{1^-}{2}} f(x) = \lim_{x\to \frac{1^-}{2}} (1-x)$ {As x→$\frac{1^-}{2}$ i.e. x < $\frac{1}{2}$, so f(x) = 1 − x }

= 1 - $\frac{1}{2}$ = $\frac{1}{2}$

R.H.L = $\lim_{x\to \frac{1^+}{2}} f(x)$ {As x→$\frac{1^+}{2}$ i.e. x > $\frac{1}{2}$. So, f(x) = x from definition of f(x)}

= $\lim_{x\to \frac{1^+}{2}} x = \frac{1}{2}$

Now from above L.H.L = R.H.L

⇨ $\lim_{x\to \frac{1}{2}} f(x) = \frac{1}{2}$ -----------(1)

From definition f ($\frac{1}{2}$) = $\frac{1}{2}$ ------------(2)

From (1) and (2)

$\lim_{x\to \frac{1}{2}} f(x)$ = f ($\frac{1}{2}$)

Hence, f(x) is continuous at x = $\frac{1}{2}$.

Q7. Test continuity of f(x) at x = 0, 1

$$f(x) = \begin{cases} 2x+1 & if\ x \le 0 \\ x & if\ 0 < x \le 1 \\ 2x-1 & if\ x > 1 \end{cases}$$

Ans. Here given that

f(x) = 2x+1 for x<0 --(1)

When x = 0, f(x) = f(0) = 2x+1 = 2X0+1 = 1 --------- (2)

When 0<x<1, f(x) = x - ---------(3)

When x = 1 , f(x) = f(1)= x = 1 ---------- (4)

When x>1 , f(x) = 2x-1 - ----------(5)

Continuity test at x = 0

L.H.L = $\lim_{x\to 0^-} f(x)$ { x→0^-=>x<0 => f(x) = 2x+1 from (1)}

= $\lim_{x\to 0^-}(2x+1)$

= (2X0)+1 = 1

R.H.L = $\lim_{x\to 0^+} f(x)$ {x→0^+=> x>0 => 0<x<1 => f(x) = x}

= $\lim_{x\to 0^+} x$ = 0

As L.H.L ≠ R.H.L

⇨ $\lim_{x\to 0} f(x)$ does not exist

Hence, f(x) is not continuous at x = 0.

Continuity test at x = 1

L.H.L = $\lim_{x\to 1^-} f(x)$ = $\lim_{x\to 1^-} x$ { x →1^- => x<1 i.e. 0<x<1 => f(x) = x from (3)}

= $\lim_{x\to 1^-} x$ = 1

R.H.L = $\lim_{x\to 1^+} f(x)$ = $\lim_{x\to 1^+} 2x-1$ { x →1^+ => x>1 , f(x) = 2x-1 from (5)}

= 2 X 1 – 1 = 1

As L.H.L = R.H.L

$\lim_{x\to 1} f(x)$ = 1

From given data f(1) = 1 {from equation (4)}

Hence, $\lim_{x\to 1} f(x)$ = f(1)

Therefore, f(x) is continuous at x = 1

Q8. Examine continuity of f(x) = [3x + 11] at x = -$\frac{11}{3}$ (2016-S)

Ans.

$\lim_{x\to -\frac{11}{3}} f(x) = \lim_{x\to -\frac{11}{3}}[3x+11]$ {Let u = 3x+11 when x→ -$\frac{11}{3}$, u = 3 X →$3\times -\frac{11}{3}$ +11 =0}

= $\lim_{u\to 0}[u]$---------------------------- (1)

Now, $\lim_{u\to 0^-}[u] = \lim_{u\to 0^-} -1$ = -1 { As u→0^- => -1<u<0 => [u] = -1

And $\lim_{u\to 0^+}[u] = \lim_{u\to 0^+} 0$ = 0 {As u→0^+=> 0<u<1 => [u] = 0 }

As, L.H.L ≠ R.H.L

⇨ $\lim_{u \to 0}[u]$ does not exist => $\lim_{x \to -\frac{11}{3}} f(x)$ does not exist.

Hence, f(x) is not continuous at x = 0.

Q9. Determine the value of K for which f(x) is continuous at x = 1.

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 1} & x \neq 1 \\ K & x = 1 \end{cases}$$

Ans.

Given function is continuous at x = 1.

=> $\lim_{x \to 1} f(x)$ = f(1)

=> $\lim_{x \to 1} f(x)$ = K ------------------ (1)

Now, let us find $\lim_{x \to 1} f(x)$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$ $\left(\frac{0}{0} form\right)$

$= \lim_{x \to 1} \frac{x^2 - 2x - x + 2}{x - 1}$

$= \lim_{x \to 1} \frac{x(x-2) - 1\,(x-2)}{x - 1}$

$= \lim_{x \to 1} \frac{(x-2)\,(x-1)}{(x-1)}$ {As x → 1, x ≠ 1, x -1 ≠ 0}

$= \lim_{x \to 1}(x - 2)$ = 1 – 2 = -1 ---------------(2)

Hence, From (1) and (2) we have K = -1 . (Ans)

Q10. If $f(x) = \begin{cases} ax^2 + b & if\ x < 1 \\ 1 & if\ x = 1 \\ 2ax - b & if\ x > 1 \end{cases}$

is continuous at x = 1, then find a and b.

Ans.

Given that f(x) is continuous at x = 1

⇨ $\lim_{x \to 1} f(x)$ = f(1)

⇨ $\lim_{x \to 1} f(x) = 1$ ------------- (1) {As f(1) = 1 given}

From (1) as $\lim_{x \to 1} f(x)$ exists

=> $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x)$----------------------- (2)

From (1) and (2) we have,

$\lim_{x \to 1^-} f(x) = 1$

=> $\lim_{x \to 1^-}(ax^2 + b) = 1$ {As x→1^-=> x< 1 => f(x) = ax^2 +b from def^n of f(x)}

=> $a \times 1^2 + b = 1$

=> $a + b = 1$-----------------(3)

Again from (1) and (2)

$\lim_{x \to 1^+} f(x) = 1$ {$x \to 1^+$ => $x > 1$, => $f(x) = 2ax - b$}

=> $\lim_{x \to 1^+} (2ax - b) = 1$

⇨ $(2 \times a \times 1) - b = 1$

⇨ $2a - b = 1$ --------------- (4)

Eq^n (3) $a + b = 1$

Eq^n (4) $2a - b = 1$

$3a = 2$

⇨ $a = \frac{2}{3}$

From (3) $a + b = 1$

⇨ $b = 1 - a = 1 - \frac{2}{3} = \frac{1}{3}$

Hence, $a = \frac{2}{3}$ and $b = \frac{1}{3}$

Q11. Find the value of 'a' such that

$$f(x) = \begin{cases} \frac{\sin ax}{\sin x} & x \neq 0 \\ \frac{1}{a} & x = 0 \end{cases}$$

is continuous at x = 0

Ans. f(x) is continuous at x = 0

=> $\lim_{x \to 0} f(x) = f(0)$

=> $\lim_{x \to 0} \frac{\sin ax}{\sin x} = \frac{1}{a}$

=> $\lim_{x \to 0} \frac{a\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin x}{x}\right)} = \frac{1}{a}$

=> $a \frac{1}{1} = \frac{1}{a}$

=> $a^2 = 1$

=> $a = \pm 1$ (Ans)

Q12. Examine the continuity of the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{at } x = 0.$$

Ans.

Let us evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

We know that $-1 \leq \sin\frac{1}{x} \leq 1$

$\Rightarrow (-1)x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \cdot 1$

$\Rightarrow -x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$

Now, $\lim_{x\to 0}(-x^2) = -0^2 = 0$

$\lim_{x\to 0} x^2 = 0^2 = 0$

Hence, by sandwich theorem

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Given $f(0) = 0$

Hence $\lim_{x\to 0} f(x) = f(0)$

Therefore, f(x) is continuous at x = 0.

Q13. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ans:-Evaluation of $\lim_{x\to 0} f(x)$ is not possible directly.

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

{when $x \to 0^-$ then $\frac{1}{x} \to -\infty \Rightarrow e^{\frac{1}{x}} \to 0$ }

$= \frac{0-1}{0+1} = -1$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$\{$when $x \to 0^+$ then $\frac{1}{x} \to \infty \Rightarrow e^{\frac{1}{x}} \to \infty \Rightarrow \frac{1}{e^{\frac{1}{x}}} \to 0$ $\}$

$= \lim_{x\to 0^+} \frac{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} - \frac{1}{e^{\frac{1}{x}}}}{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} + \frac{1}{e^{\frac{1}{x}}}} = \lim_{x\to 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}}$

$= \frac{1-0}{1+0} = 1$

From above L.H.L ≠ R.H.L

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Therefore, f(x) is not continuous at x = 0.

Q14. Discuss the continuity of the function

$f(x) = \begin{cases} x - \frac{|x|}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$ at x=0

Ans: -

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x - \frac{|x|}{x}$

$= \lim_{x\to 0^-}\{ x - \frac{(-x)}{x} \}$ $\{x \to 0^- \Rightarrow x<0 \Rightarrow |x| = -x\}$

$= \lim_{x\to 0^-}\{ x - (-1)\} = \lim_{x\to 0^-}\{ x + 1\}$

$= 0 + 1 = 1$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x - \frac{|x|}{x}$

$= \lim_{x\to 0^+}\{ x - \frac{x}{x}\}$ $\{x \to 0^+ \Rightarrow x >0 \Rightarrow |x| = x\}$

$= \lim_{x\to 0^-}\{ x - 1)\} = 0 - 1 = -1$

So, L.H.L ≠ R.H.L $\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Therefore, f(x) is not continuous at x = 0.

Exercise

Q1. Find the value of the constant K, so that the function given below is continuous at x = 0.

$$f(x) = \begin{cases} \frac{1-\cos 2x}{2x^2} & x \neq 0 \\ K & x = 0 \end{cases}$$ (5 marks)

Q2. Test the continuity of f(x) at x = 1, where

$$f(x) = \begin{cases} x^2 + 1 & if x < 1 \\ 2 & if x = 1 \\ 3x - 1 & if x > 1 \end{cases}$$ (5 marks)

Q3. Show that the function f(x) given by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & x \neq 0 \\ 2 & x = 0 \end{cases}$$ is continuous at x = 0. (5 marks)

Q4. Test continuity of f(x) at x = 1

$$f(x) = \begin{cases} \frac{x^7-1}{x-1} & x \neq 1 \\ 7 & x = 1 \end{cases}$$ (5 marks)

Q5. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} (1+2x)^{\frac{1}{x}} & if\ x \neq 0 \\ e^2 & if\ x = 0 \end{cases}$$ (2017-W) (5 marks)

Q6. Test continuity of f(x) at x = 2

$$f(x) = \begin{cases} \frac{|x-2|}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$$ (10 marks)

Q7. Find the value of K for which f(x) is continuous at x = 0.

$$f(x) = \begin{cases} \frac{8^x-4^x-2^x+1}{x^2} & x \neq 0 \\ K & x = 0 \end{cases}$$ (2016-S) (10 marks)

Q8. Test the continuity of the function f(x) at x = 0.

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan^{-1} 7x} & x \neq 0 \\ 3/7 & x = 0 \end{cases}$$ (5 marks)

Q9. Test continuity of the function f(x) at x = 1

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$ (5 marks)

Q10. Examine the continuity of the function of f(x) at x=0.

$$f(x) = \begin{cases} 2x + 1 & if x < 0 \\ 0 & if x = 0 \\ x^2 + 1 & if x > 0 \end{cases}$$ (2014-S) (5 marks)

Answers

1) K = 1,

Q no. 2, 4, 5, 8 are continuous .

6, 9, 10 are discontinuous

7.2(ln2)^2

Derivatives

Introduction

The study of differential calculus originated in the process of solving the following three problems

1. From the astronomical consideration particularly involving an attempt to have a better approximation of π as developed by Bhaskaracharya, Madhava and Nilakantha.
2. Finding the tangent to any arbitrary curve as developed by Fermat and Leibnitz.
3. Finding rate of change as developed by Fermat and Newton.

In this chapter we define derivative of a function, give its geometrical and physical interpretation and discuss various laws of derivatives etc.

Objectives

After studying this lesson, you will be able to:

(1) Define and Interpret geometrically the derivative of a function y = f(x) at x = a.
(2) State derivative of some standard function.
(3) Find the derivative of different functions like composite function, implicit function using different techniques.
(4) Find higher order derivatives of a particular function by successive differentiation method.
(5) Determine rate of change and tangent to a curve.
(6) Find partial derivative of a function with more than one variable with respect to variables.
(7) Define Euler's theorem and apply it solve different problems based on partial differentiation.

Expected background knowledge

1. Function
2. Limit and continuity of a function at a point.

Derivative of a function

Consider a function y = x^2

Table-1

x	5	5.1	5.01	5.001	5.0001
y	25	26.01	25.1001	25.010001	25.00100001

Let x = 5 and y = 25 be a reference point

We denote the small changes in the value of x as $'\delta x'$,

δx = small change in x

δy = change in y, when there is a change of δx in x.

Now, $\frac{\delta y}{\delta x}$ is called Increment ratio or Newton quotient or average rate of change of y.

Now, let us write table -1 in terms of δx, δy as

Table-2

δx	0.1	0.01	0.001	0.0001
δy	1.01	0.1001	0.010001	0.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

From table-2 δy varies as δx varies

It is clear from the table when $\delta x \to 0$
⇨ $\delta y \to 0$ and $\frac{\delta y}{\delta x} \to 10$

This $\frac{\delta y}{\delta x}$ when $\delta x \to$ 0 is the instantaneous rate of change of y at the value of x.

In above case x = 5, so $\frac{dy}{dx}$ at x = 5 is 10

Definition of derivative of a function (Differentiation)

If y = f(x) is a function. Then derivative of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$\frac{dy}{dx}$ is also denoted by f '(x)

$\frac{dy}{dx} = \frac{df(x)}{dx}$ = f ' (x) = f '
are same notations

Process of finding derivatives of dependent variable w.r.t. independent varibale is called differentiation.

Derivative of a function at a point 'a'

Derivative of y = f(x) at a point 'a' in the domain D_f is given by

$$\frac{dy}{dx}\Big]_{x=a} = f\,'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example -1

Find the derivative of f(x) = x^2 at x = 5

Ans. $\frac{dy}{dx}\Big]_{x=5} = f\,'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$

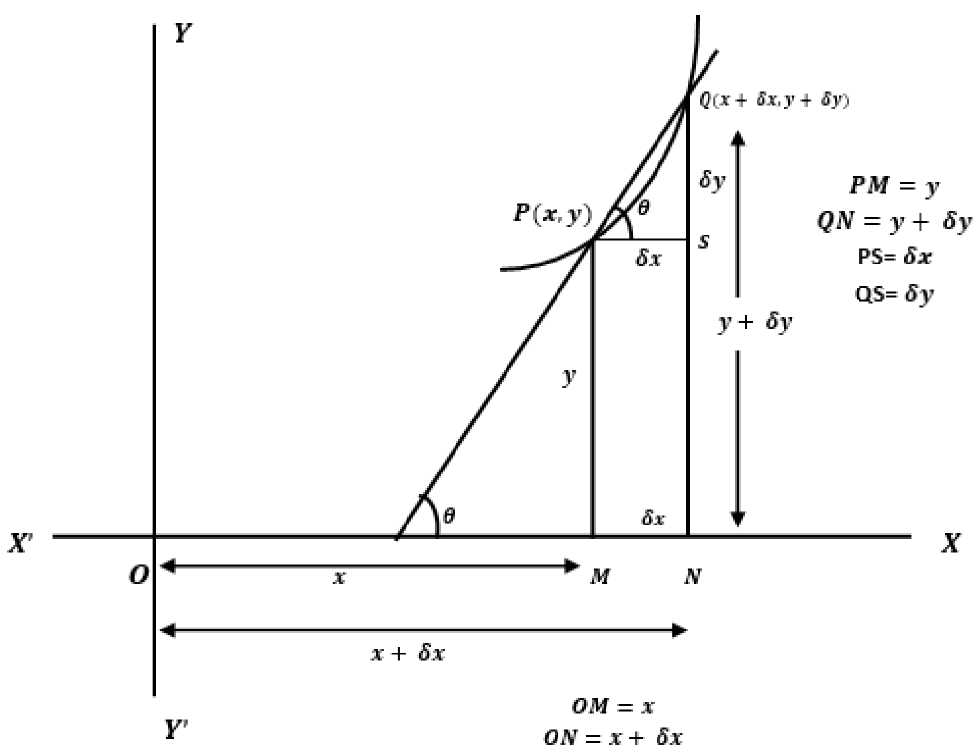
$= \lim_{h\to 0} \frac{(5+h)^2-5^2}{h}$

$= \lim_{h\to 0} \frac{(5+h+5)(5+h-5)}{h}$

$= \lim_{h\to 0} \frac{(10+h)h}{h} = \lim_{h\to 0}(10+h) = 10$

Geometrical Interpretation of $\frac{dy}{dx}$

(Fig.-1)



Let f(x) is represented by the curve in fig-1 given above.

Let Q(x+$\delta x, y+\delta y$) be the neighbourhood of P(x,y). PM and QN are drawn perpendicular to X-axis.

PS QN

Let QP Secant meets x-axis, (by extending it) and $\overrightarrow{QP}$ make angle θ with x-axis then angle QPS = θ

In ΔQPS, tan $\theta = \frac{QS}{PS} = \frac{\delta y}{\delta x}$

As QN = y + δy, NS = PM = y

=> QS = QN – NS = δy.

Similarly, ON = x+δx and OM = x => PS = MN = ON – OM = δx

When δx →0 then Q→P and QP secant becomes tangent at P.

In Δ PQS $\tan\theta = \frac{\delta y}{\delta x}$ { $\tan\theta$ gives slope of PQ line}

We know

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx} = \tan\theta$$

Now when δx → 0 the line PQ becomes tangent at P

So,

$\frac{dy}{dx} = \tan\theta$ = slope of the tangent to the curve at P.

So derivative of a function at a point represents the slope or gradient of the tangent at that point.

Example 2

Q. Find the slope of the tangent to the curve y = x^2 at x = 5.

Ans. As we have done it in example – 1.

$\frac{dy}{dx}\Big]_{x=5}$ = 10

Therefore, slope of the tangent at x = 5 is 10.

Derivative of some standard functions

1. $\frac{d}{dx}$ (c) = 0
2. $\frac{d}{dx}(x^n)$ = n.x^{n-1}
3. $\frac{d}{dx}(a^x) = a^x \log_e a$ In particular $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ In particular $\frac{d}{dx}(\log_e x) = \frac{d}{dx}$ (lnx) $= \frac{1}{x}$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x)$ = - $\sin x$
7. $\frac{d}{dx}$ (tan x) = sec^2x
8. $\frac{d}{dx}$ (cot x) = - $cosec^2x$
9. $\frac{d}{dx}$ (sec x) = sec x.tan x
10. $\frac{d}{dx}$ (cosec x) = - cosec x. cot x
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
12. $\frac{d}{dx}(\cos^{-1} x)$ = - $\frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
14. $\frac{d}{dx}(\cot^{-1} x)$ = - $\frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
16. $\frac{d}{dx}(\text{cosec}^{-1} x)$ = - $\frac{1}{|x|\sqrt{x^2-1}}$

Algebra of derivatives or fundamental theorems of derivatives

If f(x) and g(x) are both derivable functions i.e. their derivative exists then,

(i) $\frac{d}{dx}$ {cf(x)} = c f ' (x)

(ii) $\frac{d}{dx}$ (f + g) = f ' + g'

(iii) $\frac{d}{dx}$ (f - g) = f ' – g'

(iv) $\frac{d}{dx}$ {fg} = fg' + f ' g

(v) $\frac{d}{dx}\{\frac{f}{g}\} = \frac{f'g - fg'}{g^2}$

Example-3

Find the derivative of the following:

(i) $3x^3$ (ii) $6\sqrt{x}$ (iii) $9\ 3^x$ (iv) 5 cot x

Ans.

(i) $\frac{dy}{dx} = \frac{d}{dx}(3x^3) = 3\frac{d(x^3)}{dx} = 3 \text{ X } 3\ x^{3-1} = 9x^2$

(ii) $\frac{dy}{dx} = \frac{d(6\sqrt{x})}{dx} = 6\frac{d(\sqrt{x})}{dx} = 6\frac{d}{dx}(x^{\frac{1}{2}}) = 6\ \frac{1}{2}x^{\frac{1}{2}-1} = 6\ \frac{1}{2}x^{\frac{-1}{2}} = \frac{3}{\sqrt{x}}$

(iii) $\frac{dy}{dx} = \frac{d(9\ 3^x)}{dx} = 9\ \frac{d(3^x)}{dx} = 9\ 3^x \ln 3$

(iv) $\frac{d(5\cot x)}{dx} = 5\ \frac{d(\cot x)}{dx} = 5\ (-\ cosec^2 x) = -\ 5\ cosec^2 x$

Example 4

Find $\frac{dy}{dx}$

(i) $y = x^3 - x^2 + 6$

(ii) $y = \frac{1}{\sqrt{x}} + x^2(1-x) + \sin^{-1} x$

(iii) $y = \text{cosec } x - \sec^{-1} x.\cot x$

Ans.

(i) $\frac{dy}{dx} = \frac{d}{dx}(x^3 - x^2 + 6)$

$= \frac{d(x^3)}{dx} - \frac{d(x^2)}{dx} + \frac{d(6)}{dx}$

$= 3x^2 - 2x + 0$

$= 3x^2 - 2x$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(\frac{1}{\sqrt{x}} + x^2(1-x) + \sin^{-1} x)$

$= \frac{d(\frac{1}{\sqrt{x}})}{dx} + \frac{d}{dx}\{x^2(1-x)\} + \frac{d}{dx}(\sin^{-1} x)$

$= \frac{d(x^{\frac{-1}{2}})}{dx} + \{x^2\frac{d}{dx}(1-x) + \frac{d}{dx}(x^2).(1-x)\} + \frac{d}{dx}(\sin^{-1} x)$

$\{ as\ \frac{d}{dx}(fg) = fg' + f'g \}$

$= (-\frac{1}{2})\ x^{-\frac{1}{2}-1} + \{ x^2(0-1) + 2x.(1-x)\} + \frac{1}{\sqrt{1-x^2}}$

$= -\frac{1}{2x^{\frac{3}{2}}} - x^2 + 2x - 2x^2 + \frac{1}{\sqrt{1-x^2}}$

$= -\frac{1}{2x^{\frac{3}{2}}} + 2x - 3x^2 + \frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{dy}{dx} = \frac{d(\text{cosec x} - \sec^{-1} x.\cot x)}{dx}$

$= \frac{d(\text{cosec x})}{dx} - \frac{d(\sec^{-1} x.\cot x)}{dx}$

$=(-\text{ cosecx.cotx}) - \{\sec^{-1} x \,.\, (-\,cosec^2\, x) + \frac{1}{x\sqrt{x^2-1}} \cot x\}$

$= \sec^{-1} x\; cosec^2 x - \text{cosec } x \cot x - \frac{1}{x\sqrt{x^2-1}} \cot x$

Example-5

Find the derivative of following functions w.r.t x.

(i) $\frac{3x^2+2x+5}{\sqrt{x}}$ (ii) $\frac{a^x-b^x}{x}$ (iii) $\frac{\tan x}{\cos^{-1} x}$ (iv) $(\frac{x^{\frac{3}{5}}-2e^{2lnx}+\ln x^{\frac{2}{3}}}{x+1})$ (v) $x \sin x - \frac{e^x}{1+x^2}$

Ans.

(i) $\frac{dy}{dx} = \frac{\{3(2x)+2+0\}\sqrt{x} - (3x^2+2x+5)\frac{1}{2}x^{\frac{1}{2}-1}}{(\sqrt{x})^2}$

$\{ \text{As } \frac{d}{dx}(\frac{f}{g}) = \frac{f'g - fg'}{g^2} \}$

$= \frac{(6x+2)\sqrt{x} - (3x^2+2x+5)\frac{1}{2\sqrt{x}}}{x}$

$= \frac{6x^{\frac{3}{2}}+2\sqrt{x} - \frac{3}{2}x^{\frac{3}{2}} - \sqrt{x} - \frac{5}{2\sqrt{x}}}{x}$

$= \frac{\frac{9}{2}x^{\frac{3}{2}} + \sqrt{x} - \frac{5}{2\sqrt{x}}}{x}$

$= \frac{9}{2}\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{5}{2x^{\frac{3}{2}}}$

(ii) $y = \frac{a^x - b^x}{x}$

$\frac{dy}{dx} = \frac{(a^x \ln a - b^x \ln b)x - 1(a^x - b^x)}{x^2}$ $\{\frac{d}{dx}(\frac{f}{g}) = \frac{f'g - fg'}{g^2}\}$

$= \frac{xa^x \ln a - xb^x \ln b - a^x + b^x}{x^2}$

$= \frac{a^x(xln\ a - 1) + b^x(1 - x \ln b)}{x^2}$

(iii) y = $\frac{\tan x}{\cos^{-1} x}$

$$\frac{dy}{dx} = \frac{\cos^{-1} x\, sec^2 x - \tan x \left(\frac{-1}{\sqrt{1-x^2}}\right)}{(\cos^{-1} x)^2}$$

$$= \frac{\cos^{-1} x\, sec^2 x + \frac{\tan x}{\sqrt{1-x^2}})}{(\cos^{-1} x)^2}$$

(iv) y = $\frac{x^{\frac{3}{5}} - 2e^{2lnx} + \ln x^{\frac{2}{3}}}{x+1}$

$$= \frac{x^{\frac{3}{5}} - 2e^{lnx^2} + \frac{2}{3}\ln x}{x+1}$$ {As $e^{\ln x}$ = x and $lne^x = x$}

$$= \frac{x^{\frac{3}{5}} - 2x^2 + \frac{2}{3}\ln x}{x+1}$$

$$\frac{dy}{dx} = \frac{(\frac{3}{5}x^{\frac{3}{5}-1} - 4x + \frac{2}{3}\frac{1}{x})(x+1) - (x^{\frac{3}{5}} - 2x^2 + \frac{2}{3}\ln x)(1+0)}{(x+1)^2}$$

$$= \frac{(\frac{3}{5}x^{\frac{-2}{5}} - 4x + \frac{2}{3x})(x+1) - x^{\frac{3}{5}} + 2x^2 - \frac{2}{3}\ln x}{(x+1)^2}$$

$$= \frac{\frac{3}{5}x^{\frac{3}{5}} - 4x^2 + \frac{2}{3} + \frac{3}{5}x^{\frac{-2}{5}} - 4x + \frac{2}{3x} - x^{\frac{3}{5}} + 2x^2 - \frac{2}{3}\ln x}{(x+1)^2}$$

$$= \frac{\frac{2}{3} - 4x - 2x^2 - \frac{2}{5}x^{\frac{3}{5}} + \frac{3}{5x^{\frac{2}{5}}} + \frac{2}{3x} - \frac{2}{3}\ln x}{(x+1)^2}$$

(v) $\frac{dy}{dx} = \frac{d}{dx}(xsinx) - \frac{d}{dx}\left(\frac{e^x}{1+x^2}\right)$

= {x cos x+ 1. sin x} – $\left\{\frac{e^x.(1+x^2) - e^x(0+2x)}{(1+x^2)^2}\right\}$

= x cos x + sin x – $\left\{\frac{e^x + x^2e^x - 2xe^x}{(1+x^2)^2}\right\}$

= x cos x + sin x - $e^x\left\{\frac{1+x^2-2x}{(1+x^2)^2}\right\}$

= x cos x + sin x - $e^x\left(\frac{1-x}{1+x^2}\right)^2$

Example 6

Find the slope of the tangent to the curve y = $\ln x$ at x = $\frac{1}{2}$ **[2017-w]**

Ans.

Slope of tangent to the curve y = $\ln x$ at x = $\frac{1}{2}$ is $\left.\frac{dy}{dx}\right]_{x=\frac{1}{2}}$

Now, $\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$

Now $\left.\frac{dy}{dx}\right]_{x=\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

Example – 7

Find f ' ($\sqrt{3}$) if f(x) = x $\tan^{-1} x$**[2017-w]**

Ans. f(x) = $x \tan^{-1} x$

$$f'(x) = \frac{d(x \tan^{-1} x)}{dx} = x\frac{1}{1+x^2} + 1.\ \tan^{-1} x$$

$$= x\frac{1}{1+x^2} + \tan^{-1} x$$

$$f'(\sqrt{3}) = \frac{\sqrt{3}}{1+(\sqrt{3})^2} + \tan^{-1}\sqrt{3}$$

$$= \frac{\sqrt{3}}{1+3} + \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

Example-8

Find the gradient of the tangent to the curve $2x^2$-3x-1 at (1,-2).

Ans.

$\frac{dy}{dx}$ = 4x – 3

$\left.\frac{dy}{dx}\right]_{at\ (1,-2)}$ = 4X1-3 = 1

Derivative of a composite function (Chain Rule)

Composite function

A function formed by composition of more than one function is called composite function.

Example of composite functions

1) $\sin x^2$ is form by composition of two functions, one is $\sin x$ function and other is x^2.

y= $\sin x^2 = \sin u$ where u = x^2

2) Similarly y = $\sqrt{x^2 + 3x + 1}$ is written as

y= $\sqrt{u}$ where u = $x^2 + 3x + 1$

3) y= $\sqrt{\sin(x^2 + 1)}$ is form by composition of three functions.

y = $\sqrt{u}$ where u = $\sin v$ and v = (x^2+1)

Chain Rule

If y = f(u) and u is a function of x defined by u = g(x), then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Generalized chain rule

If y is a differentiable function of u, u is a differentiable function v, ….. and finally t is a differentiable function of x. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdots\cdots\cdots\cdots \frac{dt}{dx}$$

Example – 9

Find $\frac{dy}{dx}$

(i). y = $(x^2 + 2x - 1)^5$ (ii) y = $cot^3 x$

(iii) $\sqrt{\sin\sqrt{x}}$ (2016-S) (iv) $a^{\ln x}$ (v) $5^{\sin x^2}$

Ans.

(i) y = $(x^2 + 2x - 1)^5$

Here, y = u^5 and u = $x^2 + 2x - 1$

$\frac{du}{dx}$ = 2x + 2 = 2 (x+1) and $\frac{dy}{dx}$ = 5 u^4

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

= 5 u^4 2(x+1)

= 10 $(x^2 + 2x - 1)^4$ (x+1)

(ii) y = $cot^3 x$ can be written as y = u^3

where u = cot x

$\frac{du}{dx}$ = - $cosec^2 x$, $\frac{dy}{du}$ = 3 u^2

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ = 3 $u^2(-cosec^2 x)$

= - 3 $cot^2 x cosec^2 x$

(iii) y = $\sqrt{\sin\sqrt{x}}$

Here y = $\sqrt{u}$, u = $\sin v$, v = $\sqrt{x}$

So, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$, $\frac{du}{dv}$ = cos v, $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

Therefore, $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dx}$

$= \frac{1}{2\sqrt{u}}$. cos v. $\frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{\sin v}}$. cos $\sqrt{x}$. $\frac{1}{2\sqrt{x}}$

$= \frac{cos\sqrt{x}}{4\sqrt{sin\sqrt{x}}\sqrt{x}}$

(iv) y = $a^{\ln x}$

Here y = a^u where u = $\ln x$

$\frac{dy}{du}$ = $a^u \ln a$ and $\frac{du}{dx} = \frac{1}{x}$

Hence $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = a^u \ln a . \frac{1}{x}$

$= a^{lnx} \ln a \ \frac{1}{x}$

$= \frac{1}{x} \ln a \ a^{\ln x}$

(v) $y = 5^{\sin x^2}$

Here $y = 5^u$, $u = \sin v$, $v = x^2$

$\frac{dy}{du} = 5^u \ln 5$, $\frac{du}{dv} = \cos v$, $\frac{dv}{dx} = 2x$

Therefore, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = 5^u \ln 5 . \cos v . 2x$

$= 5^{\sin v} \ln 5 \cos x^2 \ 2x$

$= 2x \ ln5 \ 5^{\sin x^2} \cos x^2$

Example – 10

Differentiate the following functions w.r.t. x.

(i) $\sqrt{\cot^{-1}\sqrt{x}}$ (ii) $\frac{1}{f(x)}$ (2016-S) (iii) $\frac{1}{f(ax+b)}$ (2014-S)

(iv) $\tan^{-1}(\sec x + \tan x)$ (2017-S) (v) $\cos^{-1}(\frac{\cos x + \sin x}{\sqrt{2}})$

Ans.

(i) $\frac{d\sqrt{\cot^{-1}\sqrt{x}}}{dx} = \frac{d(\cot^{-1}\sqrt{x})}{dx}$

{Here $y = \sqrt{u}$, Then $\frac{dy}{du} = \frac{d\sqrt{u}}{du} = \frac{1}{2\sqrt{u}}$, $u = \cot^{-1}\sqrt{x} = \cot^{-1} v$, $\frac{du}{dv} = -\frac{1}{1+v^2}$}

$= \frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}} \{-\frac{1}{1+(\sqrt{x})^2}\} \frac{d\sqrt{x}}{dx}$ { $v = \sqrt{x}$, then by chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$}

$= -\frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}\,(1+x)} \frac{1}{2\sqrt{x}}$

$= -\frac{1}{4\sqrt{x}\sqrt{\cot^{-1}\sqrt{x}}\,(1+x)}$

$= -\frac{1}{4\sqrt{x}(1+x)\sqrt{\cot^{-1}\sqrt{x}}}$

(ii) $\frac{d}{dx}\{\frac{1}{f(x)}\} = -\frac{1}{\{f(x)\}^2} f'(x)$

$= -\frac{f'(x)}{\{f(x)\}^2}$

(iii) $\frac{d}{dx}\{\frac{1}{f(ax+b)}\} = -\frac{1}{\{f(ax+b)\}^2} f'(ax+b) \frac{d}{dx}(ax+b)$

$= -\frac{a f'(ax+b)}{f(ax+b)^2}$

(iv) y = $\tan^{-1}(\sec x + \tan x)$

$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$

$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

$= \tan^{-1}\left\{\frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right\}$

$= \tan^{-1}\left\{\frac{(\sin\frac{x}{2} + \cos\frac{x}{2})^2}{(\cos\frac{x}{2} - \sin\frac{x}{2})(\cos\frac{x}{2} + \sin\frac{x}{2})}\right\}$

$= \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$ {dividing numerator and denominator by $\frac{x}{2}$}

$= \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)$ $= \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}.\tan\frac{x}{2}}\right) = \tan^{-1}\{\tan(\frac{\pi}{4} + \frac{x}{2})\} = \frac{\pi}{4} + \frac{x}{2}$

Hence $\frac{dy}{dx} = \frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\} = \frac{d}{dx}(\frac{\pi}{4} + \frac{x}{2}) = \frac{1}{2}$

(v) y = $\cos^{-1}\left(\frac{\cos x + \sin x}{\sqrt{2}}\right)$

$= \cos^{-1}(\cos x \frac{1}{\sqrt{2}} + \sin x.\frac{1}{\sqrt{2}})$

$= \cos^{-1}(\cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4})$

$= \cos^{-1}(\cos(x - \frac{\pi}{4})) = x - \frac{\pi}{4}$

Hence $\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}\left(\frac{\cos x + \sin x}{\sqrt{2}}\right)\right)$

$= \frac{d}{dx}(x - \frac{\pi}{4}) = 1$

Example –11

If y = sin 5x cos 7x then find $\frac{dy}{dx}$

Ans.

$\frac{dy}{dx}$ = sin 5x $\frac{d}{dx}$(cos 7x) + $\frac{d}{dx}$(sin 5x). cos 7x

= sin 5x. (-7 sin 7x) + 5 cos 5x cos 7x

= 5 cos 5x cos 7x – 7 sin 5x sin 7x

Example –12

Find $\frac{dy}{dx}$ if y = $cosec^2(2x^2 + \log_7 x)$

Ans.

$\frac{dy}{dx}$ = 2 cosec $(2x^2 + \log_7 x)\{(-cosec(2x^2 + \log_7 x)). \cot(2x^2 + \log_7 x)\}$ (4x + $\frac{1}{x \log_e 7}$)

= - 2 $cosec^2(2x^2 + \log_7 x)$. Cot $(2x^2 + \log_7 x)$[4x + $\frac{1}{x \log_e 7}$]

Methods of differentiation

We use following two methods for differentiation of some functions.

(i) Substitution

(ii) Use of logarithms

Substitution

Sometimes with proper substitution we can transform the given function to a simpler function in the new variable so that the differentiation w.r.t to new variable becomes easier. After differentiation we again re-substitute the old variable. This can be better understood by following examples.

Example – 13

y = $\tan^{-1}(\frac{\sqrt{x} - x}{1 + x^{\frac{3}{2}}})$

Ans.

y = $\tan^{-1}(\frac{\sqrt{x} - x}{1 + x^{\frac{3}{2}}})$

(If we differentiate directly by applying chain rule , it will be very complicated. So, we have to adopt substitution technique here.)

Now y = $\tan^{-1}(\frac{\sqrt{x} - x}{1 + x^{\frac{3}{2}}})$ = $\tan^{-1}(\frac{\sqrt{x} - x}{1 + \sqrt{x}.x})$

Now Put $\sqrt{x}$ = tan $\propto$, x = tan β

Then, y = $\tan^{-1}(\frac{\tan\propto - \tan\beta}{1 + tan\alpha . tan\beta})$

= $\tan^{-1}(\tan(\propto - \beta))$

= $\propto$ - β = $\tan^{-1}\sqrt{x}$ - $\tan^{-1} x$

(As $\sqrt{x} = \tan \propto \Rightarrow \propto = \tan^{-1}\sqrt{x}$ and x = tan $\beta \Rightarrow \beta = \tan^{-1}\sqrt{x}$)

Now $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{x} - \tan x)$

$= \frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x}) - \frac{1}{1+x^2}$

$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{1+x^2}$

$= \frac{1}{(1+x)2\sqrt{x}} - \frac{1}{1+x^2}$ (Ans)

Example –14

Find $\frac{d}{dt}$ if y = $\cos^{-1}(\frac{1-t^2}{1+t^2})$ **(2015-S)**

Ans.

y = $\cos^{-1}(\frac{1-t^2}{1+t^2})$ {Put tan θ = t $\Rightarrow \theta = \tan^{-1} t$}

$= \cos^{-1}(\frac{1-tan^2\theta}{1+tan^2\theta})$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$= 2\tan^{-1} t$

$\frac{dy}{dt} = \frac{d}{dt}(2\tan^{-1} t) = \frac{2}{1+t^2}$

Note

When we apply substitution method, then we must have proper knowledge about trigonometric formulae. Because it makes the choice of new variable easy. If proper substitution is not made, then problem will be more complicated than original.

Example –15

If y = $\sec^{-1}(\frac{\sqrt{a^2+x^2}}{a})$ then find $\frac{dy}{dx}$

Ans.

y $= \sec^{-1}(\frac{\sqrt{a^2+x^2}}{a})$ Put x = a tan θ

$= \sec^{-1}(\frac{\sqrt{a^2+a^2tan^2\theta}}{a})$ $= \sec^{-1}(\frac{\sqrt{a^2(1+tan^2\theta)}}{a})$

$= \sec^{-1}(\frac{\sqrt{a^2sec^2\theta}}{a})$ $= \sec^{-1}(\frac{a\sec\theta}{a})$

$= \sec^{-1}(\sec\theta) = \theta$ $= \tan^{-1}(\frac{x}{a})$

Now $\frac{dy}{dx} = \frac{d}{dx}\{\tan^{-1}(\frac{x}{a})\} = \frac{1}{1+(\frac{x}{a})^2}\frac{d}{dx}(\frac{x}{a})$

$= \frac{1}{(1+\frac{x^2}{a^2})}(\frac{1}{a}) = \frac{1}{a}\frac{1}{\frac{a^2+x^2}{a^2}}$

$= \frac{a^2}{a(x^2+a^2)} = \frac{a}{x^2+a^2}$

Example – 16

Differentiate $sin^2(\cot^{-1}\sqrt{\frac{1+x}{1-x}})$ w.r.t x. **[2018-S]**

Ans.

y $= sin^2(\cot^{-1}\sqrt{\frac{1+x}{1-x}})$ {Put x = cos 2$\theta \Rightarrow \theta = \frac{\cos^{-1}x}{2}$}

$= sin^2(\cot^{-1}\sqrt{(\frac{1+\cos 2\theta}{1-\cos 2\theta})}) = sin^2(\cot^{-1}\sqrt{\frac{2cos^2\theta}{2sin^2\theta}})$

$= sin^2(\cot^{-1}\sqrt{cot^2\theta}) = sin^2\cot^{-1}(\cot\theta) = sin^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}(sin^2\theta)\frac{d}{dx}(\frac{\cos^{-1}x}{2})$

$= 2\sin\theta\cos\theta\frac{1}{2}(\frac{-1}{\sqrt{1-x^2}})$

$= \sin 2\theta(\frac{-1}{2\sqrt{1-x^2}}) = -\frac{\sqrt{1-cos^2 2\theta}}{2\sqrt{1-x^2}}$

$= -\frac{\sqrt{1-x^2}}{2\sqrt{1-x^2}} = -\frac{1}{2}$ (Ans)

Example – 17

Find the derivative of $\cot^{-1}(\sqrt{1+x^2}+x)$ w.r.t x

Ans.

y $= \cot^{-1}(\sqrt{1+x^2}+x)$ { Put x = cot θ =>$\theta = \cot^{-1}x$}

$= \cot^{-1}(\sqrt{1+cot^2\theta}+\cot\theta)$

$= \cot^{-1}(\sqrt{cosec^2\theta}+\cot\theta)$

$= \cot^{-1}(\operatorname{cosec}\theta + \cot\theta)$

$= \cot^{-1}(\frac{1}{sin\theta} + \frac{cos\theta}{sin\theta})$

$= \cot^{-1}(\frac{1+cos\theta}{sin\theta})$

$= \cot^{-1}(\frac{2cos^2\frac{\theta}{2}}{2sin\frac{\theta}{2}cos\frac{\theta}{2}})$

$= \cot^{-1}(\frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}})$

$= \cot^{-1}(\cot(\frac{\theta}{2})) = \frac{\theta}{2} = \frac{\cot^{-1}x}{2}$ Type equation here.

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\frac{\cot^{-1}x}{2}) = \frac{1}{2}(\frac{-1}{1+x^2}) = - \frac{1}{2(1+x^2)}$

Differentiation using logarithm

When a function appears as an exponent of another function we make use of logarithms.

Example – 18

Differentiate $(\sin x)^{\tan x}$

Ans.

$y = (\sin x)^{\tan x}$

Taking logarithms of both sides we have,

$\ln y = \ln(\sin x)^{\tan x}$

$\Rightarrow \ln y = \tan x . \ln \sin x$

Differentiating both sides w.r.t x, we have

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x \frac{1}{\sin x} . \cos x + sec^2x . \ln \sin x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x . \cot x + sec^2x . \ln \sin x = 1 + sec^2x . \ln \sin x$

$\Rightarrow \frac{dy}{dx} = y\,(1+sec^2x . \ln \sin x)$

Hence $\frac{dy}{dx} = (\sin x)^{\tan x}\,(1+sec^2x . \ln \sin x)$

Example – 19

Differentiate y = $\frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}$

Ans.

$$y = \frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}$$

Taking logarithm of both sides

$\Rightarrow \ln y = \ln(x-1)^2 + \ln\sqrt{3x-1} - \ln x^7 - \ln(6-7x^2)^{\frac{3}{2}}$ {as $\log ab = \log a + \log b$ & $\log\frac{a}{b} = \log a - \log b$}

$\Rightarrow \ln y = 2\ln(x-1) + \frac{1}{2}\ln(3x-1) - 7\ln x - \frac{3}{2}\ln(6-7x^2)$ { as $\ln x^n = n \ln x$}

Differentiating both sides w.r.t, we have

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-1}\frac{d\,(x-1)}{dx} + \frac{1}{2}\frac{1}{(3x-1)}\frac{d(3x-1)}{dx} - \frac{7}{x} - \frac{3}{2}\frac{1}{6-7x^2}\frac{d(6-7x^2)}{dx}$

$= \frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{3(14x)}{2(6-7x^2)}$

$= \frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}\left[\frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}\right]$ (Ans)

Example – 20

Find the derivative of y = $(\log x)^{\tan x}$ **(2017-W, 2015-S)**

Ans: - y = $(\log x)^{\tan x}$

Taking logarithm of both sides,

$\log y = \log(\log x)^{\tan x}$

$\Rightarrow$ logy = tanx log(logx)

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x \; \frac{1}{\log x} \cdot \frac{1}{x} + sec^2x\log(\log x)$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{\tan x}{x\log x} + sec^2x\log(\log x)\right)$

$\Rightarrow \frac{dy}{dx} = (\log x)^{\tan x}\left(\frac{\tan x}{x\log x} + sec^2x\log(\log x)\right)$

Example – 21

Differentiate $(\sin x)^{\ln x}$ w.r.t x **(2017-S)**

Ans.

y $= (\sin x)^{\ln x}$

Then log y = log $(\sin x)^{\ln x}$ = ln x log (sin x)

Differentiating w.r.t x ,

$\frac{1}{y}\frac{dy}{dx} = \; lnx \frac{1}{\sin x} \; cosx + \frac{1}{x}$ log(sin x)

$\Rightarrow \frac{dy}{dx} = y\,[\;\; lnx\ cot\ x + \frac{\log(\sin x)}{x}]$

$\therefore \frac{dy}{dx} = (\sin x)^{\ln x}\,[\;\; ln\ cot\ x + \frac{\log(\sin x)}{x}]$

Example – 22

Find $\frac{dy}{dx}$ if y = x^x

Ans. y = x^x

Taking logarithm of both sides,

$\Rightarrow$ log y = $\log x^x$ = x log x

Differentiating w.r.t x ,

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \; x.\frac{1}{x}$ + 1 . log x

$\Rightarrow \frac{dy}{dx}$ = y (1+log x) = $x^x(1 + \log x)$.

Example – 23

Differentiate $(\ln x)^x$ + $(\sin^{-1} x)^x$ w.r.t. x.

Ans:- y= $(\ln x)^x$ + $(\sin^{-1} x)^x$ = u +v

u = $(\ln x)^x$ and v = $(\sin^{-1} x)^x$

Taking logarithm of both sides,

$\Rightarrow \; \log u \; = \; \log{(\ln x)^x}$ and $\log v \; = \; \log{(\sin^{-1} x)^x}$

$\Rightarrow$ log u= x log (ln x) and log v = x log $(\sin^{-1} x)$

Differentiating w.r.t x ,

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\text{x log (ln x)})$ ()) and $\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(xlog(sin^{-1}x))$

$\Rightarrow \frac{1}{u}\frac{du}{dx}$ = [$x\frac{1}{\ln x} \cdot \frac{1}{x}$ + 1 . log (*ln x*)] and $\frac{1}{v}\frac{dv}{dx}$= $x\frac{1}{sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$ + 1. log($sin^{-1}x$)

$\Rightarrow \frac{du}{dx} = u[\frac{1}{\ln x}$ + log (ln x)] and $\frac{dv}{dx} = v[\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)]

$\Rightarrow \frac{du}{dx} = (\ln x)^x$ [$\frac{1}{\ln x}$ + log (ln x)] and $\frac{dv}{dx} = (\sin^{-1} x)^x$ [$\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)]

Now, $\frac{dy}{dx} = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$= (\ln x)^x$ [$\frac{1}{\ln x}$ + log (ln x)] + $(\sin^{-1} x)^x$ [$\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)] (Ans)

Differentiation of parametric function

Sometimes the variables x and y of a function is represent by function of another variable ‘t’, which is called as a parameter. Such type of representation of a fnction is called parametric form. For example equation of circle can be given by x = r cos t, y = r sin t.

Here x, y both are functions of parameter ‘t’.

So, this form of the function is called parametric form.

Derivative of function given in parametric form

If y = f(t) , x = g(t) , Then

$$\frac{dy}{dx} = \frac{f'(t)}{g'(t)} = \frac{\frac{df(t)}{dt}}{\frac{dg(t)}{dt}} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}$$

Example – 24

Find $\frac{dy}{dx}$ if x = at^2 and y = 2 bt

Ans.

$\frac{dx}{dt}$ = 2at , $\frac{dy}{dt}$ = 2b

Now, $\frac{dy}{dx} = (\frac{dy}{dt}) / (\frac{dx}{dt}) = \frac{2b}{2at} = \frac{b}{at}$

Example -25

Find $\frac{dy}{dx}$ if x = a(1 + $\cos\theta$) and y =b (1 - $\sin\theta$)   (2018-S)

Ans. $\frac{dx}{d\theta}$ = a (- $\sin\theta$) = -a sin θ , $\frac{dy}{d\theta}$ = b (- $\cos\theta$) = -b cosθ

Hence $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-\,b\cos\theta}{-\,a\sin\theta} = \frac{b}{a}$ cot θ

Example – 26

Find $\frac{dy}{dx}$ when x = a (cos t +t sin t) and y = a (sin t – t cos t)   **(2017-S, 2017-W)**

Ans.

$\frac{dx}{dt}$ = a (-sin t + t cos t + 1. sin t)

= a (t cos t) = a t cos t

$\frac{dy}{dt}$ = a (cos t – t (-sint) – 1. cos t)

= a t sin t ,

$\therefore$ $\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{a\,t\sin t}{a\,t\cos t}$ = tan t.

Example – 27

If sin x = $\frac{2t}{1+t^2}$ and tan y = $\frac{2t}{1-t^2}$ then find $\frac{dy}{dx}$.

Ans.

Put t = tan θ { In this case by substitution we can convert both x and y into functions of another parameter θ, which are easily differentiable w.r.t to θ .}

Then sin x = $\frac{2\tan\theta}{1+tan^2\theta}$ = sin 2θ

$\Rightarrow$ x = $\sin^{-1}(\sin 2\theta) = 2\theta$

tan y = $\frac{2t}{1-t^2} = \frac{2\tan\theta}{1-tan^2\theta}$ = tan 2θ

$\Rightarrow$ y = 2θ

Now, $\frac{dy}{d\theta}$ = 2 and $\frac{dx}{d\theta}$ = 2

Hence $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2}{2}$ = 1

Differentiation of a function w.r.t another function

Suppose we have two differentiable functions given by y = f(x) and z = g(x). Then to find the derivative of y w.r.t. z we have to follow the following formula.

$$\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})}$$

Example-28

Find the derivative of tan x w.r.t cot x **(2017-w)**

Ans.

Let y = tan x and z = cot x

$\frac{dy}{dx} = sec^2x$, $\frac{dz}{dx} = - cosec^2x$

Now, $\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})} = \frac{sec^2x}{-cosec^2x} = - sec^2x sin^2x$,Hence $\frac{d(\tan x)}{d\,(\cot x)} = - sec^2x sin^2x$

Example – 29

Find the derivative of $e^{2\log x}$ w.r.t $2x^2$ **[2018-S, 2017-w]**

Ans.

y = $e^{2\log x}$ and z = $2x^2$

$\frac{dy}{dx} = \frac{d}{dx}(e^{2\log x}) = \frac{d}{dx}(e^{\log x^2}) = \frac{d}{dx}(x^2) = 2x$

$\frac{dz}{dx} = \frac{d}{dx}(2x^2) = 4x$

Hence $\frac{dy}{dz} = (\frac{dy}{dx})/(\frac{dz}{dx}) = \frac{2x}{4x} = \frac{1}{2}$

Example – 30

Differentiate a^x w.r.t x^a **[2014-S]**

Ans. y = a^x and z = x^a

Now, $\frac{dy}{dx} = a^x\log a$ and $\frac{dz}{dx} = ax^{a-1}$

Hence $\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})} = \frac{a^x\log a}{ax^{a-1}} = \frac{a^{x-1}}{x^{a-1}}\log a$ (Ans)

Example – 31

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ **(2016-S)**

Ans. Let y = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$& z = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put x = tan t

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan t}{1+tan^2 t}\right) = \sin^{-1}(\sin 2t) = 2t$

$\Rightarrow \frac{dy}{dt} = 2$

Similarly, $z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-tan^2 t}{1+tan^2 t}\right) = \cos^{-1}(\cos 2t) = 2t$

$\Rightarrow \frac{dz}{dt} = 2$

$\frac{dy}{dz} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dz}{dt}\right)} = \frac{2}{2} = 1$

Hence $\frac{d\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)}{d\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)} = 1$

Example – 32

Find derivative of log x w.r.t $\sqrt{x}$ **[2017-W]**

Ans.

y = log x and z = $\sqrt{x}$

Now, $\frac{dy}{dx} = \frac{1}{x}$ and $\frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

Hence $\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$ (Ans)

Example –33

Differentiate $\frac{1-\cos x}{1+\cos x}$ w.r.t $\frac{1-\sin x}{1+\sin x}$

Ans.

$y = \frac{1-\cos x}{1+\cos x}$ and $z = \frac{1-\sin x}{1+\sin x}$

$\frac{dy}{dx} = \frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$ { As $\frac{d}{dx}\left\{\frac{f}{g}\right\} = \frac{f'g - fg'}{g^2}$}

$$= \frac{\sin x + \sin x cosx + \sin x - sinxcosx}{(1+\cos x)^2}$$

$$= \frac{2 \sin x}{(1+\cos x)^2}$$

$$\frac{dz}{dx} = \frac{(1+\sin x)(-\cos x) - (1-\sin x)\cos x}{(1+\sin x)^2}$$

$$= \frac{-cosx - \cos x \sin x - \cos x + \cos x \sin x}{(1+\sin x)^2}$$

$$= \frac{-2\cos x}{(1+\sin x)^2}$$

$$\frac{dy}{dz} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{\frac{2\sin x}{(1+\cos x)^2}}{\frac{-2\cos x}{(1+\sin x)^2}} = \frac{-\tan x(1+\sin x)^2}{(1+\cos x)^2}$$

Differentiation of implicit function

Functions of the form F(x,y) = 0 where x and y cannot be separated or in other words y cannot be expressed in terms of x is called Implicit function.

e.g. $x^2 + y^2 - 25 = 0$

$x^y = y^x$

$x^2y + y^2x + xy = 25$ etc

Derivative of Implicit functions can be found without expressing y explicitly in terms of x. Simply we differentiate both side w.r.t x and express $\frac{dy}{dx}$ in terms of both x and y.

Example – 34

Find $\frac{dy}{dx}$ when $x^3 + y^3 - 3xy = 0$ **[2015-S]**

Ans.

Given $x^3 + y^3 - 3xy = 0$

Differentiating both sides w.r.t x We have,

$$3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3.1.y = 0$$

$$\Rightarrow \frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y-3x^2}{3y^2-3x} = \frac{y-x^2}{y^2-x} \quad \text{(Ans)}$$

Example – 35

Find $\frac{dy}{dx}$ if $\ln\sqrt{x^2+y^2} = \tan^{-1}(\frac{y}{x})$ **[2017-w]**

Ans.

Given $\ln\sqrt{x^2+y^2} = \tan^{-1}(\frac{y}{x})$

Differentiating both sides w.r.t x,

$$\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}(2x+2y\frac{dy}{dx}) = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{x\frac{dy}{dx}-y.1}{x^2}\right)$$

$$\Rightarrow \frac{2(x+y\frac{dy}{dx})}{2(x^2+y^2)} = \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)$$

$$\Rightarrow \frac{x+y\frac{dy}{dx}}{(x^2+y^2)} = \frac{x^2(x\frac{dy}{dx}-y)}{(x^2+y^2)x^2}$$

$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$

$\Rightarrow (x-y)\frac{dy}{dx} = x + y$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x+y}{x-y}}$$

Example – 36

Find $\frac{dy}{dx}$ if $y^x = x^y$ **[2014-S, 2016-S, 2017-w]**

Ans. Given $y^x = x^y$

Taking logarithm of both sides

$\Rightarrow \ln y^x = \ln x^y$

$\Rightarrow x \ln y = y \ln x$

Differentiating both sides w.r.t x, we have

$\Rightarrow x\frac{1}{y}\frac{dy}{dx} + 1.\ln y = \frac{dy}{dx}.\ln x + y\frac{1}{x}$

$\Rightarrow \frac{x}{y}\frac{dy}{dx} + \ln y = \ln x\frac{dy}{dx} + \frac{y}{x}$

$\Rightarrow (\frac{x}{y} - \ln x)\frac{dy}{dx} = (\frac{y}{x} - \ln y)$

$\Rightarrow (\frac{x-y\ln x}{y}) \frac{dy}{dx} = \frac{y-x\ln y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(y-x\ln y)}{x(x-y\ \ln\ x)}$

$\therefore \frac{dy}{dx} = \frac{y(y-x\ln y)}{x(x-y\ln x)}$

Example – 37

Find $\frac{dy}{dx}$ if y^2 cot x = x^2 cot y

Ans. y^2 cot x = x^2 cot y

Differentiating both sides w.r.t x,

$\Rightarrow$ 2y $\frac{dy}{dx}$cot x + y^2(- $cosec^2x$) = 2x cot y + x^2(-$cosec^2y\ \frac{dy}{dx}$)

$\Rightarrow$ 2y cot x $\frac{dy}{dx}$ - y^2cosec^2x = 2x cot y - $x^2cosec^2y\ \frac{dy}{dx}$

$\Rightarrow$ (2y cot x + x^2cosec^2y)$\frac{dy}{dx}$ = 2x cot y + y^2cosec^2x

$\Rightarrow \frac{dy}{dx} = \frac{2x\cot y + y^2cosec^2x}{2y\cot x + x^2cosec^2y}$

Example – 38

Find $\frac{dy}{dx}$ if $y^x = x^{siny}$

Ans. $y^x = x^{siny}$

Taking logarithm of both sides

log y^x = log x^{siny}

$\Rightarrow$ *x log y = sin y log x*

Differentiating both sides w.r.t x,

$\Rightarrow$ 1. log y + x$\frac{1}{y}\frac{dy}{dx}$ = cos y $\frac{dy}{dx}$. $\log x + \sin y.\frac{1}{x}$

$\Rightarrow (\frac{x}{y} - \log x \cos y)\frac{dy}{dx} = \frac{\sin y}{x} - logy$

$\Rightarrow (\frac{x-y\log x \cos y}{y}) \frac{dy}{dx} = \frac{\sin y - x\log y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(\sin y - x\log y)}{x(x-y\ \log\ x\ \cos\ y)}$

Differentiation of Infinite series

Example – 39

If y = $x^{x^{x^{\cdots}}}$, find $\frac{dy}{dx}$

Ans.

y = $x^{x^{x^{\cdots}}}$

⇒ y = $x^{(x^{x^{\cdots}})}$ = x^y

Taking logarithm of both sides

⇒ log y = log x^y = y log x

Differentiating both sides

⇒$\frac{1}{y}\frac{dy}{dx}$ = $\frac{dy}{dx}$log x + y $\frac{1}{x}$

⇒ $(\frac{1}{y} - \log x)\frac{dy}{dx} = \frac{y}{x}$

⇒$\frac{dy}{dx}$ = $\frac{y}{x}\frac{1}{(\frac{1}{y}-\log x)}$ = $\frac{y^2}{x(1-y\log x)}$

Example – 2

If y = $\sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$

⇒ y = $\sqrt{\sin x + (\sqrt{\sin x + \sqrt{\sin x + \cdots}})}$

⇒ y = $\sqrt{\sin x + y}$

Squaring both sides

⇒y^2 = sin x + y

Differentiating both sides w.r.t x,

⇒ 2y $\frac{dy}{dx}$= cos x + $\frac{dy}{dx}$

⇒ (2y-1)$\frac{dy}{dx}$ = cos x

⇒ $\boxed{\frac{dy}{dx} = \frac{\cos x}{2y-1}}$

Miscellaneous examples

Example – 1

Differentiate the following functions w.r.t x

(i) $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

(ii) $|x|$ for $x \neq 0$

(iii) $\tan^{-1} e^{2x}$

(iv) $e^{\tan^{-1} x^2}$

(v) $\tan^{-1} \frac{7\,ax}{a^2-12x^2}$

(vi) $x^{\sqrt{x}}$

(vii) $\log_{10} \sin x + \log_x 10$ x > 0

(viii) $(x^e)^{e^x} + (e^x)^{x^e}$

(ix) x^{xx}

(x) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Ans.

(i) $y = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

$$= \frac{(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$$

$$= \frac{(\sqrt{a+x}+\sqrt{a-x})^2}{(\sqrt{a+x})^2-(\sqrt{a-x})^2}$$

$$= \frac{(a+x)+(a-x)+2\sqrt{a+x}\sqrt{a-x}}{(a+x)-(a-x)}$$

$$= \frac{2a+2\sqrt{(a+x)(a-x)}}{2x}$$

$$= \frac{a+\sqrt{a^2-x^2}}{x}$$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

$$= \frac{x\{0+\frac{1}{2\sqrt{a^2-x^2}}(-2x)\}-(a+\sqrt{a^2-x^2}).1}{x^2}$$

$$= \frac{-\frac{x^2}{\sqrt{a^2-x^2}}-(a+\sqrt{a^2-x^2})}{x^2}$$

$$= \frac{-x^2 - a\sqrt{a^2-x^2} - a^2 + x^2}{x^2\sqrt{a^2-x^2}}$$

$$= \frac{-x^2 - a\sqrt{a^2-x^2} - a^2 + x^2}{x^2\sqrt{a^2-x^2}}$$

$$= -\frac{(a^2 + a\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$$

(ii) y = |x|

When x < 0 , y = |x| = -x

When x > 0 , y = |x| = x

So $\frac{d|x|}{dx} = \frac{d(-x)}{dx} = -1$ when x < 0

$\frac{d|x|}{dx} = \frac{d(x)}{dx} = 1$ when x > 0

(iii) $y = \tan^{-1} e^{2x}$

$$\frac{dy}{dx} = \frac{1}{1+(e^{2x})^2}\frac{d}{dx}(e^{2x}) = \frac{2\,e^{2x}}{1+e^{4x}}$$

(iv) $y = e^{\tan^{-1} x^2}$

$$\frac{dy}{dx} = e^{\tan^{-1} x^2}\,\frac{d}{dx}(\tan^{-1} x^2) = e^{\tan^{-1} x^2}\,\frac{1}{1+(x^2)^2}\frac{d}{dx}(x^2)$$

$$= \frac{2\,x\,e^{\tan^{-1} x^2}}{1+x^4}$$

(v) $y = \tan^{-1}\frac{7\,ax}{a^2-12x^2} = \tan^{-1}\left(\frac{\frac{7\,ax}{a^2}}{\frac{a^2-12x^2}{a^2}}\right)$

$$= \tan^{-1}\left(\frac{\frac{7x}{a}}{1-\frac{12x^2}{a^2}}\right) = \tan^{-1}\left(\frac{\frac{3x}{a}+\frac{4x}{a}}{1-\frac{3x}{a}\cdot\frac{4x}{a}}\right)$$

(Putting $\frac{3x}{a} = \tan\theta_1$ & $\frac{4x}{a} = \tan\theta_2$)

$$= \tan^{-1}\left(\frac{\tan\theta 1+\tan\theta 2}{1-\tan\theta 1\tan\theta 2}\right) = \tan^{-1}\{\tan(\theta_1+\theta_2)\}$$

$$= \theta_1 + \theta_2 = \tan^{-1}\frac{3x}{a} + \tan^{-1}\frac{4x}{a}$$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{3x}{a}\right) + \frac{d}{dx}\left(\tan^{-1}\frac{4x}{a}\right)$

$$= \frac{1}{1+\left(\frac{3x}{a}\right)^2}\left(\frac{3}{a}\right) + \frac{1}{1+\left(\frac{4x}{a}\right)^2}\left(\frac{4}{a}\right) = \frac{1}{1+\frac{9x^2}{a^2}}\left(\frac{3}{a}\right) + \frac{1}{1+\frac{16x^2}{a^2}}\left(\frac{4}{a}\right)$$

$$= \frac{3a^2}{(a^2+9x^2)a} + \frac{4a^2}{(a^2+16x^2)a}$$

$$= \frac{3a}{a^2+9x^2} + \frac{4a}{a^2+16x^2}$$ (Ans)

(vi) y = $x^{\sqrt{x}}$

Taking logarithm of both sides,

ln y = ln $x^{\sqrt{x}}$

⇒ ln y = $\sqrt{x}\ lnx$

Differentiating both sides w.r.t x,

⇒ $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}} lnx + \sqrt{x}\frac{1}{x}$

⇒ $\frac{dy}{dx} = y(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}})$

⇒ $\frac{dy}{dx} = x^{\sqrt{x}}\ (\frac{\ln x+2}{2\sqrt{x}})$ (ans)

(vii) y = $\log_{10}\sin x + \log_x 10$

= $\log_{10}\sin x + \frac{1}{\log_{10} x}$ {as $\log_b a = \frac{1}{\log_a b}$}

$\frac{dy}{dx} = \frac{1}{\sin x\ \log_e 10} cosx + \{-\frac{1}{(\log_{10} x)^2} \cdot \frac{1}{x\ \log_e 10}\}$

= $\frac{\cot x}{\log_e 10}$ - $\frac{1}{x(\log_{10} x)^2 \log_e 10}$

= $cotx\ \log_{10} e$ - $\frac{\log_{10} e}{x(\log_{10} x)^2}$ (ans)

(viii) y = $(x^e)^{e^x} + (e^x)^{x^e}$

Let y = $y_1 + y_2$ where $y_1 = (x^e)^{e^x}$, $y_2 = (e^x)^{x^e}$

Now, $y_1 = (x^e)^{e^x}$

Taking logarithm of both sides

log y_1 = log $(x^e)^{e^x}$ = e^x log x^e

⇒ log y_1 = e^x e log x = $e^{x+1}logx$

Differentiating w.r.t x we have,

⇒ $\frac{1}{y_1}\frac{dy_1}{dx} = e^{x+1}logx + e^{x+1}\frac{1}{x}$

⇒ $\frac{dy_1}{dx}$ = y_1 $(e^{x+1}logx + \frac{e^{x+1}}{x})$

= $(x^e)^{e^x} e^{x+1}\ (logx + \frac{1}{x})$ ------------- (1)

Again $y_2 = (e^x)^{x^e}$

Taking log of both sides

⇒ ln y_2 = x^eln e^x = x^ex = x^{e+1}

Differentiating w.r.t x we have,

⇒ $\frac{1}{y_2}\frac{dy_2}{dx}$ = (e+1) x^{e+1-1} = (e+1)x^e

⇒ $\frac{dy_2}{dx}$ = y_2(e+1)x^e = $(e^x)^{x^e}$ (e+1)x^e ---------------------- (2)

From (1) and (2)

$\frac{dy}{dx} = \frac{dy1}{dx} + \frac{dy2}{dx}$

$= (x^{e})^{e^{x}} e^{x+1} (\log x + \frac{1}{x}) + (e^{x})^{x^{e}} (e+1) x^{e}$ (ans)

(ix) y = $(x^{x^{x}})$

Taking logarithm of both sides,

ln y = ln $x^{x^{x}}$ = x^{x}ln x

Differentiating w.r.t x we have,

$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{d(x^{x})}{dx} + x^{x}\frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y(lnx \frac{d(x^{x})}{dx} + x^{x-1})$

$= x^{x^{x}}(\ln x \frac{d(x^{x})}{dx} + x^{x-1})$ --------------- (1)

Now let z = x^{x}

Taking logarithm both sides,

⇒ log z = ln x^{x}

$\Rightarrow logz = x\ lnx$

Differentiating w.r.t x we have,

$\Rightarrow \frac{1}{z}\frac{dz}{dx} = 1. \ln x + x \frac{1}{x}$

$\Rightarrow \frac{dz}{dx} = z (\ln x + 1)$

⇒

$$\boxed{\frac{d(x^{x})}{dx} = x^{x} (\ln x + 1)}$$ - ------------(2)

From (1) and (2)

$\frac{dy}{dx} = x^{x^{x}}(\ln x\ x^{x} (\ln x + 1) + x^{x-1})$

$= x^{x^{x}}(x^{x}(\ln x)^{2} + x^{x} \ln x + x^{x-1})$

$= x^{x^{x}} x^{x-1} (x (\ln x)^{2} + x \ln x + 1)$ (Ans)

x) y = $\tan^{-1}(\frac{\sqrt{1+x^{2}} + \sqrt{1-x^{2}}}{\sqrt{1+x^{2}} - \sqrt{1-x^{2}}})$

Put = x^{2} = cos θ

Then y = $\tan^{-1}(\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}})$

$= \tan^{-1}(\frac{\sqrt{2\cos^{2}\frac{\theta}{2}} + \sqrt{2\sin^{2}\frac{\theta}{2}}}{\sqrt{2\cos^{2}\frac{\theta}{2}} - \sqrt{2\sin^{2}\frac{\theta}{2}}})$

$= \tan^{-1}(\frac{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}})$

$$= \tan^{-1}\left(\frac{\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\right)$$ {dividing numerator and denominator by $\frac{\theta}{2}$}

$$= \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2}$$

$$= \frac{\pi}{4}+\frac{\cos^{-1}x^2}{2}$$

Now $\frac{dy}{dx} = 0 + \frac{1}{2}\left(\frac{-1}{\sqrt{1-(x^2)^2}}\right)2x$

$= -\frac{x}{\sqrt{1-x^4}}$

Example – 2

If *cos y = x cos (a+y)* then show that $\frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$.

Ans. Given *cos y = x cos (a+y)*

$\Rightarrow x = \frac{\cos y}{\cos(a+y)}$

Differentiate both sides w.r.t *x* we have,

$$\Rightarrow 1 = \frac{\cos(a+y)(-\sin y)\frac{dy}{dx} - \cos y\ (-\sin(a+y))\frac{dy}{dx}}{cos^2(a+y)}$$

$$\Rightarrow 1 = \frac{dy}{dx}\left\{\frac{\sin(a+y)cosy-\cos(a+y)siny}{cos^2(a+y)}\right\}$$

$$\Rightarrow 1 = \frac{dy}{dx}\left\{\frac{\sin(a+y-y)}{cos^2(a+y)}\right\}$$

$\Rightarrow \frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$ (Proved)

Example – 3

Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1-x^2}$

Ans.

Here y = $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$, z = $\sqrt{1-x^2}$

Let x = cos θ

Then y = $\sec^{-1}\left(\frac{1}{2cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x$

$\frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1} x) = \frac{-2}{\sqrt{1-x^2}}$

$\frac{dz}{dx} = \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

Now $\frac{dy}{dz} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \frac{-2}{-x} = \frac{2}{x}$

Example – 4

If $y = 10^{\log sinx}$ find $\frac{dy}{dx}$.

Ans.

$\frac{dy}{dx} = \frac{d}{dlogsinx}(10^{logsinx}).\ \frac{d}{dx}(logsinx)$

$= 10^{\log sinx} \log_e 10 \frac{d}{dx}(logsinx)$ (As $\frac{d}{dx}((a^x) = a^x \log_e a)$

$= 10^{\log sinx}$ln $10 \frac{1}{\sin x} cosx = ln10\, cotx\, 10^{\log sinx}$

Example – 5

If $x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1} \frac{t}{\sqrt{1+t^2}}$ then find $\frac{dy}{dx}$

$x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}$ (Put $t = tan\theta$)

$= \cos^{-1}\left(\frac{1}{\sqrt{1+tan^2\theta}}\right) = \cos^{-1}\left(\frac{1}{\sec\theta}\right)$

$= \cos^{-1}(\cos\theta) = \theta = \tan^{-1} t$

$\Rightarrow \frac{dx}{dt} = \frac{1}{1+t^2}$

Similarly $y = \sin^{-1} \frac{t}{\sqrt{1+t^2}} = \sin^{-1}\left(\frac{tan\theta}{\sqrt{1+tan^2\theta}}\right) = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$

$= \sin^{-1}\left(\frac{\sin\theta}{\cos\theta\sec\theta}\right) = \sin^{-1}(\sin\theta) = \theta = \tan^{-1} t$

$\therefore \frac{dy}{dt} = \frac{1}{1+t^2}$

Hence, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/1+t^2}{1/1+t^2} = 1$

Exercise

Short Questions (2 marks)

1) Find the slope of the tangent to the curve y = x^2 at x = - $\frac{1}{2}$ **[2014-S]**

2) Find the derivative of sin x w.r.t cos x. **[2017-S]**

3) Find the derivative of cos x w.r.t $\log_e x$. **[2015-S]**

4) Find $\frac{dy}{dx}$ if x = at^2 , y = 2at. **[2017-w]**

5) Differentiate $\tan^{-1} x$ w.r.t $\cos^{-1} x$.

6) Differentiate y = $x^{\sin^{-1} x}$, w.r.t. x.

7) Differentiate cosec($\cot^{-1} x$) w.r.t sec ($\tan^{-1} x$)

8) Differentiate $sec^2(\tan^{-1} x)$ w.r.t (1-x^2)

9) Differentiate $\tan^{-1} \sqrt{\frac{1}{x} - 1}$ w.r.t. x.

10) Differentiate$\cot^{-1} x$ w.r.t. $\text{cosec}^{-1} x$

11) Find $\frac{dy}{dx}$ of each of the following

i) $(\tan^{-1} 5x)^2$

ii) $(\sin^{-1} x^4)^4$

iii) $\tan^{-1}(cos\sqrt{x})$

iv) $\log_7(\log_7 x)$

v) $\sin(e^{x2})$

Long Questions (5 marks)

12) Differentiate $\tan^{-1}(\frac{\cos x - \sin x}{\cos x + \sin x})$ w.r.t x. **[2017-w]**

13) If y = $\tan^{-1} \sqrt{\frac{1+\sin x}{1-\sin x}}$ then find $\frac{dy}{dx}$.

14) Find $\frac{dy}{dx}$ if x = y ln (xy) (2016-S)

15) Find $\frac{dy}{dx}$ if y = $(\tan x)^{\ln x}$ (2017-w)

16) If x $\sqrt{1+y}$ + y$\sqrt{1+x}$ = 0 Prove that $(1+x^2)\frac{dy}{dx} + 1 = 0$ (for $x \neq y$)

17) If y = log $\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$, then find $\frac{dy}{dx}$.

18) If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \tan^{-1} a$, Prove that $\frac{dy}{dx} = \frac{y}{x}$.

19) Find $\frac{dy}{dx}$ = ? If $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$.

20) If $e^{x+y} - x = 0$ prove that $\frac{dy}{dx} = \frac{1-x}{x}$

21) If y = $(\sqrt{x})^{\sqrt{x}^{\sqrt{x}^{\cdots}}}$ then prove that $\frac{dy}{dx} = \frac{y^2}{2-y\log x}$.

22) Find $\frac{dy}{dx}$ if x = $\frac{2at}{1+t^2}$, y = $\frac{2bt}{1-t^2}$

23) Differentiate sin^2x w.r.t $(\ln x)^2$

24) Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

25) Differentiate $\tan^{-1} x$ w.r.t $\tan^{-1}\sqrt{1+x^2}$

26) If x = $\frac{a(1-t^2)}{1+t^2}$ and y = at $\left(\frac{1-t^2}{1+t^2}\right)$ then find $\frac{dy}{dx}$

27) If sin (xy) + $\frac{x}{y} = x^2 - y$, then find $\frac{dy}{dx}$

28) If $\sqrt{1-x^2} + \sqrt{1-y^2}$ = a(x-y) , prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1-x^2}$

29) Differentiate $\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$ w.r.t. x.

30) If y = log (x+$\sqrt{x^2-1}$) then find $\frac{dy}{dx}$.

31) Find $\frac{dy}{dx}$ of each of the following

(i) $\sin^{-1}(2ax\sqrt{1-a^2x^2})$

(ii) $\left[\left(\frac{1+t^2}{1-t^2}\right)^2 - 1\right]^{\frac{1}{2}}$

(iii) $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

(iv) $x^2\cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + x^2\,\text{cosec}^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$

(v) $\sin^{-1}(3x - 4x^3)$

(vi) $\tan^{-1}\left(\frac{4x}{1-4x^2}\right)$

(vii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(viii) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$

ANSWER

(1) -1 **(2)** -cot x **(3)** $-x\sin x$ **(4)** $\frac{1}{t}$

(5) $\frac{-\sqrt{1-x^2}}{1+x^2}$ **(6)** $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$ (7)1 (8)-1

(9) - $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ **(10)** $\frac{x\sqrt{x^2-1}}{x^2+1}$

11) (i) $\frac{10\tan^{-1}5x}{1+25x^2}$ (ii) $\frac{16x^3(\sin^{-1}x^4)^3}{\sqrt{1-x^8}}$ (iii) $\frac{\sin\sqrt{x}}{2\sqrt{x}(1+cos^2\sqrt{x})}$

(iv) $\frac{1}{x\ \log_7 x(\log_e 7)^2}$ (v) $2x\ e^{x^2}\cos\ (e^{x^2})$

12) -1 **(13)** $\frac{1}{2}$ **(14)** $\frac{x-y}{x(1+\ln(xy))}$ **(15)** $(\tan x)^{\ln x}\left[\frac{1}{x}\ln\tan x + \frac{sec^2x\ lnx}{tanx}\right]$

(17) $\frac{1}{\sqrt{x}(1-\sqrt{x})}$ **(19)** $-\left(\frac{ay}{bx}\right)^{\frac{1}{3}}$ **(22)** $\frac{b(1+t^2)^3}{a(1-t^2)^3}$ **(23)** $\frac{x\sin 2x}{2\ln x}$

(24) $\frac{-4\sqrt{1-x^2}}{1+x^2}$ **(25)** $\frac{2+x^2}{x\sqrt{1+x^2}}$ **(26)** $\frac{t^4+4t^2-1}{4t}$ **(27)** $\frac{2xy^2-y-y^3\cos(xy)}{xy^2.\cos(xy)-x+y^2}$

(29) $\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}\left[2x + \frac{1}{(1+x^2)\tan^{-1}x} - \frac{x}{(1+x^2)}\right]$ **(30)** $\frac{1}{\sqrt{x^2-1}}$

31) (i) $\frac{2a}{\sqrt{1-a^2x^2}}$ (ii) $\frac{2+2t^2}{(1-t^2)^2}$ (iii) $\frac{1}{2}\frac{1}{\sqrt{1-x^2}}$ (iv) πx

(v) $\frac{3}{\sqrt{1-x^2}}$ (vi) $\frac{4}{1+4x^2}$ (vii) $\frac{2}{1+x^2}$ (viii) - $\frac{4}{(e^x-e^{-x})^2}$

Successive Differentiation

If f is a differential function of x , then the derivative of f(x) may be again differentiable w.r.t x. If $f'(x) = \frac{df}{dx}$ then $f'(x)$ is called first derivative of f.

If $f'(x)$ is differentiable , and $\frac{df'(x)}{dx} = f''(x)$, then f ″(x) is called the 2nd order derivative of f (x) w.r.t x.

The above process can be successively continued to obtained derive functions of higher orders.

Notations

1st Order derivatives → $\frac{dy}{dx}$, y′ , y_1 , Dy , *f ′(x)*

2nd Order derivatives → $\frac{d^2y}{dx^2}$, y″ , y_2 , D^2y, *f ″(x)*

3rd Order derivatives → $\frac{d^3y}{dx^3}$, y‴ , y_3 , D^3y , *f ‴(x)*

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nth Order derivatives → $\frac{d^ny}{dx^n}$, $y^{(n)}$, y_n , D^ny , $f^n(x)$

Example – 1

Find 2nd order derivatives of following function.

(i) y = $x^5 + 4x^3 - 2x^2 + 1$ (ii) y = $\log_e x$

(iii) y = $\sqrt{x^2+1}$ (iv) y = $\frac{1}{\sqrt{x}}$

(i) $y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^5 + 4x^3 - 2x^2 + 1) = 5x^4 + 12x^2 - 4x + 0$

$= 5x^4 + 12x^2 - 4x$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5x^4 + 12x^2 - 4x)$

$= 20x^3 + 24x - 4$ (Ans)

(ii) $y_1 = \frac{d}{dx}(\log_e x) = \frac{1}{x}$

$y_2 = \frac{dy_1}{dx} = \frac{d(\frac{1}{x})}{dx} = -\frac{1}{x^2}$ (ans)

(iii) $y = \sqrt{x^2+1}$

$y_1 = \frac{1}{2\sqrt{x^2+1}} \frac{d}{dx}(x^2+1)$ (Chain Rule)

$= \frac{2x+0}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

$y_2 = \frac{dy_1}{dx} = \frac{d\left(\frac{x}{\sqrt{x^2+1}}\right)}{dx}$

$= \frac{1.\sqrt{x^2+1} - x \cdot \frac{1}{2\sqrt{x^2+1}} \frac{d(x^2+1)}{dx}}{(\sqrt{x^2+1})^2}$ (applying division formula of derivative)

$= \frac{\sqrt{(x^2+1)} - \frac{x}{2\sqrt{x^2+1}} 2x}{x^2+1}$

$= \frac{(x^2+1)-x^2}{\sqrt{x^2+1}\ (x^2+1)} = \frac{1}{(x^2+1)^{3/2}}$ (Ans)

(iv) $y_1 = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2} x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-\frac{3}{2}}$

$y_2 = \frac{dy_1}{dx} = -\frac{1}{2}\left(-\frac{3}{2}\right) x^{-\frac{3}{2}-1} = \frac{3}{4} x^{-5/2} = \frac{3}{4x^{5/2}}$

Example – 2

Find y_1 and y_2 if y = log(sin x) (2018-S)

Ans.

$y_1 = \frac{d}{dx}$ log(sin x) $= \frac{1}{\sin x} cosx = cotx$

$y_2 = \frac{dy_1}{dx} = \frac{d}{dx}(cotx) = -cosec^2x$ (Ans)

Example – 3

If x = at^2, y = 2at then find $\frac{d^2y}{dx^2}$

Ans.

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}(2at)}{\frac{d}{dt}(at^2)} = \frac{2a}{2at} = \frac{1}{t}$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

$= \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{t^2}}{2at} = -\frac{1}{2at^3}$

Example – 4

If x = a ($\theta - sin\theta$) , y = a (1+cosθ) then find $\frac{d^2y}{dx^2}$

Ans.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-cos\theta)} = -\frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$ {as $\frac{dy}{dx}$ is function of θ}

$$= \frac{\frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right)}{\frac{dx}{d\theta}} = \frac{cosec^2\frac{\theta}{2}.\frac{1}{2}}{a(1-cos\theta)} = \frac{1}{2a}\frac{cosec^2\frac{\theta}{2}}{2sin^2\frac{\theta}{2}} = \frac{1}{4a}cosec^2\frac{\theta}{2}.\,cosec^2\frac{\theta}{2}$$

$$= \frac{1}{4a}cosec^4\frac{\theta}{2}$$

Example – 4

Find $\frac{d^2y}{dx^2}$ from the equation $x^2 + y^2 = a^2$

Ans.

Given $x^2 + y^2 = a^2$ ---------------------------------- (1)

Differentiate both sides,

$2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$ ---------------------------------- (2)

Again differentiating w.r.t x

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left\{\frac{y.1 - x\frac{dy}{dx}}{y^2}\right\}$ { applying division formula }

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left\{\frac{y - x\left(\frac{-x}{y}\right)}{y.y^2}\right\}$ {Form (2)}

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left\{\frac{y^2+x^2}{y.y^2}\right\} = -\frac{a^2}{y^3}$ {Form (1)}

Example – 5

If x = 3t - t^3 , y = t + 1 , find $\frac{d^2y}{dx^2}$ at t = 2.

Ans.

Given $y = t + 1$, $x = 3t - t^3$.

$\Rightarrow \frac{dy}{dt} = 1$ and $\frac{dx}{dt} = 3 - 3t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{3-3t^2} = \frac{1}{3(1-t^2)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{1}{3}\left(\frac{-1}{(1-t^2)^2}\right)(-2t)}{3-3t^2} = \frac{\frac{2}{3}\frac{t}{(1-t^2)^2}}{3(1-t^2)}$$

$$= \frac{2}{9}\frac{t}{(1-t^2)^3}$$

Now $\left.\frac{d^2y}{dx^2}\right]_{t=2} = \frac{2}{9}\frac{2}{(1-2^2)^3} = \frac{4}{9(-3)^3} = \frac{-4}{243}$

Example – 6

If y = $e^{ax}\sin bx$, then prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$ **[2017-w]**

Ans.

Given y = $e^{ax}\sin bx$ -------------------- (1)

Differentiate both sides,

$y_1 = ae^{ax}\sin bx + e^{ax}\,b\cos bx$

$\Rightarrow y_1 = ay + be^{ax}\cos bx$--------------- (2)

Differentiate w.r.t x,

$\Rightarrow y_2 = ay_1 + ba\,e^{ax}\cos bx + b\,e^{ax}b(-\sin bx)$

$\Rightarrow y_2 = ay_1 + ab\,e^{ax}\cos bx - b^2y$

$\Rightarrow y_2 = ay_1 + a(y_1 - ay) - b^2y$ {from (2)}

$\Rightarrow y_2 = ay_1 + ay_1 - a^2y - b^2y$

$\Rightarrow$ $\boxed{y_2 - 2ay_1 + (a^2 + b^2)y = 0}$ (proved)

Example – 7

If y = $e^{m\cos^{-1}x}$ then show that $(1-x^2)\frac{d^2y}{dx^2}$ - x $\frac{dy}{dx}$ - $m^2y = 0$ (2018-S)

Ans. y = $e^{m\cos^{-1}x}$

$\Rightarrow \frac{dy}{dx} = e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) = -\frac{my}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}$ = -my---------------(1)

Differentiate w.r.t x

$\Rightarrow \frac{1(-2x)}{2\sqrt{1-x^2}}\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} = -m\frac{dy}{dx}$

$\Rightarrow \frac{-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}}{\sqrt{1-x^2}} = -m\frac{dy}{dx}$

$\Rightarrow -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = -m\frac{dy}{dx}\sqrt{1-x^2}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ = m (-my) = m^2y {from (1)}

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

Example – 8

If y = ax sin x, then $x^2y_2 - 2xy_1 + (x^2+2)y = 0$ **(2016-S)**

Ans. y = ax sin x---------------------(1)

Differentiate w.r.t x,

$\Rightarrow y_1$ = a [1. sin x +x. cos x]

$\Rightarrow y_1$ = a (sin x +x cos x)-------------------- (2)

Differentiate w.r.t x,

$\Rightarrow y_2$ = a (cos x + 1. Cos x – x . sin x)

$\Rightarrow y_2$ = 2a cos x – ax sin x --------------------(3)

Now L.H.S = x^2y_2 - 2x y_1 + (x^2+2)y { applying equation (1),(2) and(3) }

= 2 a x^2 cos x - ax^3sin x – 2ax sin x – 2 a x^2 cos x + ax^3sin x + 2ax sin x

= 0 = R.H.S (Proved)

Exercise

Question with short answers (2marks)

1) Find y_2 for following

(i) $y = x^2 + \sqrt{x}$ (ii) $y = e^x \sin x$

Question with long answers (5 marks)

2) If x = 2 cos t – cos 2t , y = 2 sin t – sin 2t then find y”.

3) Find y_2 y = tan x + sec x

4) If $y = \sin^{-1} x$, then show that $(1-x^2)\, y_2 - x\, y_1 = 0$ **[2017-w]**

5) If y = A cos nx + B sin nx then show that $\frac{d^2y}{dx^2} + n^2y = 0$

6) If $y = \log(x + \sqrt{1+x^2})$, Prove that $(1 + x^2)\, y_2 + x\, y_1 = 0$

Question with long answers (10 marks)

7) If $y = \sin(m \sin^{-1} x)$ prove that $(1-x^2)\, y_2 - x\, y_1 + m^2y = 0$

8) If $y = e^{m \sin^{-1} x}$ prove that $(1-x^2)\, y_2 - x\, y_1 = m^2y$ (2017-W, 2017-S)

Ans.

1) (i) $2 - \frac{1}{4x^{3/2}}$ (ii) $2\, e^x \cos x$

2) $\frac{3}{8 \sin\frac{t}{2} cos^2 \frac{3t}{2}}$ 3) $\frac{\cos x}{(1-\sin x)^2}$

Partial Differentiation

The functions studied so far are of a single independent variable. There are functions which depends on two or more variables. Example, the pressure(P) of a given mass of gas is dependent on its volume(v) and temperature (T).

Functions of two variable

A function f : X x Y to Z is a function of two variables if there exist a unique element z = f(x,y) in Z corresponding to every pair (x,y) in X x Y.

Domain of f is $X \times Y$.

$f(X \times Y)$ is the range of f. $\{ f(X \times Y)$ C Z $\}$

Notation : - z = f(x,y) means z is a function of two variables x and y.

Limit of a function of two variable

A function f(x,y) tends to limit l as (x,y) → (a,b), .If given $\epsilon > 0$, there exist δ> 0 such that |*f(x,y)-l* | <ϵ whenever o < |(x,y) – (a,b)| <δ .

Continuity

A function f(x,y) is said to be continuous at a point (a,b) if

(i) f(a,b) is defined

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Finding limits and testing continuity of functions of two variable is beyond our syllabus so we have to skip these topics here.

Partial derivatives

Let z = f(x,y) be function of two variables.

If variable x undergoes a chance δx, while y remains constant, then z undergoes a changes written as δz

Now, δz = f(x+ δx,y) – f(x,y)

If $\frac{\delta z}{\delta x}$ exist as δx→ 0, then we write the partial derivative of z w.r.t x as

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x = \lim_{\delta x\to 0} \frac{f(x+\delta x,y)-f(x,y)}{\delta x}$$

Similarly partial derivative of z w.r.t y ,

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = z_y = \lim_{\delta y \to 0} \frac{f(x, y+\partial y) - f(x,y)}{\delta y}$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ symbols are used to notify the partial differentiation.

Note

As from above theory it is clear when partial differentiation w.r.t x is taken , then y is treated as constant and vice – versa. (All the formulae and techniques used in derivative chapter remain same here)

2nd Order Partial Differentiation

If we differentiate the $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ w.r.t x or y , then we set higher order partial derivatives as follows.

1st Order Partial Derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

2nd Order Partial Derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx} = f_{xx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{yx} = f_{yx}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy} = f_{xy}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy} = f_{yy}$$

Note: $f_{yx} = f_{xy}$ when partial derivatives are continuous .

Example -1

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

(i) z = $2x^2y + xy^2 + 5xy$.

(ii) z = $\tan^{-1}\left(\frac{x}{y}\right)$ **[2018-S]**

(iii) z = $e^y \tan x$ **[2019-W]**

(iv) z = $\log(x^2 + y^2)$ **[2015-S]**

(v) z = $\sin^{-1}\left(\frac{x}{y}\right)$ **[2014-S]**

(vi) z = f($\frac{y}{x}$) **[2017-S]**

(vii) $x^y + y^x$

Ans.

(i) z = $2x^2y + xy^2 + 5xy$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(5xy)$ (Here y is treated as constant)

$= 2y\frac{\partial}{\partial x}(x^2) + y^2\frac{\partial}{\partial x}(x) + 5y\frac{\partial}{\partial x}(x)$

$= 2y.2x + y^2.1 + 5y.1$

$= 4xy + y^2 + 5y.$

$\frac{\partial z}{\partial y} = 2x^2\frac{\partial y}{\partial y} + x\frac{\partial y^2}{\partial y} + 5x\frac{\partial y}{\partial y}$

$= 2x^2 + x.2y + 5x = 2x^2 + 2xy + 5x$

(ii) z = $\tan^{-1}(\frac{x}{y})$

$\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{x}{y})^2}\frac{\partial}{\partial x}(\frac{x}{y}) = \frac{1}{\frac{y^2+x^2}{y^2}}.\frac{1}{y}$

$= \frac{y^2}{y(x^2+y^2)} = \frac{y}{x^2+y^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{x}{y})^2}\frac{\partial}{\partial y}(\frac{x}{y}) = \frac{1}{\frac{y^2+x^2}{y^2}}(\frac{-x}{y^2})$

$= \frac{-x}{x^2+y^2}$

(iii) z = $e^y \tan x$

$\frac{\partial z}{\partial x} = e^y\frac{\partial}{\partial x}(\tan x) = \frac{e^y}{1+x^2}$

$\frac{\partial z}{\partial y} = \frac{\partial(e^y)}{\partial x}\tan x = e^y \tan x$

(iv) z = log $(x^2 + y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial x}(x^2 + y^2) = \frac{2x}{x^2+y^2}$ {$\frac{\partial}{\partial x}y^2$= 0 As y is constant}

$\frac{\partial z}{\partial y} = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial y}(x^2 + y^2) = \frac{2y}{x^2+y^2}$

(v) z = $\sin^{-1}\left(\frac{x}{y}\right)$

$$\frac{\delta z}{\delta x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}.\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}.\left(\frac{-x}{y^2}\right)$$

$$= -\frac{x}{y\sqrt{y^2-x^2}}$$

(vi) z = $f\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2} f'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = f'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$= \frac{1}{x} f'\left(\frac{y}{x}\right)$$

vii) z = $x^y + y^x$

$\frac{\partial z}{\partial x}$ = y $x^{y-1} + y^x \ln y$ (y is a constant here)

$\frac{\partial z}{\partial y}$ = x^yln x + x y^{x-1} (As x is treated as constant)

Example-2 . Find f_{xx} and f_{yx} where f(x,y) = $x^3 + y^3 + 3xy$

Ans: - $f_x = 3x^2 + 3y$, $f_y = 3y^2 + 3x$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = 6x + 0 = 6x$

$f_{yx} = \frac{\partial}{\partial y}(f_x) = 0 + 3 = 3$

Example – 3

If z = log $(x^2 + y^2)$ + $\tan^{-1}\left(\frac{y}{x}\right)$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Ans. $\frac{\partial z}{\partial x} = \frac{1}{(x^2+y^2)}2x + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$

$$= \frac{2x}{x^2+y^2} + \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{(x^2+y^2)} 2y + \frac{1}{1+\frac{y^2}{x^2}} \left(\frac{1}{x}\right)$$

$$= \frac{2y}{(x^2+y^2)} + \frac{x^2}{(x^2+y^2)\ x}$$

$$= \frac{2y+x}{(x^2+y^2)}$$

Now $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{(x^2+y^2)(2-0)-(2x-y)(2x+0)}{(x^2+y^2)^2}$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{(x^2+y^2)(2+0)-(2y+x)(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

Now $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2-2x^2+2xy+2x^2-2y^2-2xy}{(x^2+y^2)^2}$

$= \frac{0}{(x^2+y^2)^2} = 0$ (Proved)

Homogenous function and Euler's theorem

Homogenous function

A function f (x,y) is said to be homogenous in x and y of degree n iff (tx, ty) = $t^n f(x,y)$ where t is any constant.

Example – 4

Test whether the following functions are homogenous or not. If homogenous then find their degree.

(i) $2xy^2 + 3x^2y$

(ii) $\sin^{-1}\left(\frac{x}{y}\right)$

(iii) $\frac{3x^2+2y^2}{x+y}$

(iv) $x^2 + 2xy + 4x$

Ans.

(i) Let f(x, y) = $2xy^2 + 3x^2y$

f(tx, ty) = $2(tx)(ty)^2 + 3\,(tx)^2(ty)$

$= 2txt^2y^2 + 3\,t^2x^2ty$

$= t^3(2xy^2 + 3x^2y) = t^3 f(x,y)$

Hence f(x,y) is a homogenous function of degree 3.

(ii) Let $f(x, y) = \sin^{-1}\left(\frac{x}{y}\right)$

$f(tx, ty) = \sin^{-1}\left(\frac{tx}{ty}\right) = \sin^{-1}\frac{x}{y} = t^0 \sin^{-1}\left(\frac{x}{y}\right) = t^0 f(x, y)$

Hence f(x, y) is a homogenous function of degree '0'.

(iii) $f(x, y) = \frac{3x^2+2y^2}{x+y}$

$f(tx, ty) = \frac{3(tx)^2 + 2(ty)^2}{tx+ty} = \frac{t^2}{t}\frac{(3x^2+2y^2)}{x+y} = t\, f(x, y)$

Hence f (x, y) is a homogenous function of degree 1.

(iv) $f(x, y) = x^2 + 2xy + 4x$

$f(tx, ty) = (tx)^2 + 2 (tx) (ty) + 4(tx)$

$= t\,(tx^2 + 2t\,xy + 4x)$

So here f (tx, ty) cannot be expressed as $t^n f(x, y)$

Hence f(x,y) is not a homogenous function.

Note

(i) If each term in the expression of a function is of the same degree then the function is homogenous.

(ii) If z is a homogenous function of x and y of degree n , then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also homogenous of degree n-1.

(iii) If z = f(x, y) is a homogenous function of degree n , then we can write it as

$z = x^n \Phi\left(\frac{y}{x}\right)$

e.g. In example - 4(i) $2xy^2 + 3x^2y$ is homogenous function of degree 3.

Now $f(x, y) = 2xy^2 + 3x^2y = x^3\left(2\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)\right) = x^3\,\Phi\left(\frac{y}{x}\right)$

Similarly in Example – 4 (iii), f(x,y) is of degree 1.

Now $f(x, y) = \frac{3x^2+2y^2}{x+y} = \frac{x^2}{x}\left(\frac{3+2\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}\right) = x\left(\frac{3+2\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}\right) = x\,\Phi\left(\frac{y}{x}\right)$

Euler's theorem

If z is a homogenous function of degree n , then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\,z$. **[2014-S]**

Proof: -

Since z is a homogenous function of degree n , so z can be written as

$$z = x^n \Phi\left(\frac{y}{x}\right)$$

Now $\frac{\partial z}{\partial x} = n\, x^{n-1}\, \Phi(\frac{y}{x}) + x^n\, \Phi'(\frac{y}{x}) . \frac{\partial}{\partial x}(\frac{y}{x})$

$= n\, x^{n-1}\, \Phi(\frac{y}{x}) + x^n\, \Phi'(\frac{y}{x})\, (-\frac{y}{x^2})$

$= n\, x^{n-1}\, \Phi(\frac{y}{x}) - x^{n-2}\, y\, \Phi'(\frac{y}{x})$--------------(1)

Similarly, $\frac{\partial z}{\partial y} = x^n\, \Phi'(\frac{y}{x}) \frac{\partial}{\partial y}(\frac{y}{x})$

$= x^n \Phi'(\frac{y}{x})\, (\frac{1}{x}) = x^{n-1}\, \Phi(\frac{y}{x})$ ------------ (2)

Now $x \times$ Equation (1) + $y \times$ Equation (2)

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\{n\, x^{n-1}\, \Phi(\frac{y}{x}) - x^{n-2}\, y\, \Phi'(\frac{y}{x})\} + y\, x^{n-1}\, \Phi'(\frac{y}{x})$

$= n\, x^n\, \Phi(\frac{y}{x}) - x^{n-1}\, y\, \Phi'(\frac{y}{x}) + x^{n-1}\, y\, \Phi'(\frac{y}{x})$

$= n\, x^n\, \Phi(\frac{y}{x})$ = nz (proved)

Example – 5

Verify Euler's theorem for $z = \frac{y}{x}$ **[2014-S]**

Ans. $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\frac{y}{x}) = -\frac{y}{x^2}$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\frac{y}{x}) = \frac{1}{x}$

Here $z = f(x, y) = \frac{y}{x}$

$F(tx, ty) = \frac{ty}{tx} = \frac{y}{x} = f(x, y)$

Hence f(x, y) is a homogenous function of degree 0.

Statement of Euler's theorem is $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n z$ (here n=0)

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 . Z = 0$

Now we have to verify it.

From above

L.H.S $= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x(-\frac{y}{x^2}) + y . \frac{1}{x} = -\frac{y}{x} + \frac{y}{x} = 0$ = R.H.S

Hence Euler's theorem is verified.

Example – 6

Verify Euler's theorem for z = $x^2y^2 + 4xy^3 - 3x^3y$

Ans. Here z = f(x, y) = $x^2y^2 + 4xy^3 - 3x^3y$

F (tx, ty) = $t^2x^2t^2y^2 + 4txt^3y^3 - 3t^3x^3y$

= $t^4(x^2y^2 + 4xy^3 - 3x^3y) = t^4 f(x, y)$

Hence z is homogenous function of degree 4.

Here n = 4.So, the statement of Euler's theorem is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4z$$

Now we have to verify it

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y^2 + 4xy^3 - 3x^3y)$

= $2xy^2 + 4y^3 - 3\,(3x^2)y$

= $2xy^2 + 4y^3 - 9x^2y$--------------------(1)

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y^2 + 4xy^3 - 3x^3y)$

= $2x^2y + 12xy^2 - 3x^3$---------------------------(2)

L.H.S = $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

= $x(2xy^2 + 4y^3 - 9x^2y) + y(2x^2y + 12xy^2 - 3x^3)$ {from (1) and (2)}

= $2x^2y^2 + 4xy^3 - 9x^3y + 2x^2y^2 + 12xy^3 - 3x^3y$

= $4x^2y^2 + 16xy^3 - 12x^3y$

= $4(x^2y^2 + 4xy^3 - 3x^3y) = 4z$ (verified)

Example – 7

If z = $\sin^{-1}(\frac{xy}{x+y})$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$ **[2017-S, 2018-S, 2019-W]**

Ans.

Let z = $\sin^{-1}(\frac{xy}{x+y}) = \sin^{-1}u$

Now u = $\frac{xy}{x+y}$

$u = (tx\,,\,ty) = \frac{txty}{tx+ty} = \frac{t^2}{t}(\frac{xy}{x+y}) = tu$

Hence u is homogenous function of degree 1.

So by Euler's theorem

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1\,.\,u = u$ -----------(1)

As $z = \sin^{-1} u$

$\Rightarrow u = \sin z$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin z) = \cos z\,\frac{\partial z}{\partial x}$ ------------------- (2)

And $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sin z) = \cos z\,\frac{\partial z}{\partial y}$ ---------------- (3)

From (1), (2) and (3)

$\Rightarrow x\cos z\,\frac{\partial z}{\partial x} + y\cos z\,\frac{\partial z}{\partial y} = \sin z$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\sin z}{\cos z} = \tan z$ (Proved)

Example – 8

If $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Ans.

If $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$

$u\,(tx, ty) = \sin^{-1}(\frac{tx}{ty}) + \tan^{-1}(\frac{ty}{tx})$

$= \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$

$= u\,(x\,,\,y)$

Hence u is a homogenous function of degree '0'

So by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example – 9

If z = $\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = sin 2z **[2017-w]**

Ans. Let z = $\tan^{-1}$, where u = $\left(\frac{x^3+y^3}{x+y}\right)$

Now u (tx, ty) = $\frac{t^3x^3+t^3y^3}{tx+ty} = t^2\left(\frac{x^3+y^3}{x+y}\right) = t^2u$

Hence u is a homogenous function of degree 2.

So by Euler's theorem

$x\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 2u ---------- (1)

Now z = $\tan^{-1} u$

⇒ u = tan z ------------------------------ (2)

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\tan z) = sec^2z\frac{\partial z}{\partial x}$--------------------- (3)

And $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\tan z) = sec^2z\frac{\partial z}{\partial y}$-------------- (4)

From (1), (2), (3) and (4)

⇒$x\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 2u

⇒$xsec^2z\frac{\partial z}{\partial x}$+ y $sec^2z\frac{\partial z}{\partial y}$= 2 tan z

⇒$x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 2 $\frac{\tan z}{sec^2z}$ = 2 tan z cos^2z

= 2 $\frac{\sin z}{\cos z}cos^2z$ = 2 sin z cos z

= sin 2 z (proved)

Example – 10

If z is a homogenous function of x and y of degree n and $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are continuous , then show that

$x^2\frac{\partial^2 z}{\partial x^2}$ + 2xy $\frac{\partial^2 z}{\partial x\partial y}$ + $y^2\frac{\partial^2 z}{\partial y^2}$ = n(n-1)z

Proof

Given z is a homogenous function of degree n

So by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \text{------------} (1)$$

Differentiating (1) w.r.t x ,

$$1.\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = (n-1)\frac{\partial z}{\partial x} \text{------------------} (2)$$

Differentiating (1) w.r.t y ,

$$x\frac{\partial^2 z}{\partial y \partial x} + 1.\frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \text{------------------} (3)$$

{As $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are continuous}

$\{\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}\}$

Equ^n(2) X x + Equ^n(3) X y

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y} + xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x(n-1)\frac{\partial z}{\partial x} + y(n-1)\frac{\partial z}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right\}$$

$$= (n-1)\, nz$$

$$= n(n-1)z \qquad \text{(proved)}$$

Exercise

Question with short answers (2 marks)

1) if z = sin $\frac{x}{y}$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

2) If f(x,y) = $\sqrt{x^2 + y^2}$, find f_x , f_y .

3) If f(x,y) = log ($x^2 + y^2$- 2xy) find f_{xx} , f_{yx} , f_{xy}

4) If z = f(x,y) , then find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

5) Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if z = $xe^y + ye^x$

Questions with long answers (5 marks)

6) Given f(u,v) = $\frac{2u-3v}{u^2+v^2}$, find $f_u(2,1)$ and $f_v(2,1)$

7) If z = $\frac{x-y}{x+y}$, then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 0

8) If z = x^2y + 3xy^2 - $\frac{x}{y}$. Find partial derivatives of 2nd order.

9) Verify Euler's theorem for u = $x^2 \log(\frac{y}{x})$

10) If z = xy f $(\frac{y}{x})$, then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 2z

11) If u = $\sin^{-1}(\frac{x+y}{\sqrt{x}+\sqrt{y}})$ show that x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = $\frac{1}{2}$ tan u

12) If z = $\ln(\frac{x^2+y^2}{x+y})$ then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 1

13) If z = $\cos^{-1}(\frac{x^2+y^2}{x+y})$, then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = - cot z

Answers

1) $\frac{1}{y}$ cos $(\frac{x}{y})$, $\frac{-x}{y^2}$ cos $(\frac{x}{y})$ 2) $\frac{x}{\sqrt{x^2+y^2}}$, $\frac{y}{\sqrt{x^2+y^2}}$ 3) $\frac{-2}{(x-y)^2}$, $\frac{2}{(x-y)^2}$, $\frac{2}{(x-y)^2}$

4) y f′(xy) , xf′(xy) 5) e^y + y e^x , xe^y + e^x 6) $\frac{6}{25}$, $\frac{-17}{25}$

8) Z_{xx} =2y , Z_{yx} = 2x + 6y+ $\frac{1}{y^2}$, Z_{xy} = 2x + 6y+ $\frac{1}{y^2}$, Z_{yy} = 6x - $\frac{2x}{y^3}$.

INTEGRATION

Introduction

Calculus deals with some important geometrical problem related to draw a tangent of a curve and determine area of a region under a curve. In order to solve these problems we use differentiation and integration respectively.

In the previous lesson, we have studied derivative of a function. After studying differentiation it is natural to study the inverse process called integration.

Objectives

After completion of this topic you will able to

1. Explain integration as inverse process of differentiation.

2. State types of integration.

3. State integral of some standard functions like x^n , sinx, cosx,.... $sin^{-1}x$, ...a^x etc.

4. State properties of integration.

5. Find integration of algebraic, trigonometric, inverse trigonometric functions using standard integration formulae.

6. Evaluate different integrals by applying substitution method and integration by parts method.

Expected Background Knowledge

1. Trigonometry

2. Derivative

Integration (Primitive or Anti derivative)

Integration is the reverse process of differentiation.

If $\frac{df(x)}{dx}$= g(x), then the integration of g(x) w.r.t x is $\int g(x)dx = f(x) + c$

- The Symbol $\int$. is used to denote the operation of integration called as Integral sign.
- The function (here g(x)) is called the integrand.
- 'dx' denote that the Integration is to be performed w.r.t x (x is the variable of Integration).
- 'c' is the constant of Integration (which gives family of curves)
- Integrate means to find the Integral of the function and the process is known as Integration

Types of Integration

Integration are of two types:- i) Indefinite ii) definite

The integration written in the form $\int g(x)dx$ is called indefinite integral.

The integration written in the form $\int_a^b g(x)dx$ is called definite integral.

In this chapter we only discuss the indefinite integrals. The definite integrals will be discussed in the next chapter.

Algebra of Integrals

i. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int f(x)dx \pm \int g(x)dx$

ii. $\int \lambda f(x)dx = \lambda \int f(x)dx$ for any constant λ.

iii. $\frac{d}{dx}(\lambda \int f(x)dx) = \lambda \frac{d}{dx}(\int f(x)dx) = \lambda f(x)$

Simple Integration Formula of some standard functions

i) $\int k$ dx = kx + c

ii) $\int x^n$ dx = $\frac{x^{n+1}}{n+1}$ + c, n≠-1

iii) $\int \frac{1}{x}$dx =ln|x| + c

iv) $\int a^x dx = \frac{a^x}{lna} + c$

v) $\int e^x dx = e^x + c$

vi) $\int \sin x dx = -\cos x + c$

vii) $\int \cos x dx = \sin x + c$

viii) $\int sec^2 xdx = tanx + c$

ix) $\int cosec^2 xdx = -cotx + c$

x) $\int secxtanxdx = secx + c$

xi) $\int \text{cosec}\, x \cot x dx = -cosecx + c$

xii) $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + c$

xiii) $\int \frac{-1}{\sqrt{1-x^2}} dx = cos^{-1}x + c$

xiv) $\int \frac{1}{1+x^2} dx = tan^{-1}x + c$ xv) $\int \frac{-1}{1+x^2} dx = cot^{-1}x + c$

xvi) $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1} x + c$ xvii) $\int \frac{-1}{x\sqrt{x^2-1}} dx = cosec^{-1}x + c$

Methods of integration

1. Integration by using standard formula.
2. Integration by substitution.
3. Integration by parts.

1. INTEGRATION BY USING FORMULAS:-

Example -1 Evaluate the following

(i) $\int (5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}) dx$

Ans :- $\int (5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}) dx$

$= 5\int x^3dx + 2\int x^5dx - 7\int xdx + \int x^{-1/2}$ dx+5$\int \frac{dx}{x}$ { by algebra of integration}

$= 5 \times \frac{x^{3+1}}{3+1} + 2 \times \frac{x^{5+1}}{5+1} - 7 \times \frac{x^{1+1}}{1+1} + \frac{x^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + 5\ln|x| + c$

$= 5 \times \frac{x^4}{4} + 2 \times \frac{x^6}{6} - 7 \times \frac{x^2}{2} + \frac{x^{1/2}}{\frac{1}{2}} + 5\ln|x| + c$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2x^{1/2} + 5lnx + c$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2\sqrt{x} + 5lnx + c$

(ii) $\int (\frac{3x^4-5x^3+4x^2-x+2}{x^3}) dx$

Ans :- $\int (\frac{3x^4-5x^3+4x^2-x+2}{x^3}) dx$

$= \int \frac{3x^4}{x^3}dx - \int \frac{5x^3}{x^3}dx + \int \frac{4x^2}{x^3}dx - \int \frac{x}{x^3}dx + \int \frac{2}{x^3}dx$

$=3\int xdx - 5\int dx + 4\int \frac{dx}{x} - \int x^{-2}\, dx + 2\int x^{-3}\, dxdx$

$= 3 \times \frac{x^{1+1}}{1+1} - 5x + 4lnx - \frac{x^{-2+1}}{-2+1} + \frac{2x^{-3+1}}{-3+1} + c$

$= 3 \times \frac{x^2}{2} - 5x + 4lnx + \frac{1}{x} - \frac{1}{x^2} + c$

$= \frac{3x^2}{2} - 5x + 4lnx + \frac{1}{x} - \frac{1}{x^2} + c$

(iii) $\int(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}})dx$

Ans :- $\int(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}})dx$

$= 4\int \cos x dx - 3\int e^x\, dx + 2\int \frac{dx}{\sqrt{1-x^2}}$

$= 4\sin x - 3e^x + 2sin^{-1}x + c$

(iv) $\int 6x^3\,(x+5)^2 dx$

Ans :- $\int 6x^3\,(x+5)^2 dx$

$= \int 6x^3(x^2 + 10x + 25)dx$

$= \int(6x^5 + 60x^4 + 150x^3)dx$

$= 6\int x^5\, dx + 60\int x^4\, dx + 150\int x^3\, dx$

$= 6 \times \frac{x^6}{6} + 60 \times \frac{x^5}{5} + 150 \times \frac{x^4}{4} + c$

$= x^6 + 12x^5 + \frac{75}{2}x^4 + c$

(v) $\int 5tan^2x dx$

$= \int 5tan^2x dx = \int 5(sec^2x - 1)dx$

$= 5\int sec^2x dx - 5\int 1.dx$

= 5tanx – 5x + c

(vi) $\int sin^2\frac{x}{2}dx$

Ans :- $\int sin^2\frac{x}{2}dx$

$= \int(\frac{1-cosx}{2})dx = \frac{1}{2}[\int dx - \int cosx dx]$ { 1- cosx = 2 $sin^2\frac{x}{2}$ }

$= \frac{1}{2}[x - sinx] + c$

(vii) $\int \frac{sinx}{1+sinx}dx$

Ans :- $\int \frac{sinx}{1+sinx}dx$

$= \int \frac{(1-sinx)sinx}{(1+sinx)(1-sinx)}dx = \int \frac{(1-sinx)sinx}{1-sin^2x}dx$

$= \int(\frac{\sin x - sin^2x}{cos^2x})dx = \int \frac{sinx}{cos^2x}dx - \int tan^2x dx$

$= \int \frac{sinx}{cosx} . \frac{1}{cosx} dx - \int (sec^2x - 1)dx$

$= \int tanx.secxdx - \int sec^2xdx + \int dx$

$= secx - tanx + x + c$

(viii) $\int \frac{1}{sin^2x.cos^2x} dx$

Ans :- $\int \frac{1}{sin^2x.cos^2x} dx$

$= \int \frac{sin^2x+cos^2x}{sin^2x\,.cos^2x} dx$

$= \int \frac{sin^2x}{sin^2xcos^2x} dx + \int \frac{cos^2x}{sin^2xcos^2x} dx$

$= \int sec^2\, xdx + \int cosec^2xdx$

$= \tan x - cotx + c$

(ix) $\int tan^{-1} \{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\}$dx

Ans :- $\int tan^{-1} \{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\}$dx $(\therefore 1 - cos2x = 2sin^2\, x\; and\;\; 1 + cox2x = 2cos\,^2x)$

$= \int tan^{-1}\{\sqrt{\frac{2sin^2x}{2cos^2x}}\} dx \quad = \int tan^{-1} (\sqrt{tan^2x})dx$

$= \int tan^{-1} (tanx)dx \quad (\therefore tan^{-1}(tanx) = x)$

$= \int xdx = \frac{x^2}{2} + c$

(x) $\int \frac{secx}{secx+tanx} dx$

Ans:- $\int \frac{secx}{secx+tanx} dx$

$= \int \frac{secx(secx-tanx)}{(secx+tanx)(secx-tanx)} dx = \int \frac{sec^2x-secxtanx}{sec^2x- tan^2x} dx \quad \{ sec^2x - tan^2x=1\}$

$= \int sec^2x\, dx - \int secx\, tanx\, dx$ = tanx – secx + c

(xi) $\int a^xe^xdx$

Ans:- $\int a^xe^xdx = \int (ae)^xdx$ { we know $\int a^xdx = \frac{a^x}{lna}$ here ae is in place of a}

$= \frac{(ae)^x}{\ln(ae)}$ + c = $\frac{a^xe^x}{\ln(ae)}$ + c (Ans)

(xii) $\int \sqrt{1+cos2x}\,dx$ **(2017-S , 2018-S)**

Ans: - $\int \sqrt{1+cos2x}\,dx$ **=** $\int \sqrt{2cos^2x}\,dx$

$= \int \sqrt{2}cosx$dx

$= \sqrt{2}$ sinx + c

2. INTEGRATION BY SUBSTITUTION:-

When the integral $\int f(x)dx$ cannot be determined by the standard formulae then we may reduce it to another form by changing the independent variable 'x' by another variable t (as x=ɸ(t)) which can be integrated easily. This is called substitution method.

$\int f(x)dx = \int f(x)\frac{dx}{dt}dt = \int f[\phi(t)]\phi'(t)dt,$ where x=ɸ(t).

The substitution x=ɸ(t) depends upon the nature of the given integral and has to be properly chosen so that integration is easier after substitution. The following types of substitution are very often used in Integrations.

TYPE – I

$\int f(ax+b)dx$

$Put\ ax+b = t$

$adx = dt$

$=> dx=\frac{1}{a}dt$

$\therefore \int f(ax+b)dx = \int f(t)\frac{1}{a}dt = \frac{1}{a}\int f(t)dt$

TYPE – II

$\int x^{n-1} f(x^n)dx$

Put $x^n = t$

$nx^{n-1}dx = dt$

$=>x^{n-1}dx = \frac{dt}{n}$

$\therefore \quad \int x^{n-1}f(x^n)dx = \int f(t)\frac{dt}{n}= \frac{1}{n}\int f(t)dt$

TYPE – III

$\int\{f(x)\}^n.\, f'(x)dx$

Put f(x)=t

Differentiate both sites w.r.t *x,*

$f'(x) = \frac{dt}{dx}$

$=> \int\{f(x)\}^n.\, f'(x)dx \;=\; \int t^n\, dt = \frac{t^{n+1}}{n+1} + \; c$

$= \frac{[f(x)]^{n+1}}{n+1} + c \quad (\because t = f(x))$

TYPE-IV

$\int \frac{f^1(x)}{f(x)} dx$

Put f(x)=t

$=> f^1(x)dx = dt$

$\therefore \int \frac{f^1(x)}{f(x)} dx = \int \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c \;(\because f(x) = t)$

SOME USE FULL RESULTS

1. $\int \frac{dx}{ax+b}$

Ans :- Put ax+b = t

Differentiate both sites w.r.t x,

a $= \frac{dt}{dx}$

=> dx $= \frac{dt}{a}$

$\therefore \int \frac{dx}{ax+b} = \int \frac{dt/a}{t} = \frac{1}{a}\int \frac{dt}{t} = \frac{1}{a}\ln|t| = \frac{1}{a}\ln|ax+b| + c.$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c$$

2. $\int cotxdx$

Ans:- $\int cotxdx$

$= \int \frac{cosx}{sinx} dx$

Put $sinx = t$

Differentiate both sites w.r.t x,

$$cosx = \frac{dt}{dx}$$

$=> dt = cosxdx$

$\therefore \int \frac{cosxdx}{sinx} = \int \frac{dt}{t} = \ln|t| = \ln|sinx| + c$

$$\boxed{\int cotxdx = \ln|sinx| + c}$$

3. $\int tanxdx$

Ans :- $\int tanxdx$

$= \int \frac{secxtanx}{secx} dx$ (multiply & divide by sec x)

Put secx = t

Differentiate both sites w.r.t x.

sec x tan x = $\frac{dt}{dx}$

$=> secx\, tanx\, dx = dt$

$\int \frac{\sec x \tan x}{\sec x} dx = \int \frac{dt}{t} = \ln|t| = \ln|secx| + c$

$$\boxed{\int tanxdx = \ln|secx| + c}$$

4. $\int cosecxdx$

Ans :- $\int cosecxdx$

$= \int \frac{1}{sinx} dx$

$= \int \frac{1}{2sin\frac{x}{2}\cos\frac{x}{2}}$ dx

Divide numerator & denominator by $cos^2 {x}/{2}$

$$\int \frac{\frac{1}{cos^2 x/2}}{\frac{2sin\frac{x}{2}cos\frac{x}{2}}{cos^2 x/2}} dx \quad = \quad \int \frac{sec^2 x/2}{2tan\, x/2}\, dx$$

Let $\tan\frac{x}{2} = t$

$=> sec^2 {x}/{2} \times \frac{1}{2} dx = dt$

$=> sec^2 {x}/{2}\, dx = 2dt$

$= \int \frac{sec^2 x/2}{2tan\frac{x}{2}} dx$

$= \int \frac{2dt}{2t} = \int \frac{dt}{t} = \ln|t| + c$

$= ln\, |tan\frac{x}{2}| + c$

$$\boxed{\int cosecxdx = \ln|tan\frac{x}{2}| + c}$$

Now $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2sin^2\frac{x}{2}}{sinx} = \frac{1-cosx}{sinx}$ = cosecx – cotx

Hence $\int cosecxdx$= ln I cosecx-cotx I + c

5. $\int secxdx$

Ans:-$\int secxdx$

$= \int cosec(\frac{\pi}{2} + x)dx \quad (\because\ cosec\,({\pi}/{2} + x) = secx)$

$= \ln|\tan(\frac{\pi}{2} + \frac{x}{2})| + c \quad (\because\ \int cosecxdx = \ln|tan\,\frac{x}{2}| + c)$

$$\boxed{\int secxdx = \ln|[\tan(\frac{\pi}{4} + \frac{x}{2})]| + c}$$

As $\tan(\frac{\pi}{4} + \frac{x}{2})$ = secx+ tanx (we can easily verify it by applying trigonometric formulae.

Hence $\int secxdx$ **= ln I secx+tanx I + c**

BY APPLYING ABOVE FORMULA WE OBTAIN FOLLOWING

1. $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$$

Proof : - $\int \cos(ax+b)dx$

Put ax+b=θ

Differentiate both sites w.r.t x.

a = $\frac{d\theta}{dx}$

$=> adx = d\theta => dx = \frac{d\theta}{a}$

$\therefore \int \cos(ax+b)\,dx = \int \cos\theta \times \frac{d\theta}{a}$

$= \frac{1}{a}\int cos\theta d\theta = \frac{1}{a}\ sin\theta + c$

$= \frac{1}{a}\sin(ax+b)+c$

Similarly we can get the following results.

2. $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$$

3. $$\int sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$$

4. $$\int cosec^2(ax+b)dx = -\frac{1}{a}cot(ax+b)$$

5. $$\int \sec(\mathrm{ax}+\mathrm{b})\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c$$

6. $$\int cosec(ax+b)\cot(ax+b)dx = \frac{-1}{a}cosec\,(ax+b)+c$$

7. $$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$$

8. $\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a} sin^{-1}(ax+b) + c = -\frac{1}{a} cos^{-1}(ax+b) + c$

9. $$\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a} tan^{-1}(ax+b) + c = -\frac{1}{a} cot^{-1}(ax+b) + c$$

10. $$\int (ax+b)^n dx, n \neq -1 = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$$

11. $\int \frac{dx}{(ax+b)\sqrt{(ax+b)^2-1}} = \frac{1}{a} sec^{-1}(ax+b) + c$ = - $\frac{1}{a} cosec^{-1}(ax+b)$ + c

12. $\int a^{mx+b} dx = \frac{1}{m}\frac{a^{mx+b}}{lna}$+ c

The above results of substitution may be used directly to solve different integration problem.

Example – 2 integrate the following

(i) $\int xsinx^2 dx$

Ans :-$\int xsinx^2 dx$ { Let$x^2 = t$ then $2x = \frac{dt}{dx}$ =>$2xdx = dt$}

=$\int sint \frac{dt}{2}$=$\frac{1}{2}\int sintdt$ =$\frac{-1}{2} cost + c$ = $\frac{-1}{2} cosx^2 + c$ (ans)

(ii) $\int (x-2)\sqrt{(x^2-4x+7)dx}$

Ans :- $\int (x-2)\sqrt{(x^2-4x+7)dx}$

Let $x^2 - 4x + 7 = t^2$

Differentiate both sides w.r.t x

$2x - 4 = 2t\frac{dt}{dx}$

=> $(2x-4)dx = 2tdt$

=> $2(x-2)dx = 2tdt$

=> $(x-2)dx = tdt$

Now $\int (x-2)\sqrt{x^2-4x+7}\,dx = \int \sqrt{t^2}\,tdt$

$= \int t \times t\,dt = \int t^2 dt = \frac{t^3}{3} + c$

$= \frac{(x^2-4x+7)^{3/2}}{3} + c$ $\quad (\because\ t = \sqrt{x^2-4x+7} = (x^2-4x+7)^{1/2})$.

(iii) $\int (3x+5)^7 dx$

Ans: $\int (3x+5)^7 dx$

Put 3x+5 =t

Differentiate w.r.t x

$3 = \frac{dt}{dx} \Rightarrow 3dx = dt \Rightarrow dx = \frac{1}{3}dt$

$\therefore \int (3x+5)^7 dx = \frac{1}{3}\int t^7\,dt = \frac{1}{3} \times \frac{t^8}{8} + c$

$= \frac{1}{3} \times \frac{(3x+5)^8}{8} + c = \frac{(3x+5)^8}{24} + c$

(iv) $\int \frac{x^4+4x^3}{x^5+5x^4+7}\,dx$

Ans :- $\int \frac{x^4+4x^3}{x^5+5x^4+7}\,dx$

Put $x^5 + 5x^4 + 7 = t$

Differentiate w.r.t x

$5x^4 + 20x^3 = \frac{dt}{dx}$

=> $(5x^4 + 20x^3)dx = dt$

=> $5(x^4 + 4x^3)\,dx = dt$

=> $(x^4 + 4x^3)dx = \frac{dt}{5}$

$\therefore \int \frac{x^4+4x^3}{x^5+5x^4+7}\,dx = \frac{1}{5}\int \frac{dt}{t} = \frac{1}{5}\ln|t| + c$

$= \frac{1}{5}\ln|x^5 + 5x^4 + 7| + c$

(v) $\int sin^7x\ cosx\ dx$

Ans :- $\int sin^7x\ cosx\ dx$

Put $sinx = \theta$

Differentiating both sides w.r.t x

$cosx = \frac{d\theta}{dx}$

=> $cosxdx = d\theta$

$\therefore \int sin^7xcosx$ dx = $\int \theta^7\, d\theta = \frac{\theta^8}{8} + c$

$= \frac{sin^8x}{8} + c$

(vi) $\int 2e^{tan^2x}\ tanx\ sec^2x\ dx$

Ans :- $\int 2e^{tan^2x}\ tanx\ sec^2x\ dx$

Put $tan^2x = \theta$

Differentiating both sides w.r.t x,

$2\ tanx.sec^2x = \frac{d\theta}{dx}$

=> $2\ tanx\ sec^2xdx = d\theta$

$\therefore \int 2e^{tan^2x}\ tanx\ sec^2x\ dx = \int e^{\theta}d\theta = e^{\theta} + c$

$= e^{tan^2x} + c$

(vii) $\int \frac{3(lnx)^2}{x}dx$

Ans :- $\int \frac{3(lnx)^2}{x}dx$

Put $lnx = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$

$\therefore \int \frac{3(lnx)^2}{x}dx = 3\int t^2dt = 3 \times \frac{t^3}{3} + c = (lnx)^3 + c$

viii) Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ (2017-S)

Ans:- $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ (Let t = $e^x + e^{-x}$ => dt = $(e^x - e^{-x})dx$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|e^x + e^{-x}| + c$

ix) Integrate $\int \frac{1}{2-5x} dx$ (2015-S)

Ans:- $\int \frac{1}{2-5x} dx$ (Let 2 - 5x = t => - 5 dx = dt => dx = $-\frac{dt}{5}$)

$= -\frac{1}{5} \int \frac{1}{t} dt = -\frac{1}{5}\ lnt + c = -\frac{1}{5} \ln(2-5x) + c$.

x) Evaluate $\int e^x sin\ e^x dx$ (2019-W)

Ans:- $\int e^x sin\ e^x dx$ (Put $e^x = t => e^x dx = dt$)

$= \int sintdt = -cost + c = -cose^x + c$

INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS

If the integrand is of the form sinmx cosnx ,sinmx sinnx or cosmx cosnx,a trigonometric transformation will help to reduce.it to the sum of sines or cosines of multiple angles which can be easily integrated.

sinmxcosnx $= \frac{1}{2} \times 2sinmx\ cosnx$

$= \frac{1}{2} [\sin(m+n)\,x + \sin(m-n)x]$

sinmx sinnx= $\frac{1}{2} [\cos(m-n)\,x - \cos(m+n)x]$

cosmx cosnx= $\frac{1}{2} [\cos(m-n)\,x + \cos(m+n)x]$

Example – 3

i) Evaluate $\int sin3x\ cos2x\ dx$

Ans :- $\int sin3x\ cos2x\ dx$

$= \frac{1}{2} \int \sin(3x+2x) + \sin(3x-2x)\, dx$

$= \frac{1}{2} \int (sin5x + sinx)dx$

$= \frac{1}{2} \int sin5xdx + \frac{1}{2} \int sinxdx$

$= \frac{1}{2} \times \frac{-cos5x}{5} + \frac{1}{2}(-cosx)$ +c

$= \frac{-1}{10} cos5x - \frac{1}{2} cosx + c$

$= \frac{-1}{10}(cos5x - 5cosx) + c$

ii) Evaluate $\int sin2x\ sinx\ dx$

Ans :- $\int sin2x\ sinx\ dx$

$= \frac{1}{2}\int \cos(2x - x) - \cos(2x + x)dx$

$= \frac{1}{2}\int (cosx - cos3x)dx$

$= \frac{1}{2}\int cosxdx - \frac{1}{2}\int cos3xdx$

$= \frac{1}{2}\times sinx - \frac{1}{2}\times \frac{sin3x}{3} + c$

$= \frac{1}{2}sinx - \frac{1}{6}sin3x + c = \frac{1}{6}(3sinx - sin3x) + c$

iii) Evaluate $\int cos4x\ cos3x\ dx$

Ans :- $\int cos4x\ cos3x\ dx$

$= \frac{1}{2}\int (\cos(4x - 3x) + \cos(4x + 3x)dx\ )$

$= \frac{1}{2}\int (cosx + cos7x)dx$

$= \frac{1}{2}\int cosxdx + \frac{1}{2}\int cos7x$

$= \frac{1}{2}sinx + \frac{1}{2}\times \frac{sin7x}{7} + c$

$= \frac{1}{2}sinx + \frac{1}{14}sin7x + c$

$= \frac{1}{14}(sin7x + 7sinx) + c$

iv) Evaluate $\int sin^2xdx$

Ans :- $\int sin^2xdx$

$= \int (\frac{1-cos2x}{2})\ dx \qquad (\therefore sin^2x = \frac{1-cos2x}{2})$

$= \frac{1}{2}\int (1 - cos2x)dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int cos2xdx$

$= \frac{1}{2}\times x - \frac{1}{2}\times \frac{sin2x}{2} + c$

$= \frac{x}{2} - \frac{sin2x}{4} + c = \frac{1}{4}(2x - sin2x) + c$

v) Evaluate $\int cos^3x\ dx$

= $\int cos^3x\ dx$

= $\int (\frac{cos3x+3cosx}{4})\,dx$

($\therefore$ cos3x = $4cos^3x - 3cosx$ => $4cos^3x = cos3x + 3cosx$=>$cos^3x = \frac{cos3x+3cosx}{4}$)

= $\frac{1}{4}\int (cos3x + 3cosx)dx$

= $\frac{1}{4}\int cos3xdx + \frac{3}{4}\int cosxdx$

= $\frac{1}{4} \times \frac{sin3x}{3} + \frac{3}{4} \times sinx + c$

= $\frac{sin3x}{12} + \frac{3sinx}{4} + c$

= $\frac{1}{12}(sin3x + 9sinx) + c$

vi) Evaluate $\int cos^5xdx$

Ans :- $\int cos^5xdx$

= $\int cos^4x.\,cosxdx$=$\int (cos^2x)^2cosxdx$

= $\int (1 - sin^2x)^2cosxdx$

{ Put $sinx = \theta$ => cosx=$\frac{d\theta}{dx}$ =>$d\theta = cosxdx$}

= $\int (1-\theta^2)^2\ d\theta = \int (1 - 2\theta^2 + \theta^4)d\theta$

= $\int d\,\theta - 2\int \theta^2d\theta + \int \theta^4d\theta$

= $\theta - 2 \times \frac{\theta^3}{3} + \frac{\theta^5}{5} + c$

= $sinx - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c$

vii) Evaluate $\int sin^4x\ cos^3x\ dx$

Ans :- $\int sin^4x\ cos^3x\ dx$

= $\int sin^4x\ cos^2x\ cosx\ dx$

= $\int sin^4x\ (1 - sin^2x)cosxdx$

{ Put sinx =θ => cosx = $\frac{d\theta}{dx}$ =>dθ = cosxdx}

$=\int \theta^4(1-\theta^2)\, d\theta$

$=\int(\theta^4-\theta^6)d\theta = \int \theta^4 d\theta - \int \theta^6 d\theta$

$= \frac{\theta^5}{5} - \frac{\theta^7}{7} + c$

$= \frac{1}{5}sin^5 x - \frac{1}{7}sin^7 x + c$

viii) Evaluate $\int \frac{cos^3 x}{sin^4 x}dx$

Ans :- $\int \frac{cos^3 x}{sin^4 x}dx$

$= \int \frac{cos^2 x}{sin^4 x}.\, cosx\ dx$

$= \int \frac{(1-sin^2 x)}{sin^4 x}cosxdx$

$Put\ sinx = \theta$ => $cosx\, dx = d\theta$

$= \int \frac{1-\theta^2}{\theta^4}\, d\theta = \int(\theta^{-4}-\theta^{-2})\, d\theta$

$= \frac{\theta^{-3}}{-3} - \frac{\theta^{-1}}{-1} + c$

$= -\frac{1}{3\theta^3} + \frac{1}{\theta} + c = \frac{1}{sinx} - \frac{1}{3sin^3 x} + c$

$= cosec\, x - \frac{1}{3}cosec^3 x + c$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

TRIGONOMETRIC INDENTITIES

$1-sin^2\theta = cos^2\theta (or\ 1-cos^2\theta = sin^2\theta)$

$tan^2\theta + 1 = sec^2\theta\ \ (also\ cot^2\theta + 1 = cosec^2\theta)$

$sec^2\theta - 1 = tan^2\theta\ (also\ cosec^2\theta - 1 = cot^2\theta)$

➔ $the\ integrand\ of\ the\ form\ \sqrt{a^2-x^2}, \sqrt{x^2+a^2}, \sqrt{x^2-a^2}$ can be simplified by putting

X=a sin θ
X=a tanθ
X=a secθ
X=a cosθ
X=a cotθ
X=a cosec θ

Note

1. *The integrand of the form $a^2 - x^2$ can be simplify by putting x= a sin θ (or x = a cos θ)*

2. *The integrand of the form x^2+a^2 can be simplify by putting x= a tan θ (or x = a cot θ)*

3. *The integrand of the form $x^2 - a^2$ can be simplify by putting x= a se θ (or x = a cosec θ)*

Example -4

i) Integrate $\int \frac{dx}{\sqrt{a^2-x^2}}$

Ans :- $\int \frac{dx}{\sqrt{a^2-x^2}}$

Let x=a sin θ

Differentiate both sites w.r.t x

dx = a cos θd θ

And x = a sin θ=>$\theta = sin^{-1}\frac{x}{a}$

Hence $\int \frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\frac{a\cos\theta\, d\,\theta}{\sqrt{a^2-a^2sin^2\theta}}} = \sqrt{\frac{a\,cos\theta}{a\,cos\theta}}\, d\theta = \int d\theta = \theta + c = sin^{-1}\frac{x}{a} + c$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$$

ii) Integrate $\int \frac{dx}{x^2+a^2}$

Ans :- $\int \frac{dx}{x^2+a^2}$

Let $x = a\tan\theta$

diffentiating both sides w.r.t x,

$dx = a\,sec^2\theta\; d\;\theta$

And $x = tan\theta$ => $\theta = tan^{-1}\frac{x}{a}$

Hence $\int \frac{dx}{x^2+a^2} = \int \frac{asec^2\theta d\theta}{a^2tan^2+a^2}$

$= \int \frac{a sec^2\theta d\theta}{a^2(tan^2\theta+1)}$

$= \int \frac{sec^2\theta \; d\theta}{a \, sec^2\theta}$

$= \frac{1}{a} \int d\theta \; = \frac{1}{a} \theta + c = \frac{1}{a} tan^{-1} \frac{x}{a}$ + c

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} tan^{-1} \frac{x}{a} + c$$

iii) Integrate $\int \frac{dx}{\sqrt{x^2+a^2}}$

Ans :- $\int \frac{dx}{\sqrt{x^2+a^2}}$

$Let \; x \; = \; a \; tan\theta$

Diffentiating w.r.t x we have,

$dx \; = \; a sec^2\theta d\theta$

Hence $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a sec^2\theta d\theta}{\sqrt{a^2 tan^2\theta + a^2}} = \int \frac{a sec^2\theta d\theta}{\sqrt{a^2(tan^2\theta+1)}} = \int \frac{a sec^2\theta d\theta}{a sec\theta}$

= $\int sec\theta d\theta$

= $\ln|sec\theta + tan\theta| + c$ $(\therefore \int secx dx = \ln|secx + tanx| + c)$

(x = a tanθ => $tan\theta = \frac{x}{a}$ => $sec^2\theta = tan^2\theta + 1 = \frac{x^2}{a^2} + 1$ => $sec\theta = \sqrt{\frac{x^2+a^2}{a^2}}$)

= $\ln |\sqrt{\frac{x^2+a^2}{a^2}} + \frac{x}{a}|$ +c

= $\ln |\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}| + c$

= $\ln | \frac{x+\sqrt{x^2+a^2}}{a}|$+c

= $\ln|x + \sqrt{x^2 + a^2}| - ln\,|\,a\,| + c$

= $ln|x + \sqrt{x^2 + a^2}| + k$ (∴ where k = c- ln|a| is a constant)

∴

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln |x + \sqrt{x^2 + a^2}| + k$$

iv) Integrate $\int \frac{dx}{\sqrt{x^2-a^2}}$

Ans :- $\int \frac{dx}{\sqrt{x^2-a^2}}$

Let $x = a\,sec\theta$ => $dx = a\,sec\theta tan\theta d\theta$

Now $\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{asec\theta tan\theta d\theta}{\sqrt{a^2sec^2\theta - a^2}}$

$= \int \frac{asec\theta tan\theta d\theta}{\sqrt{a^2(sec^2\theta - 1)}}$

$= \int \frac{asec\theta tan\theta}{atan\theta} d\theta = \int sec\theta\ d\theta$

$= \ln|sec\theta + tan\theta| + c$

$= \ln|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}| + c$

{ As $x = a\,sec\theta => sec\theta = x/a$ => $\tan\theta = \sqrt{sec^2\theta - 1} = \sqrt{(\frac{x}{a})^2 - 1}$ }

$= \ln|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a^2}| + c = \ln|\frac{x+\sqrt{x^2-a^2}}{a}| + c$

$= \ln|x + \sqrt{x^2 - a^2}| - \ln|a| + c$

$= \ln|x + \sqrt{x^2 - a^2}| + k \quad (\therefore k = c - \ln|a| = constant)$

Hence

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2 - a^2}| + k$$

v) Integrate $\int \frac{dx}{x\sqrt{x^2-a^2}}$ **(2016-S)**

Let x = $asec\theta$ =>$dx = a\sec\theta\tan\theta\ d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

Now $\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\,sec\theta tan\theta d\theta}{asec\theta\sqrt{a^2sec^2\theta - a^2}}$

$= \int \frac{asec\theta tan\theta\ d\theta}{asec\theta\sqrt{a^2(sec^2\theta - 1)}} = \int \frac{asec\theta tan\theta\ d\theta}{asec\theta\sqrt{a^2tan^2\theta}}$

$= \int \frac{asec\theta tan\theta}{asec\theta\ atan\theta} d\theta = \int \frac{asec\theta tan\theta}{a^2sec\theta tan\theta} d\theta$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a} sec^{-1}\frac{x}{a} + c$$

$$\therefore \quad \boxed{\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} sec^{-1}\frac{x}{a} + c}$$

vi) Integrate $\int \frac{dx}{x^2-a^2}$

Ans :- $\int \frac{dx}{x^2-a^2}$

$$Let\ x = a\ sec\theta \quad => dx = a\ sec\theta\ tan\theta\ d\theta$$

Now $\int \frac{dx}{x^2-a^2} = \int \frac{asec\theta tan\theta d\theta}{a^2sec^2\theta - a^2}$

$$= \int \frac{a\ sec\theta\ tan\theta\ d\theta}{a^2(sec^2\theta - 1)}$$

$$= \int \frac{sec\theta tan\theta d\theta}{atan^2\theta}$$

$$= \frac{1}{a}\int \frac{sec\theta}{tan\theta} d\theta = \frac{1}{a}\int \frac{\frac{1}{cos\theta}}{\frac{sin\theta}{cos\theta}} d\theta$$

$$= \frac{1}{a}\int \frac{1}{sin\theta} d\theta$$

$$= \frac{1}{a}\int cosec\theta\ d\theta$$

$$= \frac{1}{a}\ln|cosec\theta - cot\theta| + c$$

{ As $x = a\ sec\theta => sec\theta = \frac{x}{a} => cos\theta = \frac{a}{x} => sin\theta = \sqrt{1 - cos^2\theta} = \sqrt{1 - (\frac{a}{x})^2} = \sqrt{\frac{x^2-a^2}{x^2}}$

$=> sin\theta = \frac{\sqrt{x^2-a^2}}{x} \quad => cosec\theta = \frac{x}{\sqrt{x^2-a^2}}\ and\ cot\theta = \frac{cos\theta}{sin\theta} = \frac{a}{\sqrt{x^2-a^2}}$ }

$$= \frac{1}{a}\ln\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c$$

$= \frac{1}{a}\ln|\frac{x-a}{\sqrt{x+a}\sqrt{x-a}}| + c$

$= \frac{1}{a}\ln|\frac{\sqrt{x-a}}{\sqrt{x+a}}| + c$

$= \frac{1}{a}\ln|\frac{x-a}{x+a}|^{\frac{1}{2}} + c$ $\quad (\therefore \log_a m^n = n\log_a m)$

$= \frac{1}{2a}\ln|\frac{x-a}{x+a}| + c$

$\therefore$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + c$$

vii) Integrate $\int \frac{dx}{a^2-x^2}$

Ans :- $\int \frac{dx}{a^2-x^2}$

$Let\ x = a\sin\theta \Rightarrow \quad dx = a\ cos\theta d\theta$

Now $\int \frac{dx}{a^2-x^2} = \int \frac{a\ cos\theta\ d\theta}{a^2-a^2sin^2\theta}$

$= \int \frac{a\ cos\theta\ d\theta}{a^2(1-sin^2\theta)} \quad = \int \frac{cos\theta d\theta}{acos^2\theta}$

$= \frac{1}{a}\int \frac{1}{cos\theta}d\theta \quad = \frac{1}{a}\int sec\theta d\theta$

{ As $x = a\ sin\theta \Rightarrow sin\theta = \frac{x}{a} \Rightarrow cos\theta = \sqrt{1-sin^2\theta} = \sqrt{1-\frac{x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$

$\Rightarrow sec\theta = \frac{1}{cos\theta} = \frac{a}{\sqrt{a^2-x^2}} \Rightarrow tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{a^2-x^2}}$ }

$= \frac{1}{a}\ln|sec\theta + tan\theta| + c$

$= \frac{1}{a}\ln|\frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}| + c$

$= \frac{1}{a}\ln|\frac{a+x}{\sqrt{a^2-x^2}}| + c$

$= \frac{1}{a}\ln|\frac{a+x}{\sqrt{a+x}\sqrt{a-x}}| + c$

$= \frac{1}{a}\ln|\frac{\sqrt{a+x}}{\sqrt{a-x}}| + c$

$= \frac{1}{a}\ln|\frac{a+x}{a-x}|^{\frac{1}{2}} + c$ $\qquad (\therefore \log m^n = n\log m)$

$= \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$

$\therefore$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$$

These 7 results deduced in Example-4 are sometimes used to find the integration of some other functions. Some examples are given below.

Example-5 :-Integrate $\int \frac{dx}{\sqrt{25-16x^2}}$

Ans :- $\int \frac{dx}{\sqrt{25-16x^2}}$ $\qquad$ (As $\int \frac{dx}{\sqrt{25-16x^2}} = \int \frac{dx}{\sqrt{16(\frac{25}{16}-x^2)}} = \frac{1}{4}\int \frac{dx}{\sqrt{(\frac{5}{4})^2 - x^2}}$)

$= \frac{1}{4}\int \frac{dx}{\sqrt{(\frac{5}{4})^2 - x^2}}$

$= \frac{1}{4} sin^{-1} \frac{x}{\frac{5}{4}} + c$ $\quad$ (using formula $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$, $here\ a = 5/4$)

$= \frac{1}{4} sin^{-1}\frac{4x}{5} + c$

Example – 6: - Integrate $\int \frac{e^x}{e^{2x}+9}dx$

Ans :- $\int \frac{e^x}{e^{2x}+9}dx$

$= \int \frac{e^x}{(e^x)^2+3^2}dx$ $\quad$ { Let $e^x = t$ => $e^x dx = dt$ }

Now $\int \frac{e^x}{(e^x)^2+3^2}dx = \int \frac{dt}{t^2+3^2} = \frac{1}{3}\tan^{-1}\frac{t}{3} + c$

{ $as \int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + c$, here a= 3}

$= \frac{1}{3}\tan^{-1}\frac{e^x}{3} + c$

Example – 7:- Integrate $\int \frac{dx}{x\sqrt{x^8-4}}$

Ans :- $\int \frac{dx}{x\sqrt{x^8-4}}$ $\qquad$ (multiplying numerator and denominator by $4x^3$)

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$ (Let $x^4 = t$ => $4x^3 = \frac{dt}{dx}$)

$= \quad \frac{1}{4}\int \frac{4x^3dx}{x^4\sqrt{(x^4)^2-4}}$

$= \frac{1}{4}\int \frac{dt}{t\sqrt{t^2-2^2}} = \frac{1}{4}\times\frac{1}{2}\sec^{-1}\frac{t}{2} + c$ (using formula $\int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a}$, here a=2)

$= \frac{1}{8}\sec^{-1}(\frac{x^4}{2}) + c$

Example –8 ; -

Integrate $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$

Ans :- $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$

$= \int \frac{x+5}{\sqrt{x^2+2.3.x+3^2-9-7}}$

$= \quad \int \frac{x+3+2}{\sqrt{(x+3)^2-16}}$ { Let x+3 = t =>dx = dt }

$= \quad \int \frac{t+2}{\sqrt{t^2-16}}dt$

$= \quad \int \frac{tdt}{\sqrt{t^2-16}} + 2\int \frac{dt}{\sqrt{t^2-16}}$

$= I_1 + I_2$………………………………….(1)

$I_1 = \int \frac{tdt}{\sqrt{t^2-16}} = \int \frac{dz}{2\sqrt{z}}$ (putting $t^2 - 16 = z$ => 2t dt = dz => t dt = $\frac{dz}{2}$)

$= \frac{1}{2}.2.z^{\frac{1}{2}} + c_1 = \sqrt{z} + c_1 = \sqrt{t^2 - 16} + c_1$

$= \sqrt{(x+3)^2 - 16} + c_1$……………………………(2)

$I_2 = 2\int \frac{dt}{\sqrt{t^2-16}}$ (applying formula $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2 - a^2}| + k$, where a=4)

$= 2\ln|t + \sqrt{t^2 - 16}| + c_2$

$= 2\ln|(x+3) + \sqrt{(x+3)^2 - 16}| + c_2$………………………(3)

From (1),(2) and (3) we have,

$\int \frac{x+5}{\sqrt{x^2+6x-7}}dx = I_1 + I_2$

$= \quad \sqrt{(x+3)^2 - 16} + c_1 + 2\ln|x + 3 + \sqrt{(x+3)^2 - 16}| + c_2$

$= \quad \sqrt{x^2 + 6x - 7} + 2\ln|x + 3 + \sqrt{x^2 + 6x - 7}| + c$ (where $c_1 + c_2 = c$ is a constant)

Example-9 : -Integrate $\int \frac{dx}{\sqrt{x}\sqrt{x-a^2}}$ **(2016-S)**

Ans: - $\int \frac{dx}{\sqrt{x}\sqrt{x-a^2}}$ (put $\sqrt{x} = t$ => $\frac{1}{2\sqrt{x}}dx = dt$ => $\frac{dx}{\sqrt{x}} = 2dt$)

$= \int \frac{2dt}{\sqrt{t^2-a^2}} = 2\ln(t + \sqrt{t^2 - a^2}) + c$ (Applying $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2 - a^2}| + k$)

$= 2\ln(\sqrt{x} + \sqrt{x - a^2}) + c$

Example-10: - Evaluate $\int \frac{dx}{2x^2+x-1}$

Ans:- $\int \frac{dx}{2x^2+x-1} = \int \frac{dx}{2(x^2+\frac{x}{2}-\frac{1}{2})} = \frac{1}{2}\int \frac{dx}{x^2+2.x.\frac{1}{4}+(\frac{1}{4})^2-\frac{1}{16}-\frac{1}{2}}$

$= \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-\frac{9}{16}} = \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-(\frac{3}{4})^2}$ (applying $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + c$)

$= \frac{1}{2}\frac{1}{2.\frac{3}{4}} ln\left|\frac{(x+\frac{1}{4})-\frac{3}{4}}{(x+\frac{1}{4})+\frac{3}{4}}\right| + c = \frac{1}{3}\ln\left|\frac{\frac{4x+1-3}{4}}{\frac{4x+1+3}{4}}\right| + c$

$= \frac{1}{3}\ln\left|\frac{4x-2}{4x+4}\right| + c = \frac{1}{3}\ln\left|\frac{2x-1}{2x+2}\right| + c$

3. INTEGRATION BY PARTS:-

If v& w are two differentiation function of x,then

$$\frac{d}{dx}(vw) = v\frac{dw}{dx} + w\frac{dv}{dx}$$

Or $v\frac{dw}{dx} = \frac{d}{dx}(vw) - w\frac{dv}{dx}$

Integrating both sites,

$$\int v\frac{dw}{dx}dx = \int \frac{d}{dx}(vw)dx - \int w\frac{dv}{dx}dx$$

$$= wv - \int w\frac{dv}{dx}dx$$

Let u = $\frac{dw}{dx}$ $then\ w = \int udx$

$Then\ the\ above\ result\ canbe\ written\ as$ $\int uv\ dx = (\int udx)v - \int(\int udx) \times \frac{dv}{dx})dx.$

This rule is called integration by parts and is used to integrate the product of two functions

Integration of the product of two functions

$= (integral\ of\ first\ function) \times second\ function - integral\ of\ (integral\ of\ first\ \times derivative\ of\ second)$

Int.of product =(int.first)×second – $\int (int.first)(der.second)dx.$)

➔ Before applying integration by parts we have follow some important things which are listed below.

1. In above formula there are two functions one is u and other one is v. The function 'u' is called the 1st function where as 'v' is called as the 2nd function.

2. The choice of 1st function is made basing on the order ETALI . The meaning of these letters is given below.

E – Exponential function

T – Trigonometric function

A – Algebraic function

L – Logarithmic function

I – inverse trigonometric function

The following table-1 gives a proper choice of 1st and 2nd function in certain cases. Here $m \epsilon N$, n may be zero or any positive integer.

Table-1

Function to be integrated	**first function**	**second function**
$x^n e^x$	e^x	x^n
$x^n sinx$	$sinx$	x^n
$x^n cosx$	$cosx$	x^n
$x^n (lnx)^m$	x^n	$(lnx)^m$
$x^n \sin^{-1} x$	x^n	$\sin^{-1} x$
$x^n\ cos^{-1} x$	x^n	$\cos^{-1} x$
$x^n\ tan^{-1} x$	x^n	$tan^{-1} x$

Example – 11

Integrate $\int x\,cosx\,dx$

Ans :- $\int x\,cosx\,dx$ { from table-1, 1st function = cosx and 2nd function = x }

$= \quad \{\int(\cos x\,dx)\}x - \int(\int cosx\,dx) \times \frac{dx}{dx}.dx$

$= \quad x\sin x - \int sinx.1.dx$

$= \quad xsinx + cosx + c$

Example – 12

Integrate $\int x^2e^xdx$

Ans:-$\int x^2e^xdx$ { 1st function = e^x and 2nd function = x^2 }

$=(\int e^xdx).x^2 - \int(\int e^xdx)\frac{d}{dx}(x^2)dx$

$=x^2e^x - \int e^x \times 2xdx$

$= \quad x^2e^x - 2\int xe^xdx$ { again by parts is applied taking e^x as 1st and x as 2nd function.}

$= \quad x^2e^x - 2[(\int e^xdx).x - \int(\int e^xdx).1.dx]$

$= x^2e^x - 2xe^x + 2e^x + c = \ (x^2 - 2x + 2)\,e^x + c$

Example –13

Integrate $\int \tan^{-1}xdx$

Ans :- $\int \tan^{-1}xdx$

{ There is no direct formula for $\tan^{-1}x$ and two functions are not multiplied with each other in this integral. This type of integration can be solved by using integration by parts by writing $\tan^{-1}x$ as 1. $\tan^{-1}x$ where '1' represent an algebraic function. }

$=\int 1.\tan^{-1}x\,dx \ =(\int 1\,dx).\tan^{-1}x - \int(\int 1.dx).\frac{d}{dx}(\tan^{-1}x)dx$

$= \quad x\tan^{-1}x - \int x.\frac{1}{1+x^2}dx$

{ Let $1 + x^2 = t \ => \ 2xdx = dt$

Now $\int x.\frac{1}{1+x^2}dx = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\,lnt + c = \frac{1}{2}\ln(1 + x^2) + c$ }

$= \quad x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + c$

Example –14

Integrate $\int lnx\, dx$ **(2016-S)**

Ans : - $\int lnx\, dx$

$= \int 1. lnx\, dx$ (Taking 1 as 1st function and lnx as 2nd function)

$= (\int 1. dx)\, lnx - \int (\int 1. dx) \frac{d}{dx}(lnx)\, dx$

$= xlnx - \int x. \frac{1}{x}\, dx \quad = xlnx - \int dx$

$= xlnx - x + c = x(lnx - 1) + c$

Example-15:-

Integrate $\int (lnx)^2 dx$

Ans :- $\int (lnx)^2 dx$

$= \int 1 \times (lnx)^2 dx$

$= (\int 1. dx). (lnx)^2 - \int (\int 1. dx) \frac{d}{dx}(lnx)^2 dx$

$= x(lnx)^2 - \int x. \frac{2lnx}{x} dx.$

$= x(lnx)^2 - 2 \int 1 \times lnx dx$

$= x(lnx)^2 - 2[x. lnx - \int x. \frac{1}{x}\, dx]$

$= x(lnx)^2 - 2xlnx + 2 \int dx$

$= x(lnx)^2 - 2xlnx + 2x + c$

$= x[(lnx)^2 - 2(lnx) + 2] + c$

Example – 16 :-

Evaluate $\int x tan^{-1}x\, dx$ **(2017-W, 2017-S)**

Ans:- $\int x tan^{-1}x\, dx$

$= (\int x dx) tan^{-1}x - \int (\int x\, dx) \frac{d}{dx} (tan^{-1}x)\, dx$

$= \frac{x^2}{2} tan^{-1}x - \int \frac{x^2}{2} \frac{1}{1+x^2}\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2}\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2}\int(1 - \frac{1}{1+x^2})\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2}(x - tan^{-1}x) + c$

$= (\frac{x^2+1}{2})\, tan^{-1}\, x - \frac{1}{2}x + c$

Note: - *When the Integrand is of the form* $e^x\{ f(x) + f'(x)\}$*, the integral is* $e^x f(x)$*,which can be verified by using integration by parts as given below.*

$\int e^x f(x)dx = e^x f(x) - \int(\int e^x dx)\, \frac{d}{dx}f(x)dx$ *(choosing* e^x *as 1st function and* $f(x)$ *as 2nd)*

$= e^x f(x) - \int e^x f'(x)dx + c$

$=> \int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x) + c$

Hence $\int e^x\{f(x) + f'(x)dx\}dx = e^x f(x) + c$

Example-17: -

Integrate $\int \frac{e^x}{x}(1 + xlnx)dx$ **(2017-S)**

Ans:- $\int \frac{e^x}{x}(1 + xlnx)dx = \int \frac{e^x}{x}dx + \int e^x(lnx)dx$

$= \int \frac{e^x}{x}dx + \int e^x(lnx)dx$ (Keeping 1st integral fixed we only simplify the 2nd one.)

$= \int \frac{e^x}{x}dx + (\int e^x dx)(lnx) - \int(\int e^x dx)\, \frac{d}{dx}(lnx)dx$

(taking e^x *as 1st and* lnx *as 2nd function.*)

$= \int \frac{e^x}{x}dx + e^x lnx - \int \frac{e^x}{x}dx + c$

$= e^x lnx + c.$

In some cases, integrating by parts we get a multiple of the original integral on the right hand side, which can be transferred and added to the given integral on the left hand side . After that we can evaluate these integrals. Some examples of such integrals are given below.

Example18: -

Integrate $\int \sqrt{x^2 + a^2}dx$

Ans:-Let $I = \int \sqrt{x^2 + a^2}dx$

$= \int \sqrt{x^2 + a^2} \times 1\, dx$

$= \{\int(1.dx)\}\sqrt{x^2+a^2} - \int(\int 1.dx).\frac{d}{dx}(\sqrt{x^2+a^2}).dx$

$= x\sqrt{x^2+a^2} - \int x \times \frac{1}{2\sqrt{x^2+a^2}} \times 2x\, dx$

$= x\sqrt{x^2+a^2} - \int x \times \frac{x}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\, dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}$

$= x\sqrt{x^2+a^2} - I + a^2 \ln|x+\sqrt{x^2+a^2}| + c$

$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2 \ln|x+\sqrt{x^2+a^2}| + c$

$\Rightarrow I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c$

$$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c$$

Example – 19

Integrate $\int sec^2\theta\sqrt{sec^2\theta+3}\, d\theta$

Ans: -$\int sec^2\theta\sqrt{sec^2\theta+3}\, d\theta$

$= \int sec^2\theta\sqrt{tan^2\theta+1+3}\, d\theta$

$= \int sec^2\theta\sqrt{tan^2\theta+4}\, d\theta \quad \{ tan\theta = t \Rightarrow sec^2\theta\, d\theta = dt\}$

$= \int \sqrt{t^2+2^2}\, dt$ (From example-18 , putting a =2)

$= \frac{t}{2}\sqrt{t^2+2^2} + \frac{2^2}{2}\ln|t+\sqrt{t^2+2^2}| + c$

$= \frac{t}{2}\sqrt{t^2+4} + \frac{4}{2}\ln|t+\sqrt{t^2+4}| + c$

$= \frac{tan\theta}{2}\sqrt{tan^2\theta+4} + 2\ \ln|tan\theta+\sqrt{tan^2\theta+4}| + c$

Example-20 : -

Integrate $\int \sqrt{x^2 - a^2}dx$ **(2014-S)**

Ans:- Let $I = \int \sqrt{x^2 - a^2}dx$

$$= \int \sqrt{x^2 - a^2} \times 1\, dx$$

$$= (\int 1.dx)\sqrt{x^2 - a^2} - \int (\int 1.dx).\frac{d}{dx}(\sqrt{x^2 - a^2})dx$$

$$= x\sqrt{x^2 - a^2} - \int x \times \frac{1}{2\sqrt{x^2 - a^2}} \times 2x\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - I - a^2 \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow 2I = x\sqrt{x^2 - a^2} - a^2 \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow \boxed{\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c}$$

Example – 21

Integrate $\int \sqrt{a^2 - x^2}\, dx$

Ans - let $I = \int \sqrt{a^2 - x^2}\, dx$

$$= \int \sqrt{a^2 - x^2} \times 1 dx$$

$$= (\int 1.dx)\sqrt{a^2 - x^2} - \int (\int 1.dx).\frac{d}{dx}(\sqrt{a^2 - x^2})dx$$

$$= x\sqrt{a^2 - x^2} - \int x.\frac{1}{2\sqrt{a^2 - x^2}}(-2x)\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}} dx + a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\frac{x}{a} + c$$

$$=> \mathrm{I} = x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\frac{x}{a} + c$$

$$=> 2\mathrm{I} = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} + c$$

$$=> \mathrm{I} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\therefore \quad \boxed{\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$$

Example – 22

Evaluate $\int a^x\sqrt{a^{2x} - 9}\, dx$

Ans:- $\int a^x\sqrt{a^{2x} - 9}\, dx$ { Put $a^x = t => a^x lna\, dx = tdt$.}

$$= \frac{1}{lna}\int \sqrt{t^2 - 3^2}\, dt \quad \{ \text{As } \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln|x + \sqrt{x^2 - a^2}| + c \}$$

$$= \frac{1}{lna}\left[\frac{t}{2}\sqrt{t^2 - 3^2} - \frac{3^2}{2}\ln|t + \sqrt{t^2 - 3^2}|\right] + c$$

$$= \frac{1}{lna}\left[\frac{t}{2}\sqrt{t^2 - 9} - \frac{9}{2}\ln|t + \sqrt{t^2 - 9}|\right] + c$$

$$= \frac{1}{lna}\left[\frac{a^x}{2}\sqrt{(a_x)_2 - 9} - \frac{9}{2}\ln|(a_x + \sqrt{(a_x)_2 - 9}|\right] + c$$

$$= \frac{1}{lna}\left[\frac{a^x}{2}\sqrt{a^{2x} - 9} - \frac{9}{2}\ln|(a^x + \sqrt{a^{2x} - 9}|\right] + c$$

Example-23 : -

Integrate $\int \sqrt{2x^2 + 3x + 4}\, dx$

Ans: - $\int \sqrt{2x^2 + 3x + 4}\, dx$

$$= \sqrt{2}\int \sqrt{x^2 + \frac{3}{2}x + 2}\, dx$$

$$= \sqrt{2}\int \sqrt{x^2 + 2.\frac{3}{4}x + (\frac{3}{4})^2 - (\frac{3}{4})^2 + 2}\ dx$$

$$= \sqrt{2}\int \sqrt{(x+\frac{3}{4})^2 - \frac{9}{16} + 2}\ dx$$

$$= \sqrt{2}\int \sqrt{(x+\frac{3}{4})^2 + \frac{23}{16}}\ dx$$ (put x+ $\frac{3}{4}$= t => dx = dt)

$$= \sqrt{2}\int \sqrt{t^2 + (\frac{\sqrt{23}}{4})^2}\ dt$$ (applying result obtained in example-18)

$$= \sqrt{2}\ [\frac{t}{2}\sqrt{t^2 + (\frac{\sqrt{23}}{4})^2} + \frac{(\frac{\sqrt{23}}{4})^2}{2}\ln|t + \sqrt{t^2 + (\frac{\sqrt{23}}{4})^2}\ |\ ] + c$$

$$= \sqrt{2}\ [\frac{x+\frac{3}{4}}{2}\sqrt{(x+\frac{3}{4})^2 + \frac{23}{16}} + \frac{\frac{23}{16}}{2}\ln|x + \frac{3}{4} + \sqrt{(x+\frac{3}{4})^2 + \frac{23}{16}}|\ ] + c$$

$$= \sqrt{2}\ [\frac{4x+3}{8}\sqrt{x^2 + \frac{3}{2}x + 2} + \frac{23}{32}\ln|x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x + 2}|\ ] + c$$

$$= \frac{4x+3}{8}\sqrt{2x^2 + 3x + 4} + \frac{23\sqrt{2}}{32}\ln|x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x + 2}| + c$$ (Ans)

Example-24: -

Integrate $\int e^{ax}\ sinbx\ dx$

Ans :- Let I = $\int e^{ax}\ sinbx\ dx$

$$= (\int e^{ax}\ dx)\ sinbx - \int(\int e^{ax}\ dx) \times \frac{d}{dx}\ (sinbx) \times\ dx$$

$$= \frac{e^{ax}}{a}sinbx - \int \frac{e^{ax}}{a} \times cosbx \times b\ dx$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\int e^{ax}cosbx\ dx$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\ [(\int e^{ax}dx)cosbx - \int(\int e^{ax}\ dx) \times \frac{d}{dx}(cosbx)dx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\ [\frac{e^{ax}}{a}cosbx - \int \frac{e^{ax}}{a} \times (-sinbx) \times b\ dx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\ [\frac{e^{ax}cosbx}{a} + \frac{b}{a}\int e^{ax} sinbxdx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a^2}\ e^{ax}cosbx - \frac{b^2}{a^2}\int e^{ax}\ sinbx\ dx$$

$$I = \frac{e^{ax}sinbx}{a} - \frac{b}{a^2}\ e^{ax}cosbx - \frac{b^2}{a^2}\ I + c$$

$=> \quad I + \frac{b^2}{a^2} I = \frac{e^{ax} sinbx}{a} - \frac{b}{a^2} e^{ax} cosbx + c$

$=> (\frac{a^2+b^2}{a^2}) I = e^{ax} [\frac{sinbx}{a} - \frac{b}{a^2} cosbx] + C$

$=> (\frac{a^2+b^2}{a^2}) I = e^{ax} [\frac{asinbx - bcosbx}{a^2}] + c$

$=> \quad I = \frac{e^{ax}}{a^2+b^2} [a\ sinbx - b\ cosbx] + c$

$\therefore \quad \boxed{\int e^{ax}\, sinbxdx = \frac{e^{ax}}{a^2+b^2} (a\ sinbx - bcosbx) + c}$

Example-25

Integrate $\int e^{ax}\ cosbx\ dx$

Ans:- By adopting the same technique as we have done in example-24 We get

$$\boxed{\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2} (a\ cosbx + b\ sinbx) + c}$$

Example-26 : - Evaluate $\int e^{2x}\, sin3x\ dx$ **(2017-S)**

Ans:- $\int e^{2x}\ sin3x\ dx$

(Proceeding in the same manner as we have done in example-24 with a=2 and b= 3)

$= \frac{e^{2x}}{2^2+3^2} [2\ sin3x - 3cos3x] + c$

$= \frac{e^{2x}}{4+9} [2\ sin3x - 3cos3x] + c$

$= \frac{e^{2x}}{13} [2\ sin3x - 3cos3x] + c$ (Ans)

Example-26 : - Evaluate $\int e^{3x}\, cos2x\ dx$ **(2016-S)**

Ans:- $\int e^{3x}\ cos2x\ dx$ (Putting a=3 and b=2 in the result obtained by Example-25)

$= \frac{e^{3x}}{3^2+2^2} [3cos2x + 2sin2x] + c$*(Note- In exam you have to proceed as example-24)*

$= \frac{e^{3x}}{9+4} [3cos2x + 2sin2x] + c$

$= \frac{e^{3x}}{13} [3cos2x + 2sin2x] + c$ (Ans)

Exercise

1. Evaluate the following Integrals (2 marks questions)

i) $\int \frac{1}{x\sqrt{x}}dx$

ii) $\int (x^{\frac{4}{7}} + \frac{1}{x^{1/3}})\,dx$

iii) $\int \frac{1-sin^3x}{sin^2x}dx$

iv) $\int \frac{cos2x}{sin^2xcos^2x}dx$

v) $\int \sqrt{1+sin2x}dx$

vi) $\int \frac{e^{2x}+1}{e^x}dx$

vii) $\int e^{2lnx}dx$

viii) $\int (\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}})\,dx$

ix) $\int \frac{x^2+\sqrt{x^2-1}}{x^3\sqrt{x^2-1}}dx$

x) $\int \frac{1-cos2x}{1+cos2x}dx$

xi) $\int \frac{dx}{1+sinx}$

xii) $\int (x^{e+}e^x + e^e)dx$

xiii) $\int (x^2+\sqrt{x})^2dx$

2. Evaluate the following (2 marks questions)

i) $\int \frac{x^2dx}{(1+x^3)2}$

ii) $\int sec^2(3x+5)dx$

iii) $\int \frac{(tan^{-1}x)^3}{1+x^2}dx$

iv) $\int \frac{sec^2\sqrt{x}}{\sqrt{x}}dx$

v) $\int tan^3xsec^2x\ dx$

vi) $\int \sqrt{1-sinx}\ cosxdx$

vii) $\int x\sqrt{x^2+3}\ dx$

viii) $\int \frac{dx}{2-3x}$ **(2017-W)**

ix) $\int \frac{xdx}{\sqrt{x^2-a^2}}$

x) $\int \frac{e^x}{(e^x-2)^2}\ dx$

xi) $\int e^{x^3}x^2\ dx$

xii) $\int e^{cos^2x}sin2x\ dx$

xiii) $\int 2x\cot(x^2+3)\ dx$

xiv) $\int e^x tane^x dx$

xv) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

xvi) $\int 3^x e^{2x}dx$

xvii) $\int \frac{sinx}{\sin(x+\propto)}dx$

xviii) $\int \frac{tanx+tan\propto}{tanx-tan\propto}\ dx$

xix) $\int sec^3x\ .tanx\ dx$ **(2016-S)**

xx) $\int \frac{e^{tan^{-1}x}}{1+x^2}\ dx$ **(2014-S)**

xxi) $\int sin^{20}x\ cos^3x\ dx$ **(2017-W)**

Question with long answers (5 and 10 marks)

(10 marks questions are indicated in right side of the question.)

3. Evaluate the following:-

$i)\ \int sin4x\ cos3x\ dx$

$ii)\ \int cos5x\ cos2x\ dx$

$iii)\ \int sin6x\ sin3x\ dx$

$iv)\ \int sin\frac{3x}{4}\ cos\frac{x}{2}dx$

$v)\ \int cos2x\ cos\frac{x}{2}dx$

$vi)\ \int sin^5x\ dx$ (10 marks)

$vii)\ \int cos^7x\ dx$ (10 marks)

$viii)\ \int sin^6x\ dx$ (10 marks)

$ix)\ \int cos^5x\ sin^3x\ dx$

$x)\ \int \frac{sin^3x}{cos^6x}dx$

$xi)\ \int sin^4x.\ cos^4x\ dx$

$xii)\ \int tan^5\ \theta.\ sec^4\theta\ d\theta$

$xiii)\ \int tan^5\theta\ d\theta$

$xiv)\ \int \frac{sin4x - sin2x}{cosx}\ dx$

4. Integrate the following :-

$i)\ \int \frac{dx}{\sqrt{11 - 4x^2}}$

$ii)\ \int \frac{e^{3x}}{\sqrt{4-e^{6x}}}dx$

$iii)\ \int \frac{dx}{x\sqrt{25-(lnx)^2}}$

$iv)\ \int \frac{cos\theta}{\sqrt{4-sin^2\theta}}d\theta$

$v)\ \int \frac{cos\theta d\theta}{\sqrt{4sin^2\theta+1}}$

$vi)\ \int \frac{dx}{\sqrt{5-x^2-4x}}$

$vii)\ \int \frac{x+3}{\sqrt{5-x^2-4x}}dx$

$viii)\ \int \frac{dx}{3x^2+7}$

$ix)\ \int \frac{e^{4x}}{e^{8x}+4}dx$

$x)\ \int \frac{\sec\theta\ tan\theta}{sec^2\theta+4}$

$xi)\ \int \frac{x^9}{x^{20}+4}dx$

$xii)\ \int \frac{dx}{x^2+6x+13}$

$xiii)\ \int \frac{dx}{\sqrt{e^{4x}-5}}$

$xiv)\ \int \frac{dx}{xlnx\sqrt{(lnx)^2-4}}$

$xv)\ \int \frac{dx}{\sqrt{4x^2-6}}$

$xvi)\ \int \frac{dx}{\sqrt{x^2+8x}}$

$xvii)\ \int \frac{x+7}{\sqrt{x^2+8x}}dx$

$xviii)$ $\int \frac{1}{4cos^2x+9sin^2x}dx$ **(2014-S)**

$xix)$ $\int \frac{1}{x\sqrt{(logx)^2-8}}dx$ **(2015-S)**

$xx)$ $\int \frac{dx}{\sqrt{25-9x^2}}$ **(2015-S)**

$xxi)$ $\int \frac{dx}{7-6x-x^2}$

5. ***Evaluate the following***

(i) $\int(1+x)e^x dx$
(ii) $\int x^3e^x dx$
(iii) $\int x\ sinx\ dx$
(iv) $\int x^2 sinax\ dx$
(v) $\int xcos^2xdx$
(vi) $\int 2x\ cos3x\ cos2x\ dx$ (10 marks)
(vii) $\int 2x^3cosx^2dx$
(viii) $\int x^7 lnx\ dx$
(ix) $\int(lnx)^3dx$
(x) $\int \frac{lnx}{x^5}dx$
(xi) $\int \sec^{-1}x\ dx$
(xii) $\int x\ \sin^{-1}x\ dx$
(xiii) $\int e^x cos^2xdx$ (10 marks)
(xiv) $\int e^{2x}cos5xdx$
(xv) $\int\sqrt{7x^2+2}dx$
(xvi) $\int e^x(tanx + lnsecx)dx$
(xvii) $\int[\frac{1}{lnx}-\frac{1}{(lnx)^2}]\ dx$
(xviii) $\int \sin(lnx)\ dx$
(xix) $\int \frac{xe^x}{(1+x)^2}dx$
(xx) $\int\sqrt{x^2-8}\ dx$
(xxi) $\int\sqrt{9-x^2}dx$
(xxii) $\int e^{2x}sinx\ dx$ **(2016-S, 2017-W)**
(xxiii) $\int e^x sinx\ dx$ **(2019-W)**

Answer

1) i) $-2x^{1/2}+c$ ii) $\frac{7}{11}x^{\frac{11}{7}}+\frac{3}{2}x^{\frac{2}{3}}+c$ iii) $-\cot x+\cos x+c$

iv) $-\cot x-\tan x+c$ v) $\sin x-\cos x+c$ vi) $e^{x}-e^{-x}+c$

vii) $\frac{x^{3}}{3}+c$ viii) $\sin^{-1}x+c$ ix) $\sec^{-1}x-\frac{1}{2x^{2}}+c$

x) $\tan x-x+c$ xi) $\tan x-\sec x+c$ xii) $\frac{x^{e+1}}{e+1}+e^{x}+e^{e}x+c$

xiii) $\frac{x^{5}}{5}+\frac{4}{7}x^{\frac{7}{2}}+\frac{x^{2}}{2}+c$

2) i) $\frac{-1}{3(1+x^{3})}+c$ ii) $\frac{1}{3}\tan(3x+5)+c$ iii) $\frac{(\tan^{-1}x)^{4}}{4}+c$

iv) $2\tan\sqrt{x}+c$ v) $\frac{\tan^{4}x}{4}+c$ vi) $-\frac{2}{3}(1-\sin x)^{3/2}+c$

vii) $\frac{1}{3}(x^{2}+3)^{\frac{3}{2}}+c$ viii) $-\frac{1}{3}\log(2-3x)+c$ ix) $\sqrt{x^{2}-a^{2}}+c$

x) $-\frac{1}{e^{x}-2}+c$ xi) $\frac{1}{3}e^{x^{3}}+c$ xii) $-e^{\cos^{2}x}+c$

xii) $\ln\left|\sin(x^{2}+3)\right|+c$ xiv) $\ln\left|\sec e^{x}\right|+c$ xv) $\ln(e^{x}-e^{-x})+c$

xvi) $\frac{3^{x}e^{2x}}{2+\ln 3}+c$ xvii) $x\cos\alpha-\sin\alpha\ln\left|\sin(x+\alpha)\right|+c$

xviii) $x\cos 2\alpha+\sin 2\alpha\ln\left|\sin(x-\alpha)\right|+c$ xix) $\frac{\sec^{3}x}{3}+c$

xx) $e^{\tan^{-1}x}+c$ xxi) $\frac{\sin^{21}x}{21}-\frac{\sin^{23}x}{23}+c$

3) i) $\frac{-1}{14}\cos 7x-\frac{1}{2}\cos x+c$ ii) $\frac{1}{14}\sin 7x+\frac{1}{6}\sin 3x+c$ iii) $\frac{1}{6}\sin 3x-\frac{1}{18}\sin 9x+c$

iv) $\frac{-2}{5}\cos\frac{5}{4}x-2\cos\frac{x}{4}+c$ v) $\frac{1}{5}\sin\frac{5}{2}x+\frac{1}{3}\sin\frac{3x}{2}+c$ vii) $-\cos x+\frac{2}{3}\cos^{3}x-\frac{1}{5}\cos^{5}x+c$

vii) $\sin x-\sin^{3}x+\frac{3}{5}\sin^{5}x-\frac{1}{7}\sin^{7}x+c$ viii) $\frac{1}{192}(60x-45\sin 2x+9\sin 4x-\sin 6x)+c$

ix) $\frac{1}{8}\cos^{8}x-\frac{1}{6}\cos^{6}x+c$ x) $\frac{1}{5}\sec^{5}x-\frac{1}{3}\sec^{3}x+c$ xi) $\frac{1}{128}(3x-\sin 4x+\frac{1}{8}\sin 8x)+c$

xii) $\frac{1}{6}\tan^{6}\theta+\frac{1}{8}\tan^{8}\theta+c$ xiii) $\frac{1}{4}\tan^{4}\theta-\frac{1}{2}\tan^{2}\theta+\ln\left|\sec\theta\right|+c$

xiv) $4\cos x-\frac{2}{3}\cos 3x+c$

4) i) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{11}}+c$ ii) $\frac{1}{3}\sin^{-1}\frac{e^{3x}}{2}+c$ iii) $\sin^{-1}\frac{\ln x}{5}+c$

iv) $sin^{-1}(\frac{sin\theta}{2}) + c$ v) $\frac{1}{2} ln\left| sin\theta + \sqrt{sin^2\theta + \frac{1}{4}} \right| + c$ vi) $sin^{-1}\frac{x+2}{3} + c$

vii) $sin^{-1}\frac{x+2}{3} - \sqrt{5 - x^2 - 4x} + c$ viii) $\frac{1}{\sqrt{21}} tan^{-1}\frac{\sqrt{3}x}{\sqrt{7}} + c$ ix) $\frac{1}{8} tan^{-1}(\frac{e^{4x}}{2}) + c$

x) $\frac{1}{2} tan^{-1}(\frac{sec\theta}{2}) + c$ xi) $\frac{1}{20} tan^{-1}(\frac{x^{10}}{2}) + c$ xii) $\frac{1}{2} tan^{-1}\frac{x+3}{2} + c$

xiii) $\frac{1}{2\sqrt{5}} sec^{-1}(\frac{e^{2x}}{\sqrt{5}}) + c$ xiv) $\frac{1}{2} sec^{-1}(\frac{lnx}{2}) + c$ xv) $\frac{1}{2} ln\left| 2x + \sqrt{4x^2 - 6} \right| + c$

xvi) $ln\left| x + 4 + \sqrt{x^2 + 8x} \right| + c$ xvii) $\sqrt{x^2 + 8x} + 3\, ln\left| x + 4 + \sqrt{x^2 + 8x} \right| + c$

xviii) $\frac{1}{6} tan^{-1}(\frac{3tanx}{2}) + c$ xix) $\log\left| logx + \sqrt{(logx)^2 - 8} \right| + c$

xx) $\frac{1}{3} sin^{-1}\frac{3x}{5} + c$ xxi) $\frac{1}{8} ln\left| \frac{7+x}{1-x} \right| + c$

5) i) $xe^x + c$ ii) $e^x(x^3 - 3x^2 + 6x - 6) + c$ iii) $sinx - xcosx + c$

iv) $\frac{1}{a^3}[(2 - a^2x^2)cosax + 2axsinax] + c$ v) $\frac{1}{8}(2x^2 + 2xsin2x + cos2x) + c$

vi) $cosx + \frac{1}{25} cos5x + x(sinx + \frac{1}{5} sin5x) + c$ vii) $x^2 sinx^2 + cosx^2 + c$

viii) $\frac{x^8}{64}(8lnx - 1)$ ix) $x[(lnx)^3 - 3(lnx)^2 + 6lnx - 6] + c$

x) $-\frac{1}{16x^4}(1 + 4lnx) + c$ xi) $xsec^{-1}x - ln\left| x + \sqrt{x^2 - 1} \right| + c$

xii) $(\frac{x^2}{2} - \frac{1}{4}) sin^{-1}x + \frac{1}{4} x\sqrt{1 - x^2} + c$ xiiii) $\frac{e^x}{10}(5 + cos2x + 2sin2x) + c$

xiv) $\frac{e^{2x}}{29}(2cos5x + 5sin5x) + c$ xv) $\frac{x}{2}\sqrt{7x^2 + 2} + \frac{1}{\sqrt{7}} ln\left| \sqrt{7}x + \sqrt{7x^2 + 2} \right| + c$

xvi) $e^x \ln(secx) + c$ xvii) $\frac{x}{lnx} + c$ xviii) $\frac{x}{2}[\sin(lnx) - \cos(lnx)] + c$

xix) $\frac{e^x}{x+1} + c$ xx) $\frac{x}{2}\sqrt{x^2 - 8} - 4ln\left| x + \sqrt{x^2 - 8} \right| + c$

xxi) $\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2} sin^{-1}\frac{x}{3} + c$ xxii) $\frac{e^{2x}}{5}(2sinx - cosx) + c$ xxiii) $\frac{e^x}{2}(sinx - cosx) + c$

DEFINITE INTEGRAL

Introduction

It was stated earlier that integral can be considered as process of summation. In such case the integral is called definite integral.

Objective

After completion of the topic you will be able to

1. Define and interpret geometrically the definite integral as a limit of sum.
2. State fundamental theorem of integral calculus.
3. State properties of definite integral.
4. Find the definite integral of some functions using properties.
5. Apply definite integral to find the area under a curve

Expected Background knowledge

1. Functional value of a function at a point.
2. Integration.

Definite Integral

Integration can be considered as a process of summation. In this case the integral is called as definite integrals.

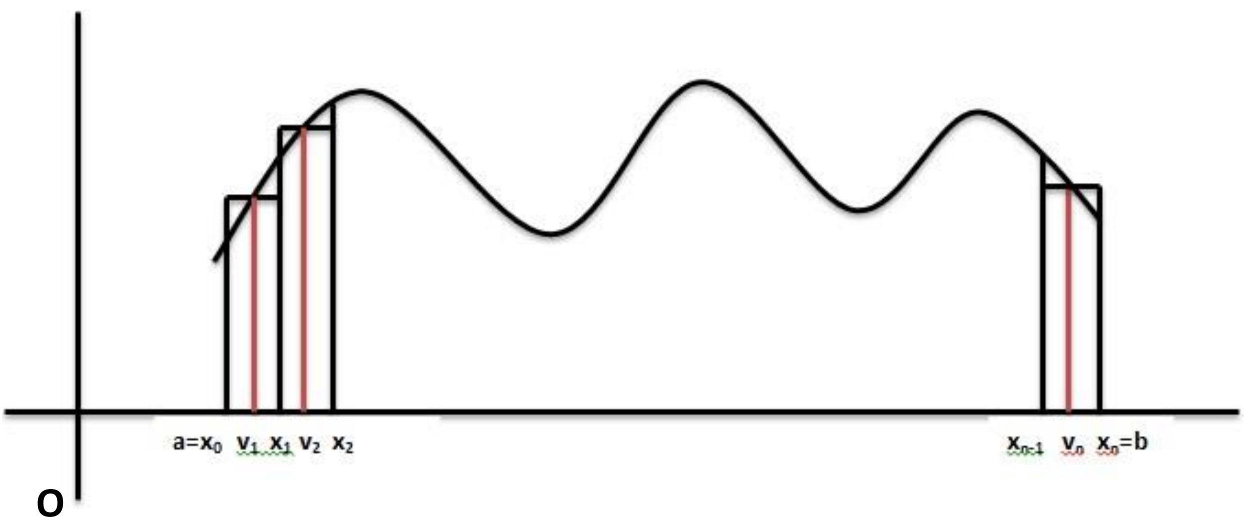

Fig-1

Definition:-

Let f(x) be a continuous function in [a,b] as shown in Fig-1 . Divide [a,b] into n sub-intervals of length h_1, h_2, ... h_n i.e. $h_1 = x_1 - x_0$, $h_2 = x_2 - x_1$,............. , $h_n = x_n - x_{n-1}$

Let v_r be any point in $[x_{r-1}, x_r]$ i.e. $v_1 \in [x_0, x_1]$, $v_2 \in [x_1, x_2]$, ..., $v_n \in [x_{n-1}, x_n]$.

Then the sum of area of the rectangles (as shown in fig) when n→∞ is defined as the definite integral of f(x) from a to b, denoted by $\int_a^b f(x)dx$

$$Here, \quad a = lower limit of integration$$
$$b = upper limit of integration$$

Mathematically,

$$\int_a^b f(x)dx = \lim_{n\to\infty}[h_1 f(v_1) + h_2 f(v_2) + \ldots\ldots\ldots + h_n f(V_n)]$$

Fundamental Theorem of Integral Calculus

If f(x) is a continuous function in [a,b] and $\int f(x)dx$ = Ø (x) + c , them

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a)$$

Note :- No arbitrary constants are used in definite integral.

Example:

1. Find $\int_1^2 x^3\, dx$

Ans.

First find $\int x^3\, dx = \frac{x^4}{4} + c$

Here, f(x) = x^3 , Φ (x) = $\frac{x^4}{4}$

By fundamental theorem

$$\int_1^2 x^3\, dx = \Phi(2) - \Phi(1)$$
$$= \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

2. Find $\int_0^1 \frac{dx}{1+x^2}$

Ans.

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1$$
$$= \tan^{-1} 1 - \tan^{-1} 0$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3. Find $\int_2^3 2xe^{x^2}$dx

Ans.

$\int_2^3 2xe^{x^2}dx$

{Let x^2 = u => 2xdx = du, when x = 2, u = x^2 = 4,

When x = 3, u = x^2 = 9,

So, lower limit changes to 4 and upper limit changes to 9}

= $\int_4^9 e^u du$

= $[e^u]_4^9$ = $e^9 - e^4$ (Ans)

Properties of Definite Integral

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Explanation

Definite integral is independent of variable.

e.g. $\int_2^3 x^2 dx = \int_2^3 u^2\, du = \int_2^3 t^2 dt$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Explanation

If limits of definite integrals are interchanged then the value changes to its negative.

e.g. $\int_2^3 xdx = -\int_3^2 xdx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ **where, a<c<b.**

Explanation

If we integrate f(x) in [a,b] and c ∈ [a,b] such that a<c<b, then the above integral is same if we integrate f(x) in [a,c] and [c,b] and then add them.

e.g. $\int_2^6 xdx = \int_2^4 xdx + \int_4^6 xdx$

verification

$\int_2^6 xdx = \left[\frac{x^2}{2}\right]_2^6$

$= \frac{6^2}{2} - \frac{2^2}{2} = \frac{36}{2} - \frac{4}{2} = 18-2 = 16$ -------------(1)

$\int_2^4 xdx + \int_4^6 xdx = \left[\frac{x^2}{2}\right]_2^4 + \left[\frac{x^2}{2}\right]_4^6 = \left[\frac{4^2}{2} - \frac{2^2}{2}\right] + \left[\frac{6^2}{2} - \frac{4^2}{2}\right]$

$= \left[\frac{16}{2} - \frac{4}{2}\right] + \left[\frac{36}{2} - \frac{16}{2}\right] = (8-2) + (18-8)$

$= 6 + 10 = 16$-----------(2)

From (1) and (2) we have,

$\int_2^6 xdx = \int_2^4 xdx + \int_4^6 xdx$ (verified)

4. $\int_0^a f(x)dx$ **=** $\int_0^a f(a-x)dx$

e.g. $\int_0^{\frac{\pi}{2}} sinxdx = \int_0^{\frac{\pi}{2}} \sin(\frac{\pi}{2} - x)\, dx$

verification

$\int_0^{\frac{\pi}{2}} sinxdx$ = [- cos x]$_0^{\frac{\pi}{2}}$

= - [cos $\frac{\pi}{2} - cos\, 0$] = - [0 - 1] = 1---------------------- (1)

$\int_0^{\frac{\pi}{2}} sin(\frac{\pi}{2} - x)$ dx = $\int_0^{\frac{\pi}{2}} \cos xdx$

= [sin x]$_0^{\frac{\pi}{2}}$ = sin $\frac{\pi}{2}$ - sin 0 = 1 – 0 = 1---------------------- (2)

From (1) and (2)

$\int_0^{\frac{\pi}{2}} sinxdx = \int_0^{\frac{\pi}{2}} \sin(\frac{\pi}{2} - x)\, dx$ (verified)

5. (i) If f(x) is an even function, then

$\int_{-a}^{a} f(x)dx$ **= 2** $\int_0^a f(x)dx$

(ii) If f(x) is an odd function, then

$\int_{-a}^{a} f(x)dx$ **= 0**

Example: - By this formula without integration we can find the integral for

f(x) is an odd function if

f (-x)= -f(x)

Sinx , x ,x^3 ……… are examples of odd functions.

f(x) is an even function if

f (-x) = f(x)

Cosx , x^2 , x^4 ……… are examples of even functions .

Example:-

$\int_{-2}^{2} x^2dx = 2\int_0^2 x^2dx$

{f(x) = $x^2$is an even function as, f(-x) = $(-x)^2 = x^2$. So, f(−x) = f(x)}

Similarly,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sinx dx = 0$$

Reason

f(x) = sinx => f(-x) = sin (-x) = -sinx

So, f(-x) = -f(x)

⇨ f(x) is an odd function.

6. (i) $\int_0^{2a} f(x)dx$ **= 2** $\int_0^{a} f(x)dx$ **if f(2a - x) = f(x)**

(ii) $\int_0^{2a} f(x)dx$ **= 0 if f(2a - x) = -f(x).**

7. $\int_a^b f(x)dx$ **=** $\int_a^b f(a+b-x)dx$

Problems

Q1. Find $\int_{-2}^{1}|x|dx$

Ans.

$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

|x| changes its definition at '0', so divide the integral into two parts (-2,0) and (0,1).

Now, $\int_{-2}^{1}|x|dx$

= $\int_{-2}^{0}|x|dx$ + $\int_{0}^{1}|x|dx${Property (3) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx$}

= $\int_{-2}^{0} -xdx + \int_0^1 xdx${when, -2 < x < 0 i.e. x < 0 then, |x| = -x }

= - $[\frac{x^2}{2}]_{-2}^{0}$ + $[\frac{x^2}{2}]_0^1${when, 0 < x < 1 i.e. 0 < x then, |x| = x }

= - $[\frac{0^2}{2} - \frac{(-2^2)}{2}] + [\frac{1^2}{2} - \frac{0^2}{2}]$

= - [0 - $\frac{4}{2}$] + [$\frac{1}{2} - 0$] = 2 + $\frac{1}{2}$ = $\frac{3}{2}$

Q2. $\int_{-6}^{6}|x+2|$ dx = ?

Ans.

$\int_{-6}^{6}|x+2|$ dx = $\int_{-4}^{8}|u|du$ {Let u = x + 2 => du = dx, when, x = -6, u = -6 +2 = -4}

{when, x = 6, u = 6 +2 = 8}

= $\int_{-4}^{0}|u|$ du + $\int_0^8 |u|du${property (3)}

= $\int_{-4}^{0} -udu + \int_0^8 udu$ { when -4<u<0 then IuI = -u and when 0<u<8 ,then IuI = u}

= - $[\frac{u^2}{2}]_{-4}^{0}$ + $[\frac{u^2}{2}]_0^8$

$= -\frac{1}{2}[u^2]_{-4}^{0} + \frac{1}{2}[u^2]_{0}^{8} = -\frac{1}{2}[0^2 - (-4)^2] + \frac{1}{2}[8^2 - 0]$

$= -\frac{1}{2}(-16) + \frac{1}{2}(64)$

$= 8 + 32 = 40$(Ans)

Q3. Find $\int_1^3 [x]dx$

Ans.

[x] is a function which changes it value at every integral point. So, we have to break the range into different integral ranges i.e. (1,3) can be breaked into (1,2) and (2,3)

$\int_1^3 [x]dx = \int_1^2 [x]dx + \int_2^3 [x]dx$ {applying property (3) i.e. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$}

$= \int_1^2 1dx + \int_2^3 2dx$ {when 1 < x< 2 then [x] = 1, when 2 <x < 3, then [x] = 2 }

$= [x]_1^2 + [2x]_2^3$

$= (2 - 1) + 2(3 - 2) = 1 + (2 \times 1) = 1 + 2 = 3$

Q4. Evaluate $\int_0^{\frac{3}{2}} [2x]dx$

Ans.

$\int_0^{\frac{3}{2}} [2x]dx = \int_0^3 [u]\frac{du}{2}$

{Put u = 2x, du = 2dx => dx = $\frac{du}{2}$ when, x = 0, u = 2x = 0 when x = $\frac{3}{2}$, u = 2x = 3}

$= \frac{1}{2}\left[\int_0^1 [u]du + \int_1^2 [u]du + \int_2^3 [u]du\right]$

$= \frac{1}{2}\left[[0]_0^1 + [u]_1^2 + 2[u]_2^3\right] = \frac{1}{2}[0 + (2 - 1) + 2(3 - 2)]$

$= \frac{1}{2}[0 + 1 + 2] = \frac{3}{2}$(Ans)

Q5. Find $\int_{-1}^{1}\{|x| + [x]\}dx$

Ans.

$\int_{-1}^{1}(|x| + [x])dx = \int_{-1}^{1}|x|dx + \int_{-1}^{1}[x]dx$

$= \int_{-1}^{0}|x|dx + \int_{0}^{1}|x|dx + \int_{-1}^{0}[x]dx + \int_{0}^{1}[x]dx$

$= \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx + \int_{-1}^{0}(-1)\,dx + \int_{0}^{1} 0\,dx$ {By def^n of |x| & [x]}

$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} - [x]_{-1}^{0} + 0$

$= -\frac{1}{2}(0^2 - (-1)^2) + \frac{1}{2}(1^2 - 0) - (0 - (-1))$

$= -\frac{1}{2}(-1) + \frac{1}{2}.1 - (1)$

$= \frac{1}{2} + \frac{1}{2} - 1 = 0$ (Ans)

Q6. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}dx$ **{ 2016-S, 2017-W}**

Ans.

In this type of problems we generally use property (4). And this type of problem can be solved by following technique.

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}dx$---------------------- (1)

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}} dx$$

{In above x is replaced by $\frac{\pi}{2} - x$. As by property(4) there is no change in integral value}

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{cosx}+\sqrt{sinx}} dx$$

$$=> I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}} dx \text{------------------ (2)}$$

Now equation (1) +(2)

⇨ $I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}} dx$

⇨ $2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}+\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}} dx$

⇨ $2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$

⇨ $2I = (\frac{\pi}{2} - 0) = \frac{\pi}{2}$

⇨ $I = \frac{\pi}{4}$

Hence, $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}} dx = \frac{\pi}{4}$

Q7. Evaluate $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)d\theta$ **{2016-S}**

Ans.

Let $I = \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)d\theta$

$= \int_0^{\frac{\pi}{4}} \log(1 + \tan(\frac{\pi}{4} - \theta))\, d\theta$ {As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$}

$= \int_0^{\frac{\pi}{4}} \log(1 + \frac{\tan\frac{\pi}{4} - tan\theta}{1 + tan\frac{\pi}{4}.\tan\theta})\, d\theta$

$= \int_0^{\frac{\pi}{4}} \log(1 + \frac{1 - tan\theta}{1 + \tan\theta})\, d\theta$

$= \int_0^{\frac{\pi}{4}} \log(\frac{1 + tan\theta + 1 - tan\theta}{1 + \tan\theta})\, d\theta$

$= \int_0^{\frac{\pi}{4}} \log(\frac{2}{1 + tan\theta})\, d\theta$

$= \int_0^{\frac{\pi}{4}} (log 2 - \log(1 + tan\theta))d\theta$ {$\log(\frac{a}{b}) = \log a - \log b$}

$= \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$

⇨ $I = \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$

⇨ $I = \log 2 \int_0^{\frac{\pi}{4}} d\theta - I$ {As $I = \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)$}

⇨ $2I = \log 2\,[\theta]_0^{\frac{\pi}{4}} = \log 2\,(\frac{\pi}{4} - 0)$

⇨ $2I = \frac{\pi}{4}\log 2$

⇨ $I = \frac{\pi}{8}\log 2$

Hence, $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta = \frac{\pi}{8}\log 2$ (Ans)

Q8. Evaluate $\int_0^1 \frac{x}{x+1} dx$

Ans.

$\int_0^1 \frac{x}{x+1} dx = \int_0^1 \frac{x+1-1}{x+1} dx$

$= \int_0^1 (\frac{x+1}{x+1} - \frac{1}{x+1}) dx$

$= \int_0^1 (1 - \frac{1}{x+1})\, dx$

$= [x - \ln(x+1)]_0^1$

$= [(1 - \ln(1+1)) - (0 - \ln(0+1))]$

$= 1 - \ln 2 - 0 + \ln 1$

$= 1 - \ln 2 - 0 + 0$

$= 1 - \ln 2$

Q9. $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Ans.

$\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^x.e^x+1}{e^x}}$

$= \int_0^1 \frac{e^x dx}{e^{x+x}+1} = \int_0^1 \frac{e^x}{e^{2x}+1} dx$

$= \int_0^1 \frac{1}{(e^x)^2+1} dx$

{ Let $t = e^x$, => $dt = e^x dx$

when, $x = 0$, $t = e^x = e^0 = 1$

when $x = 1$, $t = e^1 = e$ }

$= \int_1^e \frac{dt}{t^2+1} = [tan^{-1}t]_1^e$

$= tan^{-1}e - tan^{-1}1$

$= tan^{-1}e - \frac{\pi}{4}$ (Ans)

Q10. $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = ?$

Ans.

$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^{\sqrt{2}}$

$= \sec^{-1}\sqrt{2} - \sec^{-1} 1$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ (Ans)

Q11. Find $\int_0^4 \frac{1}{x+\sqrt{x}} dx$

Ans.

$\int_0^4 \frac{1}{x+\sqrt{x}}dx$

Let $\sqrt{x} = t$, Then, $\frac{1}{2\sqrt{x}}dx = dt$

⇨ $dx = 2\sqrt{x}\,dt = 2t\,dt$

When $x = 0$, $t = \sqrt{x} = \sqrt{0} = 0$

When $x = 4$, $t = \sqrt{4} = 2$

Now,

$\int_0^4 \frac{1}{x+\sqrt{x}}dx = \int_0^2 \frac{2tdt}{t^2+t}$

$= 2\int_0^2 \frac{tdt}{t\,(1+t)} = 2\int_0^2 \frac{dt}{1+t} = 2\,[\ln(1+t)]_0^2$

$= 2\,(\ln 3 - \ln 1) = 2\,(\ln 3 - 0) = 2\ln 3$ (Ans)

Q12. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx}dx$

Ans.

$\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx}dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot(\frac{\pi}{2}-x)}dx$ {Property 4}

$= \int_0^{\frac{\pi}{2}} \frac{1}{1+tanx}dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{cotx}}dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{\frac{cotx+1}{cotx}}dx$

$= \int_0^{\frac{\pi}{2}} \frac{cotx}{1+cotx}dx$

$= \int_0^{\frac{\pi}{2}} \frac{cotx+1-1}{1+cotx}dx$

$= \int_0^{\frac{\pi}{2}} (1 - \frac{1}{1+cotx})dx$

$= \int_0^{\frac{\pi}{2}} dx$ - $\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx}dx$ ------------------------ (1)

Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+cotx}dx$

Then from (1)

$I = \int_0^{\frac{\pi}{2}} dx - I$

⇨ $2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = (\frac{\pi}{2} - 0) = \frac{\pi}{2}$

⇨ $I = \frac{\pi}{4}$

Hence, $\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx}dx = \frac{\pi}{4}$ (Ans)

Q13. Prove that $\int_0^{\frac{\pi}{2}} \log(sinx)dx = \int_0^{\frac{\pi}{2}} \log(cosx)dx = -\frac{\pi}{2}\log 2$ { 2018-S}

Ans.

Let $I = \int_0^{\frac{\pi}{2}} \log(sinx)dx$ ----------------------- (1)

$= \int_0^{\frac{\pi}{2}} \log(\sin(\frac{\pi}{2} - x))dx$ $\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\}$

$= \int_0^{\frac{\pi}{2}} \log(cosx)dx$ - ------------------- (2)

Add (1) and (2)

$I + I = \int_0^{\frac{\pi}{2}} \log(sinx)dx + \int_0^{\frac{\pi}{2}} \log(cosx)dx$

$= \int_0^{\frac{\pi}{2}} \{\log(sinx) + \log(cosx)\}dx$

$= \int_0^{\frac{\pi}{2}} \log(sinx.cosx)dx$ $\{\log a + \log b = \log ab\}$

$= \int_0^{\frac{\pi}{2}} \log(\frac{2sinx.cosx}{2})dx$

$= \int_0^{\frac{\pi}{2}} \log(\frac{sin2x}{2})dx$

$= \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)dx$

$2I = \int_0^{\frac{\pi}{2}} \log sin2x\, dx$ - $\int_0^{\frac{\pi}{2}} \log 2\, dx$ - ----------------- (3)

Now, $\int_0^{\frac{\pi}{2}} \log sin2x\, dx$ {Put t = 2x we have dt = 2dx when x = 0, t = 0 when $x = \frac{\pi}{2}$, $t = \pi$}

$= \int_0^{\pi} \log sint \frac{dt}{2}$

$= \frac{1}{2}\int_0^{\pi} \log sintdt$ { Here f(t) = log sin(t), Then $f(\pi - t) = \log \sin(\pi - t) = \log \sin(t) = f(t)$}

$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log(sint)dt$ {By property-6 i.e.$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$,Where f(2a-x) = f(x)}

$= \int_0^{\frac{\pi}{2}} \log(sint)dt$ { We have $\int_0^{\pi} \log sintdt = 2\int_0^{\frac{\pi}{2}} \log(sint)dt$ }

$= \int_0^{\frac{\pi}{2}} \log(sinx)dx$ {Property (1) $\int_a^b f(x)dx = \int_a^b f(t)dt$}

= I (from (1))

⇨ $I = \int_0^{\frac{\pi}{2}} \log sin2xdx$ - ------------------- (4)

From (3) and (4)

⇨ $2I = I - \int_0^{\frac{\pi}{2}} \log 2\, dx$

⇨ $2I = I - \log 2\, [x]_0^{\frac{\pi}{2}}$

⇨ $I = -\log 2\, (\frac{\pi}{2} - 0)$

Hence, $\int_0^{\frac{\pi}{2}} \log(sinx)dx = \int_0^{\frac{\pi}{2}} \log cosx\, dx$

$= -\frac{\pi}{2} \log 2$ (Proved)

Exercise

Evaluate the integrals (2 marks and 5 marks questions)

(Questions with 2 marks is marked on right of the questions)

1) $\int_0^2 [x^2]\, dx$

2) $\int_0^{\frac{\pi}{2}} \frac{cosx}{sinx+cosx} dx$ **(2015-S)**

3) $\int_0^{\frac{\pi}{2}} \frac{1}{1+tanx} dx$

4) $\int_0^1 \frac{\log(1+x)}{(1+x^2)} dx$

5) $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}} dx$

6) $\int_0^{\pi} \frac{xdx}{1+sinx}$

7) $\int_0^{\frac{\pi}{2}} \log(tanx)dx$ **(2017-S) (2014-S)**

8) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx}$

9) $\int_0^{\frac{\pi}{2}} sin2x \log(tanx)\, dx$

10) $\int_0^{\pi} \frac{xsinx}{1+cos^2x} dx$

11) $\int_0^{\frac{\pi}{2}} \frac{sinx}{sinx+cosx} dx$ **(2017-W)**

12) $\int_{-1}^1 [x]dx$ **(2018-S)** (2marks)

13) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} dx$ **(2016-S) (2017-S)**

14) $\int_0^1 \frac{dx}{1+x^2} dx$ (2marks)

15) $\int_0^3 [x]dx$ **(2017-W)**

ANSWERS

1. $5-\sqrt{3} - \sqrt{2}$
2. $\frac{\pi}{4}$
3. $\frac{\pi}{4}$
4. $\frac{\pi}{8}\log 2$ {Hints Put x = tanθ}
5. $\log(\frac{5+3\sqrt{3}}{1+\sqrt{3}})$
6. π
7. 0
8. 1
9. 0
10. $\frac{\pi^2}{4}$ 11. $\frac{\pi}{4}$ 12. -1
13. $\frac{\pi}{2}$ 14. $\frac{\pi}{4}$ 15. 3

Area under plane curve

In our previous study we know that the definite integral represents the area under the curve.

Area enclosed by curve and X- axis

Area enclosed by a curve y = f(x) , X-axis , x = a and x =b is given by

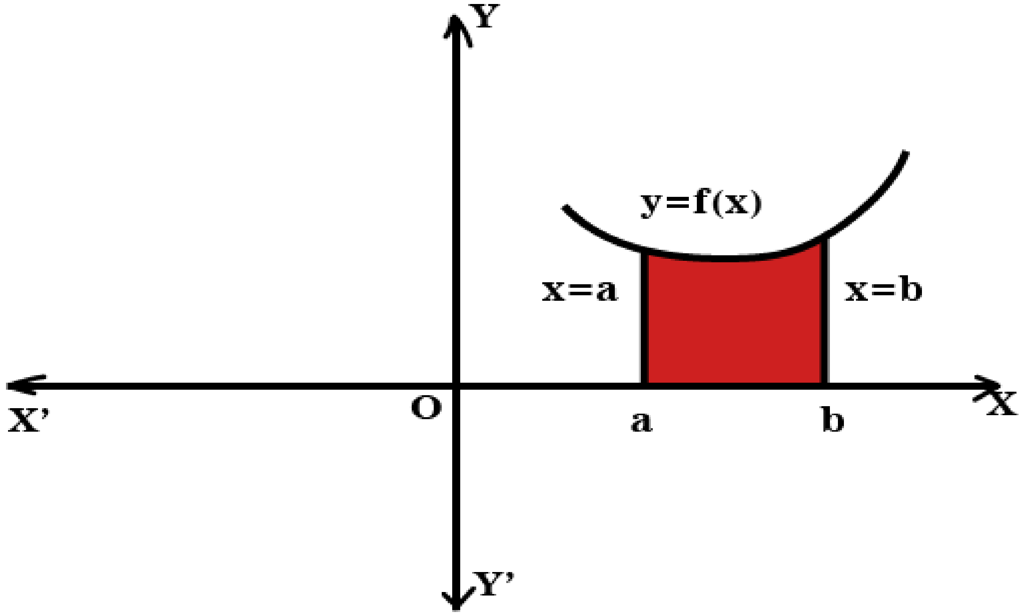

Fig-2

Area = $\int_a^b f(x)dx$

Example – 1

Find the area bounded y = e^x , X-axis x = 4 and x = 2

Ans.

Here y = e^x is the curve

Area of the curve bounded by X-axis , x = 4 and x = 2 is

Area = $\int_2^4 y\,dx = \int_2^4 e^x dx$

$= [e^x]_2^4 = e^4 - e^2$ (Ans)

Example – 2

Find the area enclosed by y = 9 -x^2 , y = 0 , x = 0 and x = 2.

Ans.

$$\text{Area} = \int_0^2 y\,dx = \int_0^2 (9 - x^2)dx$$

$$= \left[9x - \frac{x^3}{3}\right]_0^2 = [\,(9X2) - \frac{2^3}{3} - (0\text{-}0)]$$

$$= 18 - \frac{8}{3} = \frac{54-8}{3} = \frac{46}{3} \text{ (Ans)}$$

Area of a circle with centre at origin

As shown in figure-3, the circle with centre at origin is divided into four equal parts by the co-ordinate axes

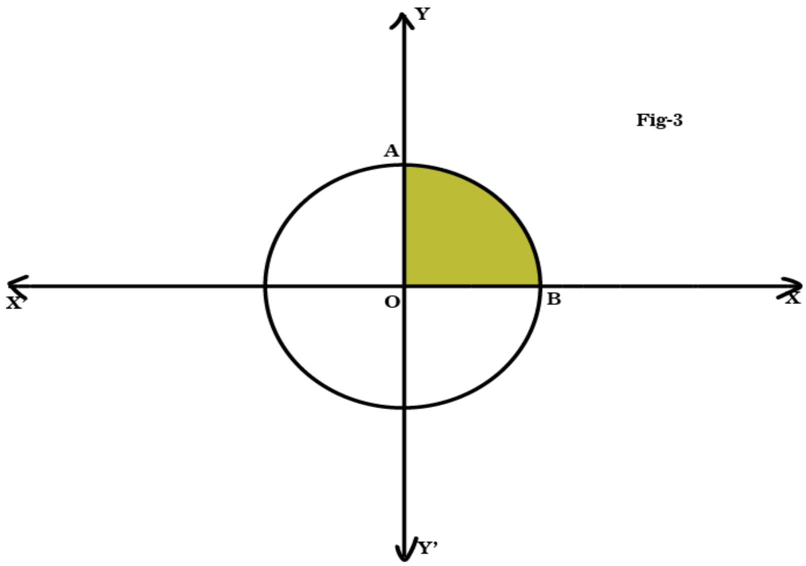

Fig-3

Hence area of the circle = 4 X area OAB

Example – 3

Find the area of the circle $x^2 + y^2 = a^2$ **(2015-S)**

Ans.

Area of circle = 4 X area OAB (see fig-4)

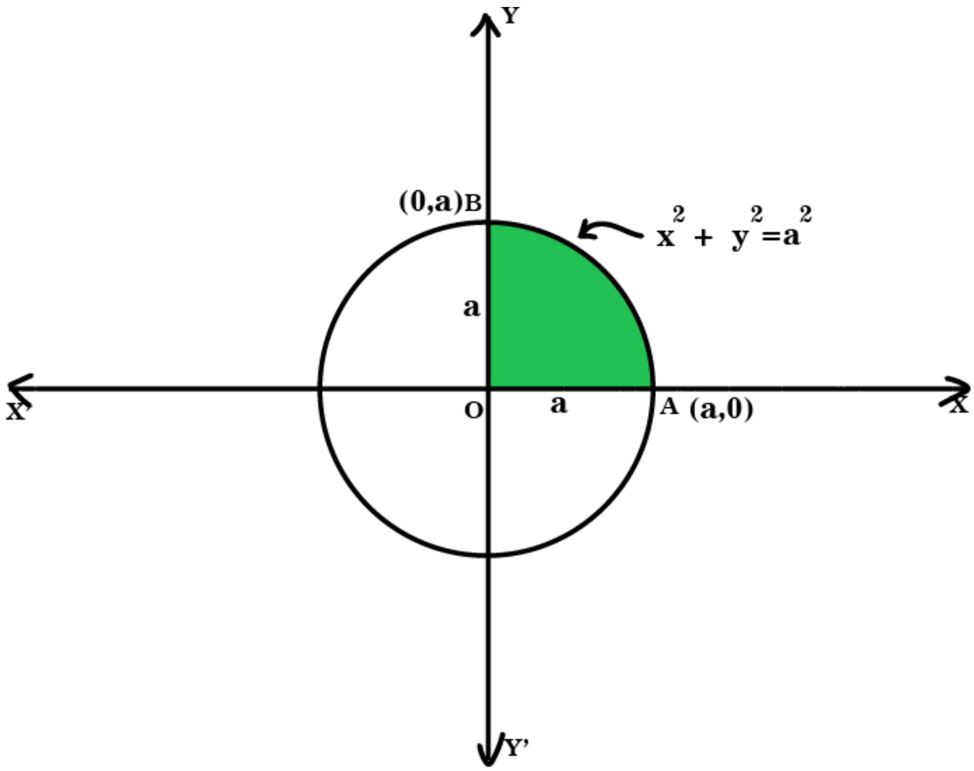

Fig-4

Now equation of circle is $x^2 + y^2 = a^2$

$\Rightarrow$ y = $\sqrt{a^2 - x^2}$ (for portion OAB)

(Actually y = $\pm\sqrt{a^2 - x^2}$) , but in 1st quadrant y is +ve)

$\Rightarrow$ y = $\sqrt{a^2 - x^2}$

Now the portion OAB is bounded by y – axis i.e. x = 0 ,

X axis and y = $\sqrt{a^2 - x^2}$

In the given region x varies from 0 to a ; as it is clear from figure the point A is (a,0)

(A = (a,0) because $x^2 + y^2 = a^2$ has radius a)

Now Area of OAB = $\int_0^a y\,dx$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - \left(0 + \frac{a^2}{2}\sin^{-1} 0\right)\right]$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 + 0 \quad = \frac{a^2}{2}\sin^{-1} 1 = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{1}{4}\pi a^2$$

Hence area of circle is = 4 X area of OAB

$$= 4 \text{ X } \frac{1}{4}\pi a^2 = \pi a^2 \text{ sq units}$$

Example – 2

Find the area bounded by the curve $x^2 + y^2 = 9$**(2017-S)**

Ans: - Area of the curve $x^2 + y^2 = 9$ i.e. circle = 4 X Area OAB { from fig-5}

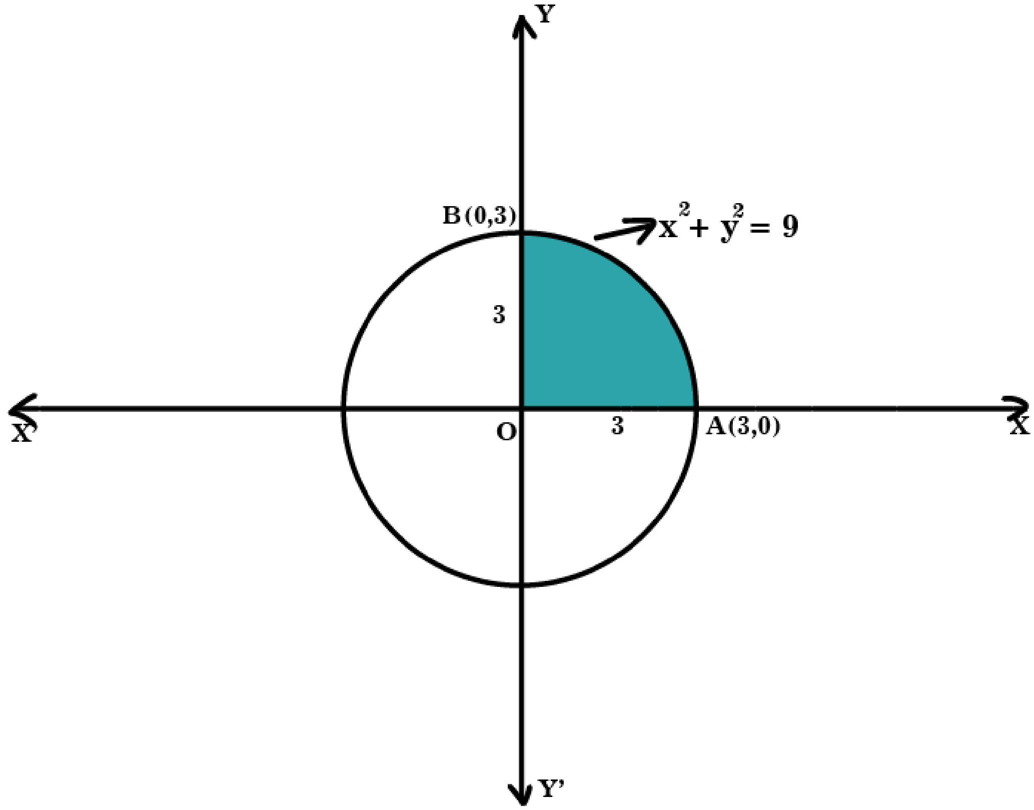

Fig-5

As $x^2 + y^2 = 9$ has radius 3

So, A is at (3,0)

Area of OAB is the area bounded by curve AB , Y axis and X axis.

Now Curve AB is $x^2 + y^2 = 9 \Rightarrow y = \sqrt{9 - x^2}$

(as in 1st quadrant y is +ve)

In the region OAB x varies from 0 to 3.

Now area of OAB = $\int_0^3 y\, dx = \int_0^3 \sqrt{9 - x^2}\, dx$

$= [\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}]_0^3$

$= [\frac{3}{2}\sqrt{9 - 9} + \frac{9}{2}\sin^{-1}(\frac{x}{3})] - [0 + \frac{9}{2}\sin^{-1} 0]$

$= 0 + \frac{9}{2}\sin^{-1} 1 - 0$

$= \frac{9}{2} X \frac{\pi}{2} = \frac{9\pi}{4}$

∴ Area bounded by the curve $x^2 + y^2 = 9$ is = 4 X Area of OAB

$= 4 X \frac{9\pi}{4} = 9\pi\ sq\ units$ (Ans)

Exercise

Q.1 Find the area bounded by the curve xy = c^2 , y = 0 , x = 2 and x = 3. (2 marks)

Q.2 Find the area bounded by the curve $x^2 + y^2 = 4$. **(2015-S)** (10 marks)

Q.3 Find the area of the circle $x^2 + y^2 = 16$. (10 marks)

Ans . (1)$c^2 \log\frac{3}{2}$

(2) 4π sq units

(3) 16 π sq units

Differential Equation

Introduction

After the discovery of calculus, Newton and Leibnitz studied differential equation in connection with problem of Physics especially in theory of bending beams, oscillation of mechanical system etc. The study of differential equation is a wide field in pure and applied mathematics, physics and engineering.

Objectives

1. Define differential equation.
2. Determine order and degree of differential equation.
3. Form differential equation from a given solution.
4. Solve differential equation by using different techniques.

Definition of Differential Equation

A differential equation is an equation involving dependent variables, independent variables and derivatives of dependent variables with respect to one or more independent variables.

Here x is an independent variable, y is dependent variable and $\frac{dy}{dx}$ is the derivative of the dependent variable w.r.t. the independent variable.

Examples: - $i)\ \frac{dy}{dx} + xy = x^2$

$ii)\ \frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = sinx$

$iii)\ \frac{dy}{dx} + sinx = cosx.$

Differential equations are of two types as follows: -

Ordinary Differential Equation

An ordinary differential equation is an equation involving one dependent variables, one independent variable and derivatives of dependent variablewith respect to independent variable.
Mathematically F $\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\right) = 0$

Examples :- $i)\ \frac{dy}{dx} + y = x^2$

$ii)\ \frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + y = \sin x$

$$iii)\ \frac{dy}{dx} + \text{ytanx} = \text{secx}$$

Partial Differential Equation

A partial differential equation is an equation involving dependent variables, independent variables and partial derivatives of dependent variable with respect to independent variables.

Examples:- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

In this chapter we only discuss about the Ordinary differential equation.

Order and Degree of Differential equation

Order

The order of the differential equation is the highest order of the derivatives occurring in it i.e. order of a differential equation is 'n' if the order of the highest order derivative term present in the equation is n.

Example1: - $\frac{dy}{dx} + y = 2x$

The highest order derivative term in the equation is $\frac{dy}{dx}$, which has order 1.

∴order of the differential equation is 1.

Example-2 : - $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + y = x$

The highest order derivative term is $\frac{d^4y}{dx^4}$, having order 4.

Hence the above differential equation has order 4.

Degree

A differential equation is said to be of degree 'n' , if the power i.e. highest exponent of the highest order derivative in the equation is 'n' after the equation has been freed from fractions and radicals as far as derivatives are concerned.

Before finding degree of a differential equation, first we have to eliminate those derivative terms present in fraction form i.e. in the denominator and derivatives with radicals i.e $\sqrt{\frac{dy}{dx}}$, $\sqrt[3]{\frac{dy}{dx}}$, $\sqrt[4]{\frac{dy}{dx}}$ terms.

Example:- Find the order and degree of following ordinary differential equations.

$i)\quad \frac{d^2y}{dx^2} = 3(\frac{dy}{dx})^4 + x \qquad ii)\quad (\frac{d^4y}{dx^4})^3 + \frac{d^3y}{dx^3} + 3(\frac{dy}{dx})^4 + \cos x = 0$

$iii)\quad \frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2}$ **(2017-W)**

$iv)\quad [1 + (\frac{dy}{dx})^2]^{\frac{3}{2}} = m\frac{d^2y}{dx^2}$ **(2016-S)**

v) $(\frac{dy}{dx})^2 + \frac{1}{\frac{dy}{dx}} = 2$ ***vi)*** $\frac{\sqrt{1+(\frac{dy}{dx})^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$

Ans: - ***i)*** $\frac{d^2y}{dx^2} = 3(\frac{dy}{dx})^4 + x$

Here $\frac{d^2y}{dx^2}$ is the highest order derivative term.

Hence order of the differential equation is 2.

Again equation does not contain any derivative term in fractional form or with radical.

Power of the highest order derivative term $\frac{d^2y}{dx^2}$ is 1.

Hence degree of differential equation is 1.

ii) $(\frac{d^4y}{dx^4})^3 + \frac{d^3y}{dx^3} + 3(\frac{dy}{dx})^4 + \cos x = 0$

From above it is clear that $\frac{d^4y}{dx^4}$ is the highest order derivative term with power 3.

Hence order = 4 and degree = 3

iii) $\frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2}$

{As the above equation contain square root, so first we have to remove square root .}

Squaring both sides we have, $(\frac{d^2y}{dx^2})^2 = 1 + (\frac{dy}{dx})^2$

Now $\frac{d^2y}{dx^2}$ is the highest order term with power 2.

$\therefore$ order = 2 and degree = 2 .

iv) $[1+(\frac{dy}{dx})^2]^{\frac{3}{2}} = m\,\frac{d^2y}{dx^2}$

As power of the left hand side derivative term is $\frac{3}{2}$, so we have to eliminate the fractional power.

Now squaring both sides we have,

$[1+(\frac{dy}{dx})^2]^3 = (m\,\frac{d^2y}{dx^2})^2$

The power of highest order derivative term $\frac{d^2y}{dx^2}$ is 2.

Hence order = 2 and degree = 2 .

v) $(\frac{dy}{dx})^2 + \frac{1}{\frac{dy}{dx}} = 2$

As $\frac{dy}{dx}$ is present in the denominator of 2nd term in L.H.S. , so we have to remove it first.

Multiplying both side by $\frac{dy}{dx}$ we have,

$(\frac{dy}{dx})^3 + 1 = 2\,\frac{dy}{dx}$

Now the only derivative term $\frac{dy}{dx}$ has power 3.

Hence order = 1 and degree = 3.

vi) $\frac{\sqrt{1+(\frac{dy}{dx})^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$

The equation contain both fractional form as well as radicals, so we have to remove it.

1st multiplying $\frac{d^2y}{dx^2}$ on both sides we have,

$\sqrt{1+(\frac{dy}{dx})2} = \sqrt[3]{\frac{d^2y}{dx^2}}\frac{d^2y}{dx^2}$

Now squaring both sides we have,

$1+(\frac{dy}{dx})^2 = (\frac{d^2y}{dx^2})^{\frac{2}{3}}(\frac{d^2y}{dx^2})^2$

Again taking cube of both sides we have

$$(1 + (\frac{dy}{dx})^2)^3 = (\frac{d^2y}{dx^2})^2 (\frac{d^2y}{dx^2})^6 = (\frac{d^2y}{dx^2})^8$$

From above the highest order derivative term $\frac{d^2y}{dx^2}$ has power 8.

Hence order = 2 and degree = 8.

Linear and Non-linear Differential Equation

A differential equation is said to be linear if it satisfies following conditions.

i) Every dependent variable and its derivatives have power '1'.

ii) The equation has neither terms having multiplication of dependent variable with its derivatives nor multiplication of two derivative terms.

Otherwise the equation is said to be non linear.

Examples:- i) $\frac{dy}{dx} + xy = x^2$

ii) $\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2} + y = \sin x$

iii) $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 4x$

iv) $(\frac{dy}{dx})^3 + 1 = 2\frac{dy}{dx}$

v) $\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + y = 4x^2$

Among the above examples (i) and (ii) satisfy all the condition of linear equation. So the 1st two equations represent linear equations.

The (iii) is non linear because of the term $y\frac{d^2y}{dx^2}$ which is a multiplication of dependent variable y and derivative term $\frac{d^2y}{dx^2}$.

The example (iv) is not linear due to the 1st term which contain $\frac{dy}{dx}$ with power 3 violating the 1st condition of linearity.

The example (v) is not linear due to 2nd term which does satisfy the 2nd linearity property.

Solutionof a differential equation:-

The relationship between the variables of a differential equation satisfying the differential equation is called a Solution of the differential equation i.e y = f(x) or F(x,y)=0 represent a Solution of the ordinary differential equation F $\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$ of order n if it satisfy it.

There are two types of Solutions i) General Solution ii) particular Solution

General Solution

The Solution of a differential equation containing as many arbitrary constants as the order of the differential equation is called as the general Solution.

Example- y= $A\cos x + B\sin x$ is a general Solution of differential equation $\frac{d^2y}{dx^2}$+y = 0

Particular Solution

The Solution obtained by giving particular values to the arbitrary constants in the general solution is called particular solution

Example: - y= 3 cosx+ 2 sinx is a particular Solution of differential equation $\frac{d^2y}{dx^2}$+y = 0.

Differential equation of first order and first degree

A differential equation of 1st order and 1st degree involves x,y and $\frac{dy}{dx}$.

Mathematically it is written as $\frac{dy}{dx} = f(x,y)$ or $F(x, y, \frac{dy}{dx}) = 0$

Solution of Differential equation of first order and first degree

The Solution of 1st order and 1st degree differential equation is obtained by following methods if they are in some standard forms as i) Variable separable form ii) Linear differential equation form.

Variable Separable form

If the differential equation is expressed in the form,

$f(x)dy + g(y)dx = 0$, then we say it variable separable form and this can be solved by integrating the terms separately as follows.

Solution is given by $\int \frac{dy}{g(y)} = -\int \frac{dx}{f(x)}$

$$=> \log|g(y)| + \log|f(x)| = \log c$$

$$=> g(y)f(x) = c$$

Where $g(y)$ and $f(x)$ are functions of y and x respectively, is called a variable and separable type equation.

Example1: - Solve $\frac{dy}{dx} = x^2 + 2x + 5$

Ans: - $\frac{dy}{dx} = x^2 + 2x + 5$

=> dy = $(x^2 + 2x + 5)dx$

Integrating both sides we have,

=> $\int \text{dy} = \int (x^2 + 2x + 5)\, dx$

=> y = $\frac{x^3}{3} + \frac{2x^2}{2}$ + 5x +C= $\frac{x^3}{3} + x^2 + 5x + c$ (Ans)

Example-2: - Solve $\frac{dy}{dx} = \frac{2y}{x^2+1}$.

Ans: - $\frac{dy}{dx} = \frac{2y}{x^2+1}$

=> $\frac{dy}{2y} = \frac{dx}{x^2+1}$

=> $\int \frac{dy}{2y} = \int \frac{dx}{x^2+1}$

=> $\frac{1}{2} \log_e y = tan^{-1}x$ + C

=> $\log_e y$ = 2$tan^{-1}x$ + K { 2C = K is a constant as C is constant}

Example-3: - Solve $\frac{dy}{dx}$ = x cosx

Ans: - $\frac{dy}{dx} = \text{x cosx}$ => $dy = xcosxdx$

Integrating both sides we have, => $\int dy = \int xcosx\, dx$

=> y = $x \int cosx\, dx - \int \{\frac{d(x)}{dx} \int cosxdx\} dx$ {integrating by parts}

=> y = x sinx – $\int 1. sinxdx$ = x sinx +cosx + C (Ans)

Example-4: - Solve $\frac{dy}{dx} = \sqrt{1 - y^2}$

Ans:- $\frac{dy}{dx} = \sqrt{1 - y^2}$

=> $\frac{dy}{\sqrt{1-y^2}}$ = dx

=> $\int \frac{dy}{\sqrt{1-y^2}} = \int dx$ { Integrating both sides}

=> $sin^{-1}y = x + c$. (Ans)

Example5: - Solve $sec^2x\ tany\ dx + sec^2y\ tanx\ dy = 0$

Ans: -$sec^2x\ tany\ dx + sec^2y\ tanx\ dy = 0$

$=> sec^2y\ tanx\ dy = - sec^2x\ tany\ dx$

$=> \frac{sec^2y\ dy}{tany} = -\frac{sec^2x\ dx}{tanx}$

$=> \int \frac{sec^2y\ dy}{tany} = -\int \frac{sec^2x\ dx}{tanx}$

Let u = tany => du = $sec^2y\ dy$ and let v= tanx => dv = $sec^2x\ dx$

$=> \int \frac{du}{u} = -\int \frac{dv}{v}$

$=> \ln u = - \ln v + \ln C => \ln u + \ln v = \ln C$

$=> \ln uv = \ln C => uv = C =>$ tany tanx = C (Ans)

Example-6: - Solve $x\ cos^2y\ dx = y\ cos^2x\ dy$

Ans:-$x\ cos^2y\ dx = y\ cos^2x\ dy$

$=> \frac{ydy}{cos^2y} = \frac{xdx}{cos^2x}$

$=> ysec^2y\ dy = xsec^2x\ dx$

Integrating both sides,

$=> \int ysec^2y\ dy = \int xsec^2x\ dx$

$=> y\int sec^2y\ dy - \int\{\frac{d(y)}{dy}.\int sec^2y\ dy\}dy = \int xsec^2x\ dx - \int\{\frac{d(x)}{dx}.\int sec^2x\ dx\}dx$

=> y tany – $\int 1.tany\ dy$ = x tanx - $\int 1.tanx\ dx$

=> y tany – log IsecyI = x tanx – log IsecxI+ C

=> y tany – log IsecyI -x tanx+ log IsecxI = C (Ans)

Example-7: - Solve $(1 + y^2)dx + (1 + x^2)\ dy = 0$ **(2015-S)**

Ans :- $(1 + y^2)dx + (1 + x^2)\ dy = 0$

$=> (1 + x^2)\ dy = - (1 + y^2)dx$

$=> \frac{dy}{1+y^2} = -\frac{dx}{1+x^2}$ $=> \int \frac{dy}{1+y^2} = -\int \frac{dx}{1+x^2}$

$=> tan^{-1}y = -tan^{-1}x + tan^{-1}C$ { as $tan^{-1}C$ can be taken as a constant}

$=> tan^{-1}y + tan^{-1}x = tan^{-1}C$

$=> tan^{-1}\frac{y+x}{1-yx} = tan^{-1}C$

$=> \frac{y+x}{1-yx} = C$ (Ans)

Example-8:- Solve $\frac{dy}{dx}$= sin(x+y)

Ans :- $\frac{dy}{dx}$= sin(x+y) { Let x+y = z differentiating w.r.t. x , $1+\frac{dy}{dx} = \frac{dz}{dx}$}

$=> \frac{dz}{dx} - 1 = \sin z$

$=> \frac{dz}{dx} = 1 + \sin z$

$=> \frac{dz}{1+\sin z} = dx$

Integrating both sides we have,

$=> \int\frac{dz}{1+\sin z} = \int dx$

$=> \int\frac{(1-\sin z)dz}{(1-\sin z)(1+\sin z)} = \int dx$

$=> \int\frac{(1-\sin z)dz}{1-\sin^2 z} = \int dx$

$=> \int\frac{(1-\sin z)dz}{\cos^2 z} = \int dx$

$=> \int(\sec^2 z - \frac{\sin z}{\cos z}\frac{1}{\cos z})dz = \int dx$

$=> \int(\sec^2 z - \tan z\sec z)dz = \int dx$

$=> \tan z - \sec z = x + C$

$=> \tan(x+y) - \sec(x+y) - x = C$

Example-9: - Find the particular solution of $\frac{dy}{dx} = \cos^2 y$, y = $\frac{\pi}{4}$ when x = 0.

Ans: -$\frac{dy}{dx} = \cos^2 y$

$=> \frac{dy}{\cos^2 y}$ = dx $=> \sec^2 y\, dy = dx$

Integrating both sides we have,

$=> \int \sec^2 y\, dy = \int dx$

=> $\tan y = x + C$------------(1) (general Solution)

Now putting $x=o$ and y = $\frac{\pi}{4}$ in equation (1) we have,

=> $\tan\frac{\pi}{4}$ = 0+ C => C = 1-------------------------(2)

From (1) and (2) we have,

$\tan y = x+1$ (Ans)

Example-10:- Find the particular solution of $(1+x)y\,dx + (1\text{-}y)x\,dy = 0$, Given $y=2$ at $x=1$.

Ans:-$(1+x)y\,dx + (1\text{-}y)x\,dy = 0$

=> $(1\text{-}y)x\,dy = -\,(1+x)y\,dx$

=> $\frac{1-y}{y}\,dy = -\frac{(1+x)}{x}\,dx$

=> $(\frac{1}{y} - 1)\,dy = -(\frac{1}{x} + 1)\,dx$

=> $\int(\frac{1}{y} - 1)\,dy = -\int(\frac{1}{x} + 1)\,dx$

=> $\log y - y = -\,(\log x + x) + C$ (general solution)-------------(1)

Putting x=1 and y =2 in Equation(1) we have,

=> log2 – 2 = - (log1 +1) +C

=> log2 – 2 = - (0 +1) +C =-1 + C

=> C = log2 -1---------------------(2)

From (1) and (2) we have,

$\log y - y = -\log x - x + \log 2 - 1$

=> $\log y - y + \log x + x = \log 2 - 1$ (Ans)

Linear Differential Equation

A differential equation is said to be linear, if the dependent variable and its derivative occurring in the equation are of first degree only and are not multiplied together.

Example: -i)$\frac{dy}{dx} + y = sinx$

ii)$\frac{dy}{dx} + ytanx = secx$ etc.

General form of linear differential equation

The general form of linear differential equation is given by,

$\frac{dy}{dx} + Py = \boldsymbol{Q}$ is linear in y and $\frac{dy}{dx}$.

Where P and Q are the functions of x only or constants.

This type of differential equation are solved when they are multiplied by a factor, which is called integrating factor (I.F.).

I.F. = $e^{\int P\,dx}$

Then the solution is given by y (I.F) = $\int Q.(I.F.)dx + C$.

If equation is given in the form

$\frac{dx}{dy} + Px = \boldsymbol{Q}$ **,** where P and Q are functions of y only or constants and is linear in x and $\frac{dx}{dy}$, then

I.F. = $e^{\int P\,dy}$

Then the solution is given by x (I.F) = $\int Q.(I.F.)dy + C$.

This can be better understood by following examples.

Example-11: - Solve $(1 + x^2)\frac{dy}{dx} + 2xy = x^3$ **(2014-S, 2016-S, 2017-W).**

Ans: - $(1 + x^2)\frac{dy}{dx} + 2xy = x^3$

$=> \frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{x^3}{1+x^2}$

By comparing with the general form of linear differential equation $\frac{dy}{dx} + Py = \boldsymbol{Q}$.

Here P = $\frac{2x}{(1+x^2)}$ & Q = $\frac{x^3}{1+x^2}$

Now integrating factor I.F. = $e^{\int P\,dx} = e^{\int \frac{2x}{(1+x^2)}dx}$ (putting $1 + x^2 = t => 2xdx = dt$)

$= e^{\int \frac{dt}{t}} = e^{lnt}$ = t = $1 + x^2$

Solution is given by

$$y \times I.F. = \int Q.(I.F.)dx + C = \int \frac{x^3}{1+x^2}(1 + x^2)dx + C = \int x^3 dx + c$$

⇨ $y(1+x^2) = \frac{x^4}{4} + c$

$\therefore y = \frac{x^4}{4(1+x^2)} + \frac{c}{1+x^2}$

Example-12: - Solve $\frac{dy}{dx} + y = e^{-x}$

Ans: - By comparing the given equation with general form of linear differential equation we have, P= 1 and Q = e^{-x}

I.F. = $e^{\int P\,dx} = e^{\int 1.dx} = e^{x}$

Solution is $y.(I.F.) = \int Q.(I.F.)dx + C$

$= \int e^{-x} e^{x} dx + c$

=> $ye^{x} = \int 1.dx + c = x + c$

=> y $= xe^{-x} + ce^{-x}$

Example-13: - Solve $(1-x^2)\frac{dy}{dx} - xy = 1$ **(2017-S)**

Ans:- $(1-x^2)\frac{dy}{dx} - xy = 1$

=> $\frac{dy}{dx} - \frac{x}{1-x^2} y = \frac{1}{1-x^2}$

Comparing with general form we have,

P = $\frac{x}{1-x^2}$ and Q = $\frac{1}{1-x^2}$

Now I.F. = $e^{\int P\,dx}$ = $e^{\int -\frac{x}{1-x^2}dx}$ (Put $1-x^2 = t$ => $-2xdx = dt$ => $-xdx = \frac{dt}{2}$)

$= e^{\int \frac{dt}{2t}} = e^{\frac{1}{2}lnt} = e^{ln\sqrt{t}} = \sqrt{t}$

$= \sqrt{1-x^2}$

Solution is $y.(I.F) = \int Q.(I.F.)dx + C$

=> $y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}dx + C = \int \frac{dx}{\sqrt{1-x^2}} + c$

$= sin^{-1}x + c$

Hence solution of the differential equation is given by

$y = \frac{sin^{-1}x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$ (Ans)

Example-14 ; - Solve $\frac{dy}{dx} + y\, cotx = cosx$.

Ans:- By comparing the given equation with general form,

P = $cotx \quad and\ Q = cosx$

I.F. = $e^{\int P\, dx} = e^{\int cotx\, dx} = e^{lnsinx} = sinx$

solution is given by $y.(I.F.) = \int Q.(I.F.)dx + C$

$$=> y.sinx = \int cosx\, sinx\, dx + c \quad (\text{ put } sinx = t => cosxdx = dt)$$

$$= \int tdt + c = \frac{t^2}{2} + c = \frac{sin^2 x}{2} + c$$

Hence $y = \frac{sinx}{2} + \frac{c}{sinx}$ (Ans)

Example-15 ; - Solve $\frac{dy}{dx} + y\, secx = tanx$. **(2017-W).**

Ans: - Comparing the given equation with general from of linear equation.

P = secx and Q = tanx

I.F. = $e^{\int Pdx} = e^{\int secx\, dx}$

= $e^{\ln(secx+tanx)}$

$= secx + tanx$

The solution is given by, $y.I.F. = \int Q.(I.F.)dx + C$

⇨ $y(secx + tanx) = \int tanx.(secx + tanx)dx + C$

$= \int (secxtanx + tan^2x)dx + C$

$= \int (secxtanx + sec^2x - 1)dx + C$

$= secx + tanx - x + c$

Hence $y = 1 - \frac{x}{secx+tanx} + \frac{c}{secx+tanx}$ (Ans)

Example16: - Solve $(x + y + 1)\frac{dy}{dx} = 1$.

Ans:- $(x + y + 1)\frac{dy}{dx} = 1$

$=> (x + y + 1)dy = dx$

$=> (x + y + 1) = \frac{dx}{dy}$

=> $\frac{dx}{dy} = (x+y+1)$

=> $\frac{dx}{dy} - x = y+1$

Now comparing with the general form $\frac{dx}{dy} + Px = Q$, we have,

P = -1 & Q = y+1

Now I.F. = $e^{\int Pdy} = e^{\int -1dy} = e^{-y}$.

The solution is given by

$x\,(I.F.) = \int Q.(I.F.)dy + c$

⇨ $xe^{-y} = \int (y+1).e^{-y} + c$

$= (y+1)\int e^{-y}dy - \int (\int e^{-y}dy)\,(\frac{d}{dy}(y+1))\,dy + c$

$= (y+1)(-e^{-y}) - \int -e^{-y}.1.dy + c$

$= -e^{-y}(y+1) + (-e^{-y}) + c$

$= -e^{-y}(y+1+1) + c = -e^{-y}(y+2) + c$

Hence $x = -y - 2 + ce^{y}$ (Ans)

Exercise

Question with short answers (2 marks)

1. Find the order and degree of the differential equations.

i) $a\frac{d^2y}{dx^2} = [1 + (\frac{dy}{dx})^3]^{3/2}$ **(2016-S)**

ii) $\frac{d^2y}{dx^2} = (\frac{dy}{dx})^{2/3}$ **(2019-W)**

iii) $\frac{d^3y}{dx^3} = \sqrt{x + (\frac{dy}{dx})^5}$ **(2015-S,2017-W)**

iv) $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1+\frac{dy}{dx}}}$ **(2014-S)**

v) $2\frac{d^2y}{dx^2} + 3\sqrt{1 - (\frac{dy}{dx})^2} - y = 0.$ **(2017-S)**

vi) $(\frac{dy}{dx})^2 + y^3 = \frac{d^3y}{dx^3}$

2) Solve the following

i) $\frac{dy}{dx} = e^{x+y}$ **(2019-W)**

ii) $\frac{dy}{dx} = \frac{2y}{x^2+1}$

iii) $\frac{dy}{dx} = tany$

iv) $cos^2x\, dy = cos^2y dx$

v) $ydx = xdy$

Questions with long answers (5 marks)

3) Solve the following

i) $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ **(2019-W)**

ii) $\frac{dy}{dx} + ytanx = secx$ **(2015-S)**

iii) $x(1 + y^2)dx - y(1 + x^2)dy = 0$ **(2015-S)**

iv) $\frac{dy}{dx} - y = e^x$

v) $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$.**(2016-S)**

vi) $\frac{dy}{dz} = \frac{\sqrt{1-y^2}}{\sqrt{1-z^2}}$ **(2014-S)**

vii) $\frac{dy}{dx} + ycotx = 4xcosecx$ **(2017-S)**

viii)$(x + 2y^3)\frac{dy}{dx} = y$.

ix) $(e^y + 1)cosxdx + e^y sinxdy = 0$.

x) $sinx\ siny\ dy = cosx\ cosy\ dx$

xi) $\frac{dy}{dx} = \frac{xy+y}{xy+x}$

Questions with long answers (10 marks)

4) Solve the following.

i) $e^x\sqrt{1 - y^2}\,dx + \frac{y}{x}dy = 0$ **(2014-S)**

ii) $cos^2x\frac{dy}{dx} + y = tanx$. **(2017-S)**

iii) $\cos(x + y)\,dy = dx$.

Answers

1) i) 2,2 ii)2,3 iii) 3,2 iv) 2,2 v) 2,2 vi) 3,1

2) i) $-e^y = e^x + c$ ii) $lny = 2tan^{-1}x + c$ iii) $\log(siny) = x + c$

iv) tany-tanx = c v) $\frac{y}{x} = c$

3) i) $y(1 + x^2) = \frac{4x^3}{3} + c$ ii) $ysecx = tanx + c$ iii) $\frac{1+y^2}{1+x^2} = c$

iv) $y = xe^x + ce^x$ v) $y(x^2 - 1) = x + c$ vi) $sin^{-1}(y\sqrt{1 - z^2} - z\sqrt{1 - y^2}) = c$

vii)$ysinx = 2x^2 + c$ viii) $x = y^3 + cy$ ix) $sinx(e^y + 1) = c$

x) $secy\ cosecx = c$ xi) $cx = ye^{y-x}$

4) i) $\sqrt{1 - y^2} = e^x(x - 1) + c$ ii) $ye^{tanx} = e^{tanx}(tanx - 1) + c$

iii) $y = \tan(\frac{x+y}{2}) + c$

Multiple Choice Questions

Q.1 The value of α for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \alpha\hat{j} + 3\hat{k}$ are perpendicular is

a) 10 b) -10 c) 15 d) -15

Q2. If the vectors $\vec{a} = \alpha\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ are parallel then the value of α is

a) -3 b) 3 c)-4 d) 4

Q3. A unit vector in the direction $(\vec{a} + \vec{b})$ where $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = = \hat{i} - 2\hat{j} + 3\hat{k}$ is equal to

a) $\frac{1}{2}\hat{i} + \frac{1}{2}\hat{k}$ b) $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$ c) $\frac{2}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{k}$ d) None of these

Q4. If points A and B have the following co-ordinates A(3,0,2) , B(-2,1,4) , then the vector AB is

a) $5\hat{i} + \hat{j} + 2\hat{k}$ b) $-5\hat{i} - 2\hat{j} + 2\hat{k}$ c) $-5\hat{i} + \hat{j} + 2\hat{k}$ d) $-5\hat{i} - \hat{j} - 2\hat{k}$

Q5. A unit vector parallel to the sum of the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $-2\hat{i} + \hat{j} - 3\hat{k}$ is

a) $\frac{1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\hat{k}$ b) $\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$ c) $\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{k}$ d) $\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j}$

Q6. The scalar projection of $\vec{a} = (\hat{i} - 2\hat{j} + \hat{k})$ on $\vec{b} = (4\hat{i} - 4\hat{j} + 7\hat{k})$ is

a) $\frac{19}{6}$ b) $\frac{19}{3}$ c) $\frac{19}{9}$ d) $\frac{19}{7}$

Q7. The unit vector along $\hat{i} + \hat{j} + \hat{k}$ is

a) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ b) $\frac{\hat{i} + \hat{j} + \hat{k}}{3}$ c) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{5}}$ d) None of these

Q8. The value of $(\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k})$ is

a) -1 b) 1 c) 2 d) -2

Q9. Two forces act on a particle at a point. If they are $(4\hat{i} + \hat{j} - 3\hat{k})$ and $(3\hat{i} + \hat{j} - \hat{k})$, then their resultant is

a) $7\hat{i} + 2\hat{j} - 4\hat{k}$ b) $7\hat{i} - 2\hat{j} - 4\hat{k}$ c) $-7\hat{i} + 2\hat{j} - 4\hat{k}$ d) $-7\hat{i} - 2\hat{j} - 4\hat{k}$

Q10. The magnitude of $5\hat{i} + 3\hat{j} - 2\hat{k}$ is

a) $\sqrt{35}$ b) $\sqrt{37}$ c) $\sqrt{38}$ d) $\sqrt{40}$

Q11. The value of $\lim_{x \to 0^+} \frac{|x|}{x}$ is

a)-1 b) 1 c) 0 d) None of these

Q12. The value of $\lim_{x\to\infty}\frac{2x}{3+4x}$ is

a)-1 b) 1 c) $\frac{1}{2}$ d) - $\frac{1}{2}$

Q13. The value of $\lim_{x\to b}\frac{x^2-b^2}{x-b}$ is

a) b b) $\frac{1}{2}b$ c) 2b d) –b

Q14. $\lim_{x\to 2}\frac{x^3-8}{x-2}$ is equal to

a) 4 b) 3 c) 12 d) 6

Q15. The value of $\lim_{x\to 1}\frac{\sqrt{x-1}}{x-1}$ is

a) 0 b) -1 c) 1 d) does not exists

Q16. The value of $\lim_{x\to\frac{5}{2}}[x]$ is

a) 2 b) 3 c) 2.5 d) None of these

Q17. The value of $\lim_{\theta\to 0}\frac{\sin 7\theta}{\sin 5\theta}$ is

a) $\frac{5}{7}$ b) $\frac{7}{5}$ c) 1 d) 0

Q18. If f(x) = $\frac{x^2-16}{x-4}$, x $\neq 4$ is continuous at x = 4, then the value of f(4) is

a) 8 b) 4 c) 10 d) 16

Q19. The value of k for which f(x) = $\frac{\sin^2 Kx}{x^2}$, x $\neq 0$, f(0) = 1 is continuous at x = 0 is

a) ± 2 b) $\pm\frac{1}{2}$ c) 0 d) ± 1

Q20. The value of $\lim_{x\to 1}\frac{2x^2-3x+1}{x^2-1}$ is

a) $\frac{1}{2}$ b) $\frac{3}{2}$ c) 2 d)1

Q21. The slope of the curve $y=\frac{5}{3}x^2$ at x = 2

a) $\frac{10}{3}$ b) $\frac{20}{3}$ c) $\frac{5}{3}$ d) $\frac{25}{3}$

Q22. If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, then $\frac{dy}{dx}$ is

a) tan x b) cot x c) $sec^2\, x$ d) - $cosec^2 x$

Q23. The derivative of x w.r.t tan x is

a) $sec^2 x$ b) $cos^2 x$ c) $-tan^2 x$ d) $-cot^2 x$

Q24. If $x = 4t$, $y = t^2$, then $\frac{dy}{dx}$ is

a) $\frac{2}{x}$ b) $\frac{t}{2}$ c) $\frac{2}{t}$ d) $\frac{x}{2}$

Q25. If $y = \sqrt{1 - \cos 2x}$, then $\frac{dy}{dx}$ is

a) 2sin 2x b) $\sqrt{2}$ cos x c) - $\sqrt{2}$ sin x d) None of these

Q26. If $y = \sqrt{1 + sin2x}$, then $\frac{dy}{dx}$ is

a) cos x b) sin x c) 0 d) None of these

Q27. The derivative of $\sin x^0$ is

a) $\cos x^0$ b) $\frac{\pi}{180} \cos x^0$ c) $\pi \sec x^0$ d) None of these

Q28. If y = log (tan x) , then $\frac{dy}{dx}$ is

a) $\frac{1}{\sin 2x}$ b) $\frac{1}{\cos 2x}$ c) $\frac{2}{\sin 2x}$ d) $\frac{2}{\cos 2x}$

Q29. If $y = \sin^{-1} x + \cos^{-1} x$, then $\frac{dy}{dx}$ is

a) 1 b) -1 c) 0 d) 2

Q30. If $x^2 + y^2 = a^2$, then $\frac{dy}{dx}$ is

a) 2x b) 0 c) $\frac{x}{y}$ d) $-\frac{x}{y}$

Q31. If $y = \log_e x$ then y_2 is

a) $\frac{1}{x}$ b) $-\frac{1}{x^2}$ c) x d) $-x^2$

Q32. $\frac{d^2y}{dx^2}$ when $y = e^x \sin x$, is

a) $2e^x \cos x$ b) $2e^x \sin x$ c) $e^x \cos x$ d) $e^x \sin x$

Q33. If $y = \ln(\sin x)$, then y_2 is

a) cot x b) tan x c) sec^2x d) $-cosec^2x$

Q34. If $y = e^{\sin^{-1} x}$, then $(1 - x^2)\, y_2 - xy_1$ is equal to

a) y b) -y c) 0 d) None of these

Q35. If $y = \tan^{-1} x$, then $(1 + x^2)\, y_2 + 2xy_1$ is

a) $\frac{1}{1+x^2}$ b) $-\frac{1}{1+x^2}$ c) 1 d) 0

Q36. The function whose 2nd derivative is itself is

a) x b) log x c) e^x d) non of these

Q37. If $f(x,y) = e^{xy}$, then $y.\frac{\partial f}{\partial y} - x.\frac{\partial f}{\partial x}$ is

a) $2x\, e^{xy}$ b) $2ye^{xy}$ c) 0 d) None of these

Q38. If $z = \tan^{-1}(\frac{y}{x})$ then $\frac{\partial z}{\partial x}$ is

a) $\frac{x}{x^2+y^2}$ b) $-\frac{y}{x^2+y^2}$ c) $\frac{1}{x^2+y^2}$ d) $-\frac{1}{x^2+y^2}$

Q39. If $z = f(\frac{x}{y})$, then $\frac{\partial z}{\partial y}$ is

a) $-\frac{x}{y^2}f'(x/y)$ b) $\frac{x}{y^2}f'(x/y)$ c)) $-\frac{1}{y^2}f'(x/y)$ d) None of these

Q40. If $z = x^2y + xy^2$, then $\frac{\partial z}{\partial y}$ is

a) $2xy + y^2$ b) $x^2 + 2xy$ c) 2xy d) 4xy

Q41 . If $z = \sin(x/y)$ then $\frac{\partial z}{\partial x}$ is

a) $\frac{1}{y}\cos(x/y)$ b) $-\frac{x}{y}\cos(x/y)$ c) $\cos(x/y)$ d)) None of these

Q42. $\int sin^2 \frac{x}{2}\, dx$ is equal to

a) $\frac{1}{2}[x - \cos x] + c$ b) $\frac{1}{2}[x - \sin x] + c$ c) $(x - \sin x) + c$ d) $(x - \cos x) + c$

Q43. Evaluation of $\int \sqrt{1 - \sin 2x}\; dx$ is

a) $(\sin x + \cos x) + c$ b) $\sin x - \cos x + c$ c) $-\sin x + \cos x + c$ d) None of these

Q44. $\int \frac{x^2}{x^2+1}\,dx$ is

a) $x + \tan^{-1} x + c$ b) $\tan^{-1} x + c$ c) $2\tan^{-1} x + c$ d) $x - \tan^{-1} x + c$

Q45. $\int \log e^x dx$ is

a) $\frac{x^2}{2} + c$ b) $2x^2 + c$ c) $x^2 + c$ d) $(x^2 + 1) + c$

Q46. The value of 'n' for which $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ is not true is

a) n = 1 b) n = -1 c) n = 0 d) None of these

Q47. $\int \sin\frac{x}{2} . \cos\frac{x}{2}\, dx$ is

a) $\frac{1}{2}\cos x + c$ b) $\frac{1}{2}\sin x + c$ c) $-\frac{1}{2}\cos x + c$ d) $-\frac{1}{2}\sin x + c$

Q48. The value of $\int e^{(\sin^{-1} x + \cos^{-1} x)}\, dx$ is

a) $e^{\frac{\pi}{2}} x + c$ b) $x + c$ c) $e^{\frac{\pi}{2}} + c$ d) None of these

Q49. The value of $\int |x| dx$, when x < 0 is

a) $\frac{x^2}{2} + c$ b) $-\frac{x^2}{2} + c$ c) $x^2 + c$ d) $-x^2 + c$

Q50. $\int 2^{x+2} dx$ is

a) $2.\frac{2^x}{\log 2} + c$ b) $4.\frac{2^x}{\log 2} + c$ c) $8.\frac{2^x}{\log 2} + c$ d) None of these

Q51. The value of $\int sin^2 x\, d(\sin x)$ is

a) $\frac{sin^3 x}{3} + c$ b) $\frac{sin^2 x}{2} + c$ c) $\frac{cos^2 x}{2} + c$ d) $\frac{cos^3 x}{3} + c$

Q52. Evaluation of $\int_0^{\frac{\pi}{2}} \sin x\, dx$ is

a) -1 b) 1 c) 0 d) $\frac{1}{2}$

Q53. The value of $\int_1^2 x^3\, dx$ is

a) $\frac{17}{3}$ b) $\frac{15}{4}$ c) $\frac{17}{4}$ d) None of these

Q54. The value of $\int_{-3}^{4} |x| dx$ is

a) $\frac{25}{2}$ b) $\frac{7}{2}$ c) $\frac{9}{2}$ d) $\frac{23}{2}$

Q55. The value of $\int_1^3 [x]dx$ is

a) 1 b) 2 c) 3 d) 4

Q56. The value of $\int_0^1 \frac{dx}{1+x^2}$ is

a) $\frac{\pi}{4}$ b) 0 c) $\frac{\pi}{2}$ d) None of these

Q57. The value of $\int_0^4 \frac{dx}{\sqrt{x}}$ is

a) 6 b) 4 c) 8 d) 10

Q58. The value of $\int_0^1 sin^2x\ dx + \int_0^1 cos^2 x\ dx$ - $\int_0^1 dx$ is

a) 0 b) -1 c) 1 d) None of these

Q59. The area bounded by y = x, x = 0 & x = 1 is

a) 1 sq. Unit b) $\frac{1}{2}$ sq. Unit c) $\frac{1}{3}$ sq. Unit d) None of these

Q60. The area bounded by the curve $xy = k^2$, the x-axis and x = 2, x = 3 is

a) $k^2 \log\frac{2}{3}$ sq. unit b) $k^2 \log 2$ sq. Unit c) $k^2 \log 3$ sq. Unit d) $k^2 \log\frac{3}{2}$ sq. Unit

Q61. The order and degree of the differential equation $(\frac{dy}{dx})^4 + y^5 = \frac{d^3y}{dx^3}$ is

a) 3 and 1 b) 3 and 4 c) 1 and 4 d) 1 and 3

Q62. The order and degree of the differential equation $\frac{d^2y}{dx^2} = k[1 + (\frac{dy}{dx})^2]$ is

a) 2,2 b) 2, 1 c) 1,2 d) 1,1

Q63. The order and degree of the differential equation $\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$

a) 2,2 b) 2, 1 c) 1,2 d) 1,1

Q64. The degree of the differential equation $\frac{dy}{dx} = \frac{3}{\frac{dy}{dx}}$ is

a) 3 b) 2 c) 1 d) None of these

Q65. The solution of $\frac{dy}{dx} = \frac{x}{y}$ is

a) $x^2 + y^2 = c$ b) $\frac{x^2}{2} + y^2 = c$ c) $y^2 - x^2 = c$ d) $-y^2 - x^2 = c$

Q66. The solution of $\sqrt{4+\frac{dy}{dx}}=2$ is

a) $y = x + c$ b) $y = c$ c) $y + x = c$ d) $x = c$

Q67. The solution of $\frac{dy}{dx} = sec^2 x$ is

a) $y = 2 \sec x + c$ b) $y = \cot x + c$ c) $y = \tan x + c$ d) $y = \text{cosec } x + c$

Q68. The integrating factor of the linear differential equation $\frac{dy}{dx} + (\sec x)y = \tan x$ is

a) sec x + tan x b) cosec x – cot x c) tan x + cot x d) None of these

Q69. The I.F of the linear differential equation $\frac{dy}{dx} + \frac{3}{x}.y = x$ is

a) x^2 b) x^3 c) x^4 d) x

Q70. The solution of the differential equation $\frac{dx}{dy} + \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} = 0$ is

a) $\sin^{-1} x + \sin^{-1} y = c$ b) $\cos^{-1} x - \cos^{-1} y = c$

c) $\sin^{-1} x - \sin^{-1} y = c$ d) None of these

Answers: -

Q1. (d)	Q2. (a)	Q3. (b)	Q4. (c)	Q5. (a)	Q6. (c)
Q7. (a)	Q8. (b)	Q9. (a)	Q10. (c)	Q11. (b)	Q12.(c)
Q13. (c)	Q14. (c)	Q15. (d)	Q16. (a)	Q17. (b)	Q18. (a)
Q19. (d)	Q20. (a)	Q21.(b)	Q22. (c)	Q23. (b)	Q24. (b)
Q25. (b)	Q26.(d)	Q27. (b)	Q28. (c)	Q29. (c)	Q30. (d)
Q31.(b)	Q32.(a)	Q33. (d)	Q34. (a)	Q35.(d)	Q36.(c)
Q37.(c)	Q38. (b)	Q39. (a)	Q40.(b)	Q41.(a)	Q42.(b)
Q43. (a)	Q44.(d)	Q45. (a)	Q46. (b)	Q47.(c)	Q48.(a)
Q49.(b)	Q50.(b)	Q51.(a)	Q52.(b)	Q53. (b)	Q54.(a)
Q55. (c)	Q56. (a)	Q57.(b)	Q58.(a)	Q59.(b)	Q60.(d)
Q61. (a)	Q62. (b)	Q63.(a)	Q64.(b)	Q65.(c)	Q66.(b)
Q67.(c)	Q68.(a)	Q69.(b)	Q70.(a)		