LEARNING MATERIAL

OF

MATHEMATICS - II

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MATRICES AND DETERMINANT

MATRICES:

[History of the Matrix: The matrix has a long history of application in solving linear equations. They were known as arrays until the 1800's. The term "matrix" (Latin for "womb", derived from *mater*—mother) was coined by James Joseph Sylvester in 1850, who understood a matrix as an object giving rise to a number of determinants today called minors, that is to say, determinants of smaller matrices that are derived from the original one by removing columns and rows. An English mathematician named Cullis was the first to use modern bracket notation for matrices in 1913and he simultaneously demonstrated the first significant use of the notation A=(ai,j)to represent a matrix where ai,j refers to the element found in the *i*th row and the *j*th column. Matrices can be used to compactly write and work with multiple linear equations, referred to as a system of linear equations.]

Definition

Matrix (whose plural is matrices) is a rectangular array of numbers (or other mathematical objects), arranged in rows and columns, for which operations such as addition and multiplication are defined. The numbers are called the elements, or entries, of the matrix. Generally the capital letters of the alphabets are used to denote matrices and the matrices are commonly written in box brackets or parentheses ([ ] , ( ))

Example:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

If there are m rows and n columns in a matrix, it is called a " m by n" matrix or a matrix of order m×n, where m is the number of rows and n is the number of columns..

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

A is a matrix of order 2×3 (matrix with two rows and three columns)

$$B = \begin{bmatrix} 5 & 2 \\ 6 & 1 \\ 7 & 3 \end{bmatrix}$$

B is a matrix of order 3×2 (matrix with three rows and two columns)

Types of matrices:

1. **Row matrix:** Matrix with a single row is called a row matrix

$$A = [\ a \ \ b \ \ c \ ]$$

A is a row matrix (1×3) with one row and 3 columns

$$B = [a_{11} \ldots \ldots \ldots . . a_{1n}]$$

B is also a row matrix (1×n) with 1 row and n columns.

2. **Column matrix:** Matrix with a single column is called a column matrix

$$A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

A is a column matrix (3×1) with 3 rows and 1 column

$$B = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \end{bmatrix}$$

B is column matrix(n×1) with n rows and 1 column.

3. **Null matrix:** a matrix is said to be a null matrix or zero matrix if all its entries are zero it is noted by $O_{m\times n,}$ if it has m rows and n columns.

Example:

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. **Square matrix:** if the number of rows and columns of a matrix are equal the it is said to be a square matrix..

Example:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

A is a square matrix(3x3) of order 3 where number of rows and columns are each 3.

5. **Diagonal matrix :** a square matrix of which the non -diagonal elements are all zero is called a diagonal matrix.

Example:

$$A = \begin{bmatrix} a_{11} & 0 \ldots \ldots 0 \\ 0 & a_{22} \ldots \ldots 0 \\ . & \\ . & \\ 0 & 0 \ldots \ldots a_{nn} \end{bmatrix}$$

A is a diagonal matrix of order n

Example:

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix of order 3

6. **Scalar matrix:** if the diagonal elements of a diagonal matrix are all equal it is called a scalar matrix.

Example:

$\begin{bmatrix} \mathrm{a} & 0 & 0 \\ 0 & \mathrm{a} & 0 \\ 0 & 0 & \mathrm{a} \end{bmatrix}$ is a scalar matrix of order 3

7. **Identity matrix:** if the diagonal elements of a diagonal matrix are all unity (1) , it is called a unit matrix.

Example:

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3

A unit matrix is also called identity matrix. A unit matrix of order n is denoted by I_n or I.

8. **Transpose of a matrix**: Transpose of a matrix is obtained just by changing its rows into columns and columns into rows. It is denoted by A^T OR A^I

Example:

If $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $then\ A^T$ or $A^I = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

(A is (2x3) matrix whereas A^T is (3x2) matrix)

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$ (both are (3x3) matrices)

Algebra of matrices

a) Equality of matrices: Two matrices A and B are said to be equal if and only if

i. The order of A is equal to that of B

ii. Each element of A is equal to the corresponding element of B.

Example:

$[\mathrm{x} \quad \mathrm{y}] = [1 \quad 2] \Rightarrow$ x= 1 and y= 2

But, $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} \neq [1 \quad 2]$

As the order of $\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix}$ is 2×1 where as order of $[1 \quad 2]$ is 1×2.

b) Addition of matrices: The sum of two matrices A and B is the matrix such that each of its elements is equal to the sum of the corresponding elements of A and B. The sum is denoted by A+B .Thus the addition of matrices is defined if they are of same order and is not defined when they are of different orders.

Example:

If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ 0 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

$A + B$ is defined as the order of A and B are same (3×3)

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 0+1 \\ 3+0 & 1-2 & 5+2 \\ 0+1 & -2+2 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 3 & -1 & 7 \\ 1 & 0 & 0 \end{bmatrix}$$

Which is also of order (3×3)

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Then $A + B + C = \begin{bmatrix} 2+2+4 & 3+1+3 & 4+3+1 \\ 1+2+2 & 2+2+3 & 3+1+4 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 8 \\ 5 & 7 & 8 \end{bmatrix}$

Three matrices of order(2×3) are added and the sum is a matrix of the order (2 ×3) .

If $A = [1 \quad 2]$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

Then A+B is not defined as the order of A and B are not same

Properties:

1. The addition of matrices is **commutative**

If A and B are two matrices of same order, then A+B = B+A

Proof:

Let A =(a_{ij}) and B =(b_{ij}) be two matrices of same order

Then, $A + B = \left(a_{ij} + b_{ij}\right) = \left(b_{ij} + a_{ij}\right) = B + A$

Example:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

Then

$$A + B = \begin{bmatrix} 1+2 & 2+1 \\ 3+2 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 6 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 2+1 & 1+2 \\ 2+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 6 \end{bmatrix}$$

Hence $A + B = B + A$

2. The matrix addition is **associative**

If A,B and C are three matrices of same order, then $A + (B + C) = (A + B) + C$

Proof:

Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ be three matrices of same order.

Then,

$$A + (B + C) = (a_{ij} + (b_{ij} + c_{ij})) = ((a_{ij} + b_{ij}) + c_{ij}) = (A + B) + C$$

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \; C = \begin{bmatrix} 2 & 1 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1+(3+2) & 3+(2+1) \\ 2+(1+3) & 2+(1+3) \\ 3+(2+1) & 1+(3+2) \end{bmatrix}$$

$$= \begin{bmatrix} (1+3)+2) & (3+2)+1) \\ (2+1)+3) & (2+1)+3) \\ (3+2)+1) & (1+3)+2) \end{bmatrix} = (A + B) + C$$

3. Zero matrix (O) is the **identity matrix** for addition

$$A + O = A$$

Proof:

Let $A = (a_{ij})$, Then, $A + O = (a_{ij} + 0) = (a_{ij}) = A$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A + O = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

4. Additive inverse of a matrix

The matrix in which each element is the negative of the corresponding element of a given matrix A, is called the negative of A and is denoted by (–A). The matrix **–**A is called the **additive inverse** of the matrix A.

i.e If $A = (a_{ij}), Then - A = (-a_{ij})$

and $A + (-A) = (a_{ij} + (-a_{ij})) = (0) = O$

Example:

If $A = \begin{bmatrix} 2 & 3 \\ 5 & -6 \end{bmatrix}$ $then,$ $-A = \begin{bmatrix} -2 & -3 \\ -5 & 6 \end{bmatrix}$

and $A + (–A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

i.e $A + (-A) = O$

Again if $A + B = O$. Then A is the additive inverse of B and B is the additive inverse of A.

c) Subtraction

The subtraction of two matrices A and B of the same order is defined by

$$A – B = A + (–B)$$

Example:

If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & -3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 5 & -1 \\ 3 & 4 & -2 \end{bmatrix}$

$Then \quad A – B = A + (–B)$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & -3 & -2 \end{bmatrix} + \begin{bmatrix} -2 & -1 & 3 \\ 0 & -5 & 1 \\ -3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -4 & 1 \\ 1 & -7 & 0 \end{bmatrix}$$

d) Product of a matrix and a scalar

The product of a scalar m and a matrix A , denoted by mA is the matrix whose elements is m times the corresponding elements of A . Thus if ,

If $A = (a_{ij}), Then\ mA = (ma_{ij})$

Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ then \quad mA = \begin{bmatrix} ma & mb \\ mc & md \end{bmatrix}$$

Example:

$$If\ A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} then, 2A = \begin{bmatrix} 2 \times 2 & 2 \times 1 \\ 2 \times 3 & 2 \times -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & -4 \end{bmatrix}$$

e) Matrix multiplication

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is a ($m$xp) matrix and B is a (pxn) matrix, then their matrix product AB is the (mxn) matrix whose entries are given by product of the corresponding row of A and the corresponding column of B .i,e the elements in the ith row and jth column of AB is the sum of the products formed by multiplying each element in the ith row of A by the corresponding element in the jth column of B.

i.e

If $A =$ (a_{ij}) is a (mxp) matrix

and $B =$ (b_{ij}) is a (pxn) matrix

Then, $AB =$ (c_{ij}) is a (mxn) matrix, where $c_{ij}=\sum_{k=1}^{k=p} a_{ik}b_{kj}$

Example:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3}$ $and\ B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times3}$

The product, AB is defined because number of column of A = number of rows of B.

In this case the order of AB is (2×3)

We have,

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Applying $c_{ij}=\sum_{k=1}^{k=3} a_{ik}b_{kj}$, we get,

$$=\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + \ a_{23}b_{31} \ \ a_{21}b_{12} + a_{22}b_{22} + \ a_{23}b_{32} a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

Example:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

Then the product AB is defined and given as

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$=\begin{bmatrix} (1 \times -1) + (2 \times 2) + (3 \times 3) \\ (-2 \times -1) + (1 \times 2) + (-1 \times 3) \end{bmatrix}$$

$$=\begin{bmatrix} -1 + 4 + 9 \\ 2 + 2 - 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix}_{2\times1}$$

Properties:

1. The multiplication of matrices is **Not always commutative** that is if A and B are any two matrices then $AB \neq BA$

case-1: If $A\ and\ B$ are matrices of different orders such that the product AB is defined, but BA is not defined or $if\ BA\ is\ defined\ but\ AB\ is\ not\ defined$**.**

Example:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $then\ AB\ is\ defined, but\ BA\ is\ not\ defined,$

Therefore, $AB \neq BA$

case-2: If $A\ and\ B$ are square matrices of same order, then the product $AB\ and\ BA$ are both defined.
But $AB \neq BA$, in general

Example-1:

$$If\ \ A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad Then$$

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -1 & 4 \end{bmatrix}$$

$\therefore \quad AB \neq BA$

But in some cases matrix multiplication is commutative i.e. $AB = BA$

Example-2

$$If\ A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\ (scalar\ matrix)\ and\ B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.Then\ we\ have$$

$$AB = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$$

Hence $AB = BA$

Example-3: $if\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\ \ and\ \ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ (unit\ matrix).Then$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$Hence\ AB = BA$

2. The multiplication of matrix is **associative**

If A, B, C are three matrices such that the products (AB)C and A(BC)are defined,
Then $(AB)C = A(BC)$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 5 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}\ and\ C = \begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix}$.Then

$$AB = \begin{bmatrix} 1+0+6 & 5-6+3 \\ 4+0+2 & 20+0+1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 6 & 21 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 7 & 2 \\ 6 & 21 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 0-6 & 14+8 & -7+4 \\ 0-63 & 12+84 & -6+42 \end{bmatrix} = \begin{bmatrix} -6 & 22 & -3 \\ -63 & 96 & 36 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 & 5 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 0-15 & 2+20 & -1+10 \\ 0+9 & 0-12 & 0-6 \\ 0-3 & 4+4 & -2+2 \end{bmatrix} = \begin{bmatrix} -15 & 22 & 9 \\ 9 & -12 & -6 \\ -3 & 8 & 0 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -15 & 22 & 9 \\ 9 & -12 & -6 \\ -3 & 8 & 0 \end{bmatrix} = \begin{bmatrix} -15+18-9 & 22-24+24 & 9-12+0 \\ -60+0-3 & 88+0+8 & 36+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 22 & -3 \\ -63 & 96 & 36 \end{bmatrix}$$

Therefore, $(AB)C = A(BC)$

3. **Identity matrix of multiplication**

The identity matrix of multiplication for the set of all square matrices of a given order is the unit matrix of the same order.

Example-1:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then we have,

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

Therefore, $AI = IA = A$

Example-2:

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, then\ we\ have$

$$AI = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$$

i.e $AI = IA = A$

DETERMINANT

To every square matrix A of order n, we can associate a number (real or complex) called determinant of the matrix A, written as det $A\ or\ |A|$. In the case of a 2×2 matrix the determinant may be defined as

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then, |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Notes:

i. Only square matrices have determinants.
ii. For a matrix $A, |A|$ is read as determinant of A and not, as modulus of A.

Determinant is used in the solution of linear algebraic equations.

Consider the two equations,

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

Solving this system of equations, we get

$x = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$

and $y = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$

This solution exists, provided $a_1b_2 - a_2b_1 \neq 0$

The quantity $(a_1b_2 - a_2b_1)$ determines whether a solution of the linear equation exists or not and is denoted by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, which is called a determinant of order 2.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

The determinant is also sometimes denoted by the symbol Δ .

Similarly, the system of equation

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$

$a_3x + b_3y + c_3z + d_3 = 0$

admits a solution if, $(a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1) \neq 0$

The above expression can be denoted by

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, which is called a determinant of order 3

i.e $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1$

Minor and cofactor

Minor: Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting ith row and j th column, and it is denoted by M_{ij}.

In a determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$ and so on.

Cofactor: The cofactor of an element a_{ij} is defined as $(-1)^{i+j}M_{ij}$ where M_{ij} is the minor of a_{ij} . It is denoted by C_{ij}

i.e C_{ij} = cofactor of $a_{ij} = (-1)^{i+j}M_{ij}$

$C_{11} = (-1)^{1+1}M_{11} = M_{11}$

$C_{12} = (-1)^{1+2}M_{12} = -M_{12}$

C_{13} =$(-1)^{1+3}M_{13}$ = M_{13} and so on..

Expansion of determinants

Example:

(a) For determinants order 2

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$= $a_{11}a_{22}$- $a_{21}a_{12}$

(b) For higher order determinants:

A determinant is evaluated by expanding the determinant by the elements of any row (or any column) as the sum of products of the elements of the row (column)with the cofactors of the respective elements of the same row (column). Thus There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, $C_2$and C_3) and each way gives the same value.

$\Delta=\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix}$=$a_1(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ +$b_1(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$+$(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

= $a_1(b_2c_3-b_3c_2)$ – $b_1(a_2c_3-c_2a_3)$+$c_1(a_2b_3- b_2a_3)$

=a_1C_{11} + b_1C_{12} + c_1C_{13}

(where C_{ij} is the cofactor of the element corresponding to ith row jth column)

The above expansion has been made using the elements of the 1st row

Example:

Let $\Delta=\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & -1 & 2 \end{vmatrix}$=$2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$-$3\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix}$+$4\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix}$

= 2(4+3) -3(-2-12) +4(1-8)

= 2(7) -3(-14) + 4(-7)

=14 +42 -28 =56-28 =28

Properties of determinant:

Property 1: The value of the determinant is not altered by changing the rows into columns and the columns into rows.

i.e $|A'| = |A|$, where $A' = transpose\ of\ matrix\ A$**.**

Example:

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ = $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

L.H.S = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ = $a_1b_2 - a_2b_1$

R.H.S= $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ = $a_1b_2 - a_2b_1$

Example:

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$=$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$, as

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ = 10–12=–2

$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ = 10–12=–2

Property 2: If two adjacent rows or columns of a determinant are interchanged then the sign of the determinant changes without changing its numerical value.
Example:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$ (changing 1st and 2nd row)

Or $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$$ (changing 1st and 2nd column)

As,

$$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1$$

$$\Delta_1=\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}= a_2b_1-a_1b_2= -\ (a_1b_2-a_2b_1)=-\Delta$$

Example:

$$\Delta=\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}= 2-12 = -10$$

$$\Delta_1=\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}=12-2 = 10 = -\Delta$$

Property 3: if two rows or two columns of a determinant are identical then the value of the determinant is zero.

Example: (Two rows are equal)

$$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix}=a_1b_1-\ a_1b_1= 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_2 & c_2 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_2 & c_2 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix}$$

$$=a_1(b_2c_2 - b_2c_2) - b_1(a_2c_2 - a_2c_2)\ +\ (a_2b_2 - a_2b_2) = 0$$

Example: (Two columns are equal)

$$\begin{vmatrix} a_1 & a_1 \\ b_1 & b_1 \end{vmatrix}=a_1b_1-\ a_1b_1= 0$$

$$\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_2 \\ b_3 & b_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + b_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$=a_1(b_2b_3 - b_2b_3) - b_1(a_2b_3 - a_3b_2)\ + b_1(a_2b_3 - a_3b_2) = 0$$

Property 4: If each elements of any row or any column is multiplied by the same factor then the determinant is multiplied by that factor

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = m\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

As, $\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = mad - mbc = m(ad - bc) \quad = m\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

As,

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = ma_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - mb_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + mc_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= m\left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\}$$

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example :

$$\begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} 50 & 100 & 150 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50\begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} - 100\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 150\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}$$

$$= 50(15 - 1) - 100(12 - 2) + 150(4 - 10)$$

$$= (50 \times 14) - (100 \times 10) + (150 \times -6)$$

$$= 700 - 1000 - 900 = -1200$$

$$\text{R.H.S} = 50\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 50\left[1\begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 50\{14 - 20 - 18\} = 50 \times -24 = -1200$$

Note:

If any two rows or any two columns in a determinant are proportional, then the value of the determinant is also zero.

Property 5 : If elements of a row or a column in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants of the same order.

Example:

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 \\ a_2 + \alpha_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a_1 + \alpha_1 & b_1 \\ a_2 + \alpha_2 & b_2 \end{vmatrix} = (a_1 + \alpha_1)b_2 - (a_2 + \alpha_2)b_1$$

$$= a_1b_2 + \alpha_1 b_2 - a_2b_1 - \alpha_2 b_1$$

$$= (a_1b_2 - a_2b_1) + (\alpha_1 b_2 - \alpha_2 b_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 \\ \alpha_2 & b_2 \end{vmatrix} = \text{R.H.S}$$

Property 6: If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

Example:

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

L.H.S= $\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} kc & kd \\ c & d \end{vmatrix}$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k\begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k \times 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{R.H.S}$$

Notes:

I. If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

II. If value of determinant 'Δ' becomes zero by substituting x = α, then x – α is a factor of 'Δ'.

III. If all the elements of a determinant above or below the main diagonal are zeros, then the value of the determinant is equal to the product of diagonal elements.

Adjoint of a Matrix :

If A is a square matrix, then the transpose of the matrix of which the elements are cofactors of the corresponding elements on A is called the adjoint of A and denoted by Adj A.

Example:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$cof\ A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ where c_{ij} is the cofactor corresponding to the element a_{ij}.

Then $Adj\ A =$ (cof $A)^T$ (Transpose of the cofactor matrix)

$$= \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Example 1:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

Here C_{11} =1, C_{12}= –3, C_{21}=–2, C_{22}=1

$Cof\ A = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$Adj\ A = (Cof\ A)^T = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

Example 2:

$Let\ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

Here

$C_{11}= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$ = 1-2 = -1, C_{12} = - $\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$ = -(2-4) = 2, C_{13} = $\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$ = 0,

C_{21}= -$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ = -(2-1) = -1, C_{22}= $\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$ =1-2 = -1, C_{23}= - $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ = -(1-4) = 3,

C_{31}= $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ = 4-1 = 3, C_{32}= - $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$ = 0, C_{33}= $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ = 1-4 = -3

$$Cof\ A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 3 \\ 3 & 0 & -3 \end{bmatrix}$$

$$Adj\ A = (Cof A)T = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & 3 \\ 3 & 0 & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 3 \\ 2 & -1 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

Theorem-1

$If\ A\ is\ a\ square\ matrix\ then\ A.(AdjA) = |A|\ I = (Adj\ A).A$

Proof;-

$$Let\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\ then$$

$$Adj\ A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

$$A.(Adj\ A) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11} + a_{12}c_{12} & a_{11}c_{21} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{12} & a_{21}c_{21} + a_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\ I$$

$Hence\ \ A.(Adj\ A) = |A|\ I$

If $|A| = 0\ then\ A.(Adj\ A)$ is zero matrix. In this case the matrix A is said to be a singular matrix.

i.e $Matrix\ A\ is\ singular\ if\ |A| = 0\ and\ non-singular\ if\ |A| \neq 0.$

Inverse of a Matrix:

If $A\ and\ B$ are two square matrices of the same order such that $AB = BA = I$

Then B is called the multiplicative inverse of A.

$B\ is\ written\ as\ A^{-}1\ or\ B = A^{-}1$

$Also, A\ is\ called\ the\ inverse\ of\ B\ and\ is\ written\ as\ B^{-}1\ or\ A = B^{-}1$

If A is a non-singular matrix, then A^{-1} exists and the inverse is given by

$$A^{-1} = \frac{1}{|A|}(Adj\ A)$$

Proof:

From theorem-1, we have

$$A.(Adj\ A) = |A|\ I$$

Or, $A\left(\frac{Adj A}{|A|}\right)$ =I

Therefore, $A^{-1}=\frac{1}{|A|}\,Adj\;A$

Example1

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

A is a non-singular matrix. Hence A^{-1} exists.
We know that

$$A^{-1} = \frac{1}{|A|}(\,Adj\;A)$$

Here $C_{11} = 1,\qquad C_{12} = -1$

$C_{21} = -1,\qquad C_{22} = 2$

$$Cof\;A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

$$Adj\;A = (Cof\;A)^{T} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{T} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\,Adj\;A) = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Example 2:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

Hence A^{-1} exists.

Here $C_{11} = 1\;, C_{12} = -3$

$C_{21} = -2,\quad C_{22} = 1$

$$Adj\;A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

We know that

$$A^{-1} = \frac{1}{|A|}(\,Adj\;A) = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{-1}{5} \end{bmatrix}$$

Example 3

Let, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.\,Then$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

=1(1-6) -1(2-2)+1(6-1) = -5-0+5 = 0

$Here\ |A| = 0$.

i.e A is a singular matrix.

Hence, inverse of A does not exists.

Example 4

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

=1(2-1) – 2(4-1)+ 3 (2-1)

=1-6+3= -2 ≠ 0

Hence A^{-1} exists

Now

$C_{11} = 1, \quad C_{12} = -3, \quad C_{13} = 1$

$C_{21} = -1, \quad C_{22} = -1, \quad C_{23} = 1$

$C_{31} = -1, \quad C_{32} = 5, \quad C_{33} = -3$

$$Adj\ A = (cof A)^{T} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{1}{|A|}(Adj\ A)$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -1 & -1 \\ -3 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{-5}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$$

Cramer's Rule:-

Cramer's rule is used in the solution of simultaneous linear equations

Consider the equations in two variables

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

Solving these two equations by using cross multiplication method we have

$$\frac{x}{d_1b_2 - d_2b_1} = \frac{y}{a_1d_2 - a_2d_1} = \frac{1}{a_1b_2 - a_2b_1}$$

i.e $\frac{x}{D_x} = \frac{y}{D_y} = \frac{1}{D}$

Where,

D =$a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

$D_x = d_1 b_2 - d_2 b_1 = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$, $D_y = a_1 d_2 - a_2 d_1 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

Therefore, x = $\frac{D_x}{D}$, y = $\frac{D_y}{D}$

Consider the equations in three variables

a_1x + b_1y + c_1 z $= d_1$

a_2x + b_2y + $c_2 z = d_2$

a_3x + b_3y + $c_3 z = d_3$

Here,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now multiplying D by x, we have

$$xD = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} \quad (C_1 \to C_1 + yC_2 + zC_3)$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_x$$

Or xD $= D_x$

Or, $\frac{x}{D_x} = \frac{1}{D}$

Similarly, we can show that $\frac{y}{D_y} = \frac{1}{D}$ and $\frac{z}{D_Z} = \frac{1}{D}$

Where,

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} , \quad D_Z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Therefore, $x = \frac{D_X}{D}, \ y = \frac{D_Y}{D}, \ z = \frac{D_Z}{D}$

Example 1

Consider, $x + 2y = 5$

$3x + y = 7$

Here D= $\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$ = 1-6 = -5 ≠ 0

The system admits a solution

$$D_x = \begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} = 5 - 14 = -7$$

$$D_y = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 7 - 15 = -8$$

By Cramer's rule,

$$x = \frac{D_x}{D} = \frac{-7}{-5} = \frac{7}{5} \ , \ y = \frac{D_y}{D} = \frac{-8}{-5} = \frac{8}{5}$$

Example 2

Consider, $x + y = 3$

$2x + 2y = 7$

Here D= $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$

The system does not possess a solution.

Example 3

Consider, $x - 2y + z = 2$

$6x - 9y + z = 1$

$-9x + 12y + z = 4$

Here, D= $\begin{vmatrix} 1 & -2 & 1 \\ 6 & -9 & 1 \\ -9 & 12 & 1 \end{vmatrix}$

=1$\begin{vmatrix} -9 & 1 \\ 12 & 1 \end{vmatrix}$+2 $\begin{vmatrix} 6 & 1 \\ -9 & 1 \end{vmatrix}$ +1 $\begin{vmatrix} 6 & -9 \\ -9 & 12 \end{vmatrix}$

$= -21 + 30 - 9 = 0$

Similarly, $D_X = 0$, $D_Y = 0$, $D_Z = 0$

So, the system has infinite number of solution.

Example 4

Consider, $x + 2y + 3z = 6$

$2x + 4y + z = 7$

$3x + 2y + 9z = 1$

Here D= $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$

=1$\begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix}$ -2 $\begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix}$ +3$\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

=1(36-2) -2(18-3) +3 (4 -12)

= 34 – 32 – 24

= - 20 ≠ 0

The system of equations admits a solution

Similarly, we have,

$$D_x = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = -20$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = -20$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = -20$$

By Cramer's rule.

$$x = \frac{D_X}{D} = \frac{-20}{-20} = 1\ , \quad y = \frac{D_Y}{D} = \frac{-20}{-20} = 1,\ \ z = \frac{D_Z}{D} = \frac{-20}{-20} = 1$$

Solution of simultaneous linear equations by matrix inverse method

Let us consider two linear equations with two variables

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

The above system of equations can be written as

AX = B

Where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Or, $X = A^{-1}B$

$= \frac{1}{|A|}(\mathrm{Adj}A)\ B,$

if $|A| \neq 0$ Then the system admits solution.

Similarly, for three linear equation with three variables

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The above system of equations can be written in matrix form as

$$AX = B, \quad \text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} and\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then,

$$X = A^{-1}B$$

Or, $X = \frac{1}{|A|}(Adj\ A)B,$

If $|A| \neq 0$ Then the system admits solution

Example 1

Consider the following system of linear equations

3x-4y = 1

2x + y = 8

The above system can be written as

$AX = B$

Where $A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

Or, $X = A^{-1}B = \frac{1}{|A|}(\text{AdjA})\text{ B}$

Here, $|A| = \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix}$ =3-(-8)= 11 ≠ 0

So, the system of equations admits solution.

Here, $C_{11} = 1$, $C_{12} = -2$

$C_{21} = 4$, $C_{22} = 3$

$Cof\ A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$Adj\ A = (Cof A)^{T} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

So that, we have

$$\text{X} = \frac{1}{|A|}(\text{AdjA})\text{B} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 1+32 \\ -2+24 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 33 \\ 22 \end{bmatrix} = \begin{bmatrix} 33/11 \\ 22/11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Hence, $x = 3$ and $y = 2$

Example 2

Let us consider the system of equations,

$x - y + z = 2$

$2x + y - 3z = 5$

$3x - 2y - z = 4$

The system of equations can be written in matrix form as

$$AX = B, where\ A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Or, $X = A^{-1}B = \frac{1}{|A|}(\text{AdjA})\text{B}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & -1 \end{vmatrix} = 1(-1-6) - 1(-2+9) + 1(-4-3) = -7+7-7 = -7 \neq 0$$

So, A^{-1} exist and the system admits solution.

We have,

$C_{11} = -7, \quad C_{12} = -7, \quad C_{13} = -7$

$C_{21} = -3, \quad C_{22} = -4\,, \quad C_{23} = -1$

$C_{31} = 2, \quad C_{32} = 5, \quad C_{33} = 3$

So that,

$$Cof\ A = \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix}$$

$$Adj\ A = (CofA)^T = \begin{bmatrix} -7 & -7 & -7 \\ -3 & -4 & -1 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{bmatrix}$$

So that, we have,

$$X = \frac{1}{|A|}(AdjA)\,B = (1/-7)\begin{bmatrix} -7 & -3 & 2 \\ -7 & -4 & 5 \\ -7 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

$$= (1/-7)\begin{bmatrix} -14-15+8 \\ -14-20+20 \\ -14-5+12 \end{bmatrix} = (1/-7)\begin{bmatrix} -21 \\ -14 \\ -7 \end{bmatrix} = \begin{bmatrix} -21/-7 \\ -14/-7 \\ -7/-7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Hence, $x = 3\,,\ y = 2,\ z = 1$

Some Solved Problems

Q-1: Find the minors and cofactors of all the elements of the matrices

(i) $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

Sol:

(i) Given $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$

Let M_{ij} and $C_{ij} = (-1)^{i+j}M_{ij}$ are the minor and cofactors of the element a_{ij}, then

$M_{11} = 4$, $C_{11}=(-1)^{1+1}M_{11}=4$

$M_{12}= -1$, $C_{12}=(-1)^{1+2}M_{12}=1$

$M_{21}=3$, $C_{21}=(-1)^{2+1}M_{21}=-3$

$M_{22} =2$, $C_{22}=(-1)^{2+2}M_{22}=2$

(ii) Given $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

Let M_{ij} and $C_{ij} = (-1)^{i+j}M_{ij}$ are the minor and cofactors of the element a_{ij}.Then,

$M_{11} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10,$ $C_{11}=(-1)^{1+1}M_{11}= -10$

$M_{12} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1,$ $C_{12}=(-1)^{1+2}M_{12}= -1$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7,$ $C_{13}=(-1)^{1+3}M_{13}= 7$

$M_{21} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0,$ $C_{21}=(-1)^{2+1}M_{21}= 0$

$M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1,$ $C_{22}=(-1)^{2+2}M_{22}= 1$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2,$ $C_{23}=(-1)^{2+3}M_{23}= -2$

$M_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5,$ $C_{31}=(-1)^{3+1}M_{31}= 5$

$M_{32} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1,$ $C_{32}=(-1)^{3+2}M_{32}= -1$

$M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3,$ $C_{33}=(-1)^{3+3}M_{33}= -3$

Q- 2: Evaluate the value of the determinant $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ using properties

Sol:

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

$$=\begin{vmatrix} 265-240 & 240-219 & 219 \\ 240-225 & 225-198 & 198 \\ 219-198 & 198-181 & 181 \end{vmatrix} (C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$$

$$= \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

$$= \begin{vmatrix} 25-21 & 21 & 219-210 \\ 15-27 & 27 & 198-270 \\ 21-17 & 17 & 181-170 \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2 and C_3 \leftarrow C_3 - 10C_2)$$

$$= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 4-4 & 21-17 & 9-11 \\ -12+12 & 27+51 & -72+33 \\ 4 & 17 & 11 \end{vmatrix} \ (\mathrm{R_1 \to R_1 - R_3,\ R_2 \to R_2 + 3R_3})$$

$$=\begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}$$

Expanding with respect to 1st column

$$= \ 4\begin{vmatrix} 4 & -2 \\ 78 & -39 \end{vmatrix}$$

$= 4(-156 + 156) = 4 \times 0 = 0$

Q- 3: Solve, $\begin{vmatrix} x & a & a \\ m & m & m \\ b & x & b \end{vmatrix} = 0$

Sol:

Given, $\begin{vmatrix} x & a & a \\ m & m & m \\ b & x & b \end{vmatrix} = 0$

Or, $m\begin{vmatrix} x & a & a \\ 1 & 1 & 1 \\ b & x & b \end{vmatrix} = 0$ (since $m \neq 0$)

Or, $\begin{vmatrix} x-a & 0 & a \\ 0 & 0 & 1 \\ b-x & x-b & b \end{vmatrix} = 0 \quad C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$

Or, (-1)$\begin{vmatrix} x-a & 0 \\ b-x & x-b \end{vmatrix} = 0$

Or, $(x-a)(x-b) = 0$

Or, $x = a$ or b

Q- 4: Expand $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$, by using properties of determinant

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1-1 & 1-1 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix} (C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x-y & y-z \\ x^3-y^3 & y^3-z^3 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)(y-z)(y^2+yz+z^2-x^2-xy-y^2)$$

$$= (x-y)(y-z)(z-x)(x+y+z)$$

Q- 5 : Factorize $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

Solution:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$

$=\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$ (Taking xyz common factor from R_3)

($R_1 \leftrightarrow R_2$, $R_2 \leftrightarrow R_3$)

$=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix} (C_1 \leftarrow C_1 - C_2 \text{and } C_2 \leftarrow C_2 - C_3)$

$= \begin{vmatrix} x^2-y^2 & y^2-z^2 \\ x^3-y^3 & y^3-z^3 \end{vmatrix}$

$=\begin{vmatrix} (x-y)(x+y) & (y-z)(y+z) \\ (x-y)(x^2+xy+y^2) & (y-z)(y^2+yz+z^2) \end{vmatrix}$

$= (x-y)\ (y-z) \begin{vmatrix} x+y & y+z \\ (x^2+xy+y^2) & (y^2+yz+z^2) \end{vmatrix}$

$= (x-y)\ (y-z) \begin{vmatrix} x-z & y+z \\ x^2+xy-yz-z^2 & y^2+yz+z^2 \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2)$

$=(x-y)\ (y-z) \begin{vmatrix} x-z & y+z \\ (x-z)(x+y+z) & y^2+yz+z^2 \end{vmatrix}$

$= (x-y)\ (y-z)\ (z-x) \begin{vmatrix} 1 & y+z \\ x+y+z & y^2+yz+z^2 \end{vmatrix}$

Now by expanding we get

$= (x-y)\ (y-z)(z-x)(xy+yz+zx)$

Q- 6: Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Proof:

$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2 + R_3$

$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -b-c-a & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$

$(C_1 \leftarrow C_1 - C_2 and C_2 \leftarrow C_2 - C_3)$

$=(a+b+c)\begin{vmatrix} a+b+c & -b-c-a \\ 0 & a+b+c \end{vmatrix}$

$=(a+b+c)(a+b+c)^2= (a+b+c)^3$

Q- 7 Solve $\begin{vmatrix} 2x+1 & 3 \\ x & 2 \end{vmatrix} = 5$

Solution:

Given $\begin{vmatrix} 2x+1 & 3 \\ x & 2 \end{vmatrix} = 5$

Or, 2(2x+1)-3x = 5

Or, 4x+2 -3x = 5

Or, x+2 = 5

Or, x = 3

Q- 8: Verify that $[AB]^T = B^T A^T$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$

Sol:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+15 & 2+8+18 & 3+4+3 \\ 6+21+40 & 12+28+48 & 18+14+8 \\ 6-9+20 & 12-12+24 & 18-6+4 \end{bmatrix} = \begin{bmatrix} 22 & 28 & 10 \\ 67 & 88 & 40 \\ 17 & 24 & 16 \end{bmatrix}$$

$$[AB]^T = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix}$$

Again, $A^T = \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$ *and* $B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

$$B^T A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+15 & 6+21+40 & 6-9+20 \\ 2+8+18 & 12+28+48 & 12-12+24 \\ 3+4+3 & 18+14+8 & 18-6+4 \end{bmatrix} = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix}$$

Hence, $[AB]^T = B^T A^T$

Q- 9: Write down the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, if $a_{ij} = 2i+3j$

Sol:

$$\begin{bmatrix} 2+3 & 2+6 & 2+9 \\ 4+3 & 4+6 & 4+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \end{bmatrix}$$

Q-10: Construct a 2×3 matrix having elements $a_{ij} = i+j$

Sol:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

EXERCISE

1. 02 Marks Questions

I. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

II. Solve $\begin{vmatrix} a & b & c \\ b & a & b \\ x & b & c \end{vmatrix} = 0$.

III. Find the minor and cofactor of the elements 4 and 0 in the determinant $\begin{vmatrix} 1 & 2 & -3 \\ 4 & 5 & 0 \\ 2 & -1 & 1 \end{vmatrix}$.

IV. Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

V. What is the maximum value of $\begin{vmatrix} \sin x & \cos x \\ -\cos x & 1+sinx \end{vmatrix}$.

VI. Without expanding evaluate $\begin{vmatrix} sin^2\theta & cos^2\theta & 1 \\ cos^2\theta & sin^2\theta & 1 \\ -10 & 12 & 2 \end{vmatrix}$,

VII. Without expanding, find the value of $\begin{vmatrix} 1/a & 1 & bc \\ 1/b & 1 & ca \\ 1/c & 1 & ab \end{vmatrix}$.

VIII. If X +$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, Then find X.

IX. Find x and y, if $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

2. 05 Marks Questions

I. Solve $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0$

II. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

III. Prove that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$.

IV. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

V. Prove that $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

VI. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

VII. Prove that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & \mathrm{a}^2+\mathrm{b}^2 \end{vmatrix} = 4\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

VIII. Prove that $(AB)^T = B^T A^T$, where $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}$.

IX. Find the adjoint of the matrix $\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$

X. If = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 1 \\ 3 & 1 & -2 \end{bmatrix}$, find adjoint of A.

XI. Find the inverse of the matrices $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$

XII. Solve by Cramer's rule: $2x - 3y = 8\,,\quad 3x + y = 1$

XIII. Solve by matrix method : $5x - 3y = 1,\quad 3x + 2y = 12$

3. 10 Marks Questions

I. Prove that $\begin{vmatrix} sin^2 A & \cot A & 1 \\ sin^2 B & \cot B & 1 \\ sin^2 C & \cot C & 1 \end{vmatrix} = 0$, where A, B and C are the angles of a triangle.

II. Prove that $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0$, where $A + B + C = \pi$.

C. Two-dimensional Geometry

Introduction:

Coordinate geometry is a branch of mathematics which deals with the systematic study of geometry by use of algebra. It was first initiated by French Mathematician Rene Descartes (1596-1665) in his book 'La Geometry', Published in 1637. Hence it is also known as Cartesian Co-Ordinate Geometry.

Fundamental concept:

In two-dimensional co- ordinate geometry, the position of point on the plane is defined with the help of an 'order pair of the numbers also known as 'Co-ordinates'. After determining the Coordinates of the point on a line or curve on the plane, we will find out distance between two points, internal and external division, area of closed figure (Triangle), slope of the lines, consistency of lines ,equations of line and circle using algebra.

Coordinate system:

A system in a plane which involves two mutually perpendicular lines which intersect at the origin and measured with equal units to form a orthogonal system called Cartesian co-ordinate system.
This system is used to specify the location of a point in 2D. (Fig 3.1)

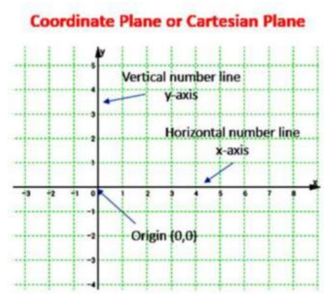


Fig 3.1

Coordinate axes:

The intersecting lines are called coordinates axes.
The horizontal line is called x-axis.
The vertical line is called y- axis
The point of intersection of axes is called the origin.

Origin:

The point where both the axis meet/intersect is called origin and its coordinates are (0,0). From origin towards right through x -axis, ox is measured as +ve units and towards left from origin ox' is –ve. Similarly, from origin towards up through y –axis, oy is +ve and towards down oy' is –ve.

Coordinates:

A pair of numbers which locates the points on the coordinate plane is called its coordinates.
It is denoted as an order pair (x , y) .
'x' is the distance of a point from the Y-axis is known as **abscissa** or x-coordinate.
'y' is the distance of a point from the x-axis is known as **ordinate** or y-coordinate.

Quadrant:

The coordinate axes divide the plane into four equal parts, called quadrants named as
xoy (1st Quadrant) $x > 0, y > 0$
x'o y (2nd Quadrant) $x < 0, y > o$

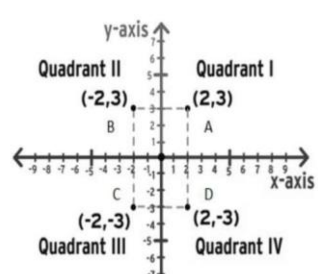


Fig 3.2

$x'oy'$ (3^{rd} Quadrant) $x < 0, y < 0$
xoy' (4^{th} quadrant) $x > 0, y < 0$

Representation of any point (x, y) on the Cartesian plane:
Coordinates of any point on x -axis are (x,0)
Coordinates of any point on y -axis are (0, y)
Example:
Any point A(2,3) is located at 2 unit distance from y -axis measured on ox (positive direction of x-axis right to origin) and 3 units distance from x-axis measured on OY (positive direction of Y axis) .so its lies in 1^{st} quadrant . (figure3.2)
Similarly points B(-2,3),C(-2,-3) and D(2,-3) are located in 2^{nd} ,3^{rd} and 4^{th} quadrant respectively as shown in figure.

Distance formula:
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in the coordinate plane. (Fig 3.3)

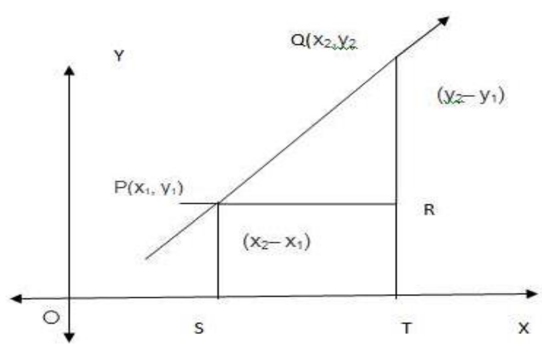


fig 3.3

$\triangle PQR$ is a right-angle triangle.
By Pythagoras theorem,

$$PQ^2 = PR^2 + QR^2$$

Or, $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Or, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is the required distance between two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$.
Or, $PQ = \sqrt{(\text{Difference of abscissa})^2 + (\text{Difference of ordinates})^2}$

Distance between a point from the origin:
Distance of a point P(x, y) from the origin O (0, 0) is

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

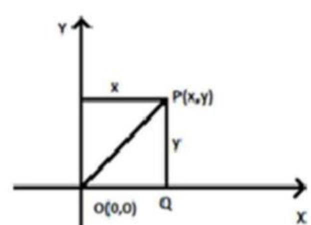


fig 3.4

Some Solved Problems

Q-1: Find the distance between the points P (1, 2) and Q (2, -3).
Sol:
The distance between the points P (1, 2) and Q (2, -3) is

|PQ|= $\sqrt{(2-1)^2+(-3-2)^2}$ =$\sqrt{1+25}$ = $\sqrt{26}$ units.

Q-2: If the distance between the points (3, a) and (6, 1) is 5 find 'a'.
Sol: Given the distance between the points (3, a) and (6, 1) = 5
Using distance formula,

$\sqrt{(6-3)^2+(1-a)^2}$ =5

Or, 9+ $(1-a)^2$=25 (Squaring both sides)

Or, $(1-a)^2 = 16$

Or, 1-a = ±4

Or, a=5, -3

Q-3: If O(0, 0), A(1,0), B(1.1) are the vertices of the triangle, what type of triangle is ΔOAB?
Sol:
Given O(0, 0), A(1,0), B(1, 1) are the vertices of the triangle ΔOAB.
Using distance formula,

|OA|=$\sqrt{(1-0)^2+(0-0)^2}$ = 1

|OB|=$\sqrt{(1-0)^2+(1-0)^2}$ =$\sqrt{2}$

|AB|=$\sqrt{(1-1)^2+(1-0)^2}$ =1

Therefore, OA =AB and $|OA|^2$+$|AB^2|$=$|AB|^2$
Hence, ΔOAB is a right-angle isosceles triangle.

Division/Section formula
Internal division
Let A (x_1, y_1) and B(x_2, y_2) be two given points. Suppose P (x, y) is a point on AB which divides the line AB in the ratio m:n internally i.e. $AP : PB = m : n$.

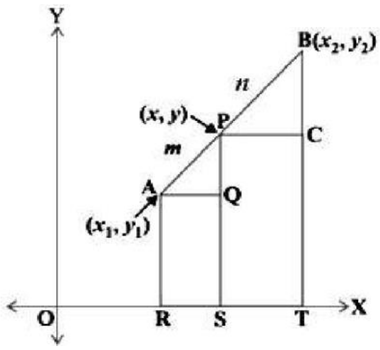


Fig 3.5

From the figure, ΔAPQ and triangle ΔPCB are similar.

Hence, $\frac{AQ}{PC} = \frac{AP}{PB} = \frac{PQ}{BC}$, Or, $\frac{x-x_1}{x_2-x} = \frac{m}{n} = \frac{y-y_1}{y_2-y}$

Now, $\frac{x-x_1}{x_2-x} = \frac{m}{n}$,

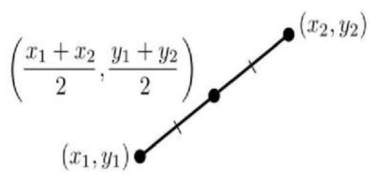
Or, $mx_2 - mx = nx - nx_1$

Or, $mx + nx = mx_2 + nx_1$

Or, $x = \frac{mx_2+nx_1}{m+n}$

and $\frac{m}{n} = \frac{y-y_1}{y_2-y}$

Or, $my_2 - my = ny - ny_1$



Or, $my + ny = my_2 + ny_1$

Or, $y = \frac{my_2 + ny_1}{m+n}$

Hence, the coordinates of point P are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

Note: Midpoint formula

If R is the midpoint of the line joining P (x_1, y_1) and Q(x_2, y_2),

Then the co-ordinates of R are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

External division:

If A (x_1, y_1) and B(x_2, y_2) be two given points.

Let P(x, y) be any point, which divides the line AB in the ratio m:n externally, then $\frac{AP}{BP} = \frac{m}{n}$

Then the co-ordinates of the point P are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

(Similar type of proof as with internal division).

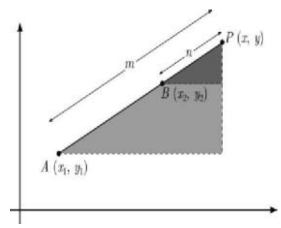


Fig. 3.7

Some Solved Problems

Q-1: Find the coordinates of the point which divides the line joining the points P (1, 2) and Q (3, 4) in the ratio 2:1 internally.

Sol:

Given P (1, 2) and Q (3, 4) be two points.

Let R be the point which divides PQ internally in the ratio 2:1.

Using internal division formula, the co-ordinates of point R are

$$\left(\frac{(2\times3)+(1\times1)}{2+1}, \frac{(2\times4)+(1\times2)}{2+1}\right) = \left(\frac{7}{3}, \frac{10}{3}\right)$$

Q-2: Find the coordinates of the point which divides the line joining the points P(2,3) and Q (–3, 1) in the ratio 3:2 externally.

Sol:

Let R be the point which divides PQ, joining P(2,3) and Q(–3, 1), externally in the ratio 3:1.

Using external division formula,

The co-ordinates of point R are $\left(\frac{(3\times-3)-(2\times2)}{3-2}, \frac{(3\times1)-(2\times3)}{3-2}\right) = (-13, -3)$

Q-3: Find midpoint of the line joining P(2, 3) and Q(4, 5).

Sol:

Let R be the midpoint of the line joining P(2, 3)and Q(4,5).

Using the mid-point formula, the co-ordinates of R are$\left(\frac{2+4}{2}, \frac{3+5}{2}\right) = (3, 4)$

Q-4: In what ratio does the point $(-1, -1)$ divide the line segment joining the points $(4,4)$ and $(7,7)$?

Sol:

Let the point $C(-1, -1)$ divides the line segment joining the points $A(4,4)$ and $B(7,7)$ in the ratio $k: 1$.

Then the co-ordinates of point C are $\left(\frac{7k+4}{k+1}, \frac{7k+4}{k+1}\right)$.

Therefore, $\frac{7k+4}{k+1} = -1$

Or, $7k + 4 = -k - 1$

Or, $8k = -5$

Or, $k = -\frac{5}{8}$

Hence, the point C divides AB externally in the ratio 5:8.

Q-5: In what ratio does the x-axis divide the line segment joining the points (2, –3) and (5, 6)?

Sol:

The co-ordinates of the point which divides the line segment joining the points (2, –3) and (5, 6) internally in the ratio K:1 are $\left(\frac{5k+2}{k+1}, \frac{6k-3}{k+1}\right)$.

As, this point lies on x-axis, where y-co-ordinate of every point is zero.

Therefore, $\frac{6k-3}{k+1} = 0$, Or, $6k - 3 = 0$, Or, $k = \frac{1}{2}$

Hence, the required ratio is 1 : 2.

Centroid of a triangle:

Let Δ ABC with vertices are given by A (x_1, y_1), B(x_2, y_2) and C(x_3, y_3).

Let D, E, F are the midpoints of the side BC, AC and AB respectively.

∴ Coordinates of D, E and F are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ and $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ respectively.

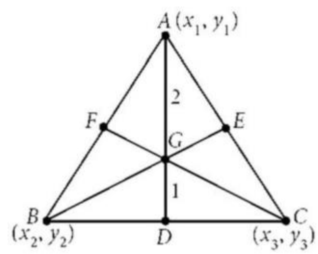


Fig.3.8

Let G_1 be a point which divides the median AD internally in the ratio 2:1.

So, the co-ordinates of G_1 are $\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+1.x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+1.y_1}{2+1}\right)$

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly, Let G_2 and G_3 be points which divide the median BE and CF internally in the ratio 2:1.

So, the co-ordinates of G_2 and G_3 are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Since the co-ordinates of the points are G_1, G_2 and G_3 same i.e. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$,

So, the points G_1, G_2 and G_3 are not different points but the same point.

Hence, the point having co-ordinates $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ common to AD, BE and CF and divides in the ratio 2:1., which is known as the centroid.

Some Solved Problems

Q-1: Find the co-ordinates of centroid of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Sol:

We know that, the co-ordinates of the centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Therefore, The co-ordinates of the centroid of the triangle with vertices (0, 6), (8, 12) and (8, 0) are $\left(\frac{0+8+8}{3}, \frac{6+12+0}{3}\right) = \left(\frac{16}{3}, 6\right)$

Q-2: Two vertices of a triangle are (1, 2), (3, 5) and its centroid is at the origin. Find the co-ordinates of the third vertex**.**

Sol:

Let the co-ordinates of the third vertex of the triangle be (x, y).

So, the coordinates of the centroid of a triangle with vertices (1, 2). (3, 5) and (x, y) are given by $\left(\frac{1+3+x}{3}, \frac{2+5+y}{3}\right) = \left(\frac{4+x}{3}, \frac{7+y}{3}\right)$,

Given the centroid is at the origin, (0, 0).

Therefore, $\frac{4+x}{3} = 0$ and $\frac{7+y}{3} = 0$, Or, $x = -4$ and $y = -7$.

Hence, the co-ordinates of the third vertex are (–4, –7).

Area of a triangle

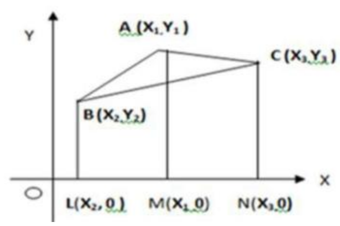


Fig-3.9

Let ABC be a triangle with vertices $A(x_1, y_1), B(x_2, y_2)$ and (x_3, y_3) .

Now, Area of Δ ABC= Area of trapezium ABLM

+ Area of trapezium AMNC

– Area of the trapezium BLNC

We know that, Area of a trapezium=$\frac{1}{2}$ (Sum of parallel sides)(Distance between them)

$\therefore$ Area of ΔABC= $\frac{1}{2}(BL + AM)LM + \frac{1}{2}(AM + CN)MN - \frac{1}{2}(BL + CN)LN$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

This can also be conviniently expressed in the determinant form as

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Notes:

1. If the vertices are taken in anti-clockwise sense, then the area calculated of the triangle will be positive, where as if the points are taken in clockwise, then the area calculated will be negative. But, if the vertices are taken arbitrarily, the area calculated may be positive or negative.
   In case, the area calculated is negative, we consider the numerical /absolute i.e. positive value.
2. To find the area of a polygon, we divide the polygon into some triangles and take the sum of numerical values of area of each triangle.

Co-linearity of three points:

Points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are collinear, if they lie on a straight line i.e. area of Δ ABC = 0,

i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Some Solved Problems

Q-1: Find the area of triangle whose vertices are A (4, 4), B (3, –2), and C (–3, 16).

Sol:

Area of the Δ ABC = $\frac{1}{2}\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix}$

$$= \frac{1}{2}[4(-2-16) - 4(3+3) + 1(48-6)]$$

$$= \frac{1}{2}[-72 - 24 + 42] = \frac{1}{2}(-54) = -27$$

$\therefore$ Area of triangle= $|-27| = 27$ square units

Q- 2: Find the value of 'a', so that area of the triangle having vertices A (0, 0)$_1$ B (1, 0) and C (0, a) is 10 units.

Sol:

Given Area of Δ ABC =10 units

Or, $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = 10$

Or, 1(a – 0) = 20

Or, a = 20 units

Q-3: Find the value of a so that A (1, 4), B (2, 7) and C (3, a) are Collinear**.**

Sol:

Given A (1, 4), B (2, 7) and C (3, a) are Collinear

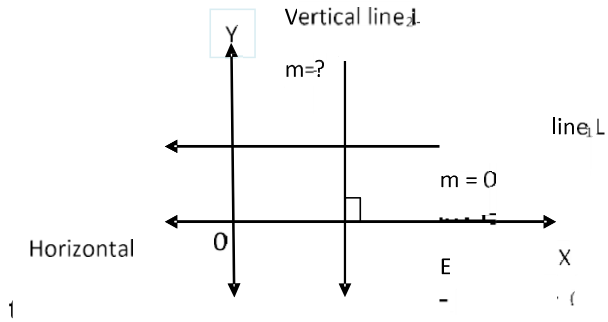
Therefore, Area of Δ ABC = 0

Or, $\frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 3 & a & 1 \end{vmatrix} = 0$

Or, 1 (7 – a) – 4 (2 – 3) + 1 (2a – 21) = 0

Or, 7 – a + 4 + 2a – 21 = 0

Or, a = 10



Theorem: Every first-degree equation in x and y represents a straight line.

Proof:

Let $ax + by + c = 0$ be a first-degree equation in x and y, where a, b and c are constants.
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$.
Then, $ax_1 + by_1 + c = 0$ and $x_2 + by_2 + c = 0$.
Let R be any point on the line segment joining P and Q. Let R divides PQ in the ratio K:1.
Then, the co-ordinates of R are $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right)$.

So, we have, $a\left(\frac{kx_2+x_1}{k+1}\right) + b\left(\frac{ky_2+y_1}{k+1}\right) + c$

$$= \frac{1}{k+1}(akx_2 + ax_1 + kby_2 + by_1 + ck + c)$$
$$= \frac{1}{k+1}[k(ax_2 + by_2 + c) + (ax_1 + by_1 + c)]$$
$$= \frac{1}{k+1}[k(0) + 0] = 0.$$

$\therefore$ $R\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right)$ lies on the curve represented by $ax + by + c = 0$, and hence every point on the line segment joining P and Q lies on $ax + by + c = 0$.
Hence, $ax + by + c = 0$ represents a straight line.

Slope(Gradient) of a line:

The tangent of the angle made by a line with the positive direction of the x –axis in anticlockwise sense is called slope or gradient of the line.
Generally, the slope of a line is denoted by the letter 'm'.
Hence, $m = tan\theta$, where 'θ' is the angle made by the line with positive direction of x – axis in anticlockwisedirection... (Fig: 3:9)

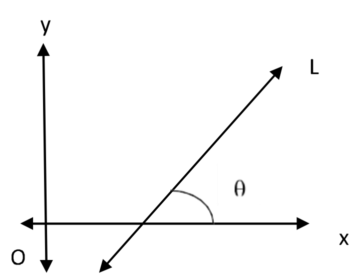


Fig. 3.9

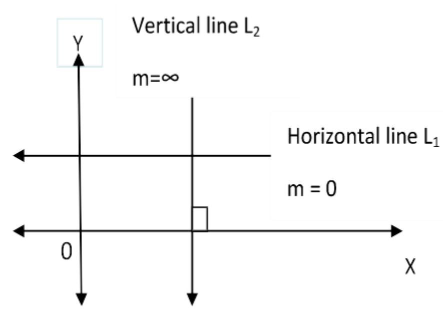
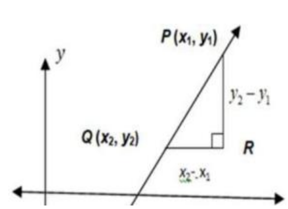


Fig. 3.10

Note: **1.** In (fig 3.10) L_1 is the line parallel to *x*–axis So $\theta = 0^o \Rightarrow m = \tan 0^o = 0$
So, slope of the line parallel to *x*–axis is zero.

Note 2. The line L_2 is perpendicular to *x*-axis or parallel to *y*-axis, so $\theta = 90^o \Rightarrow m = \tan 90^o$, is not defined
So slope of the line parallel to *y*-axis (vertical line) is not defined.

Note 3: Slope of a line equally inclined to both the axes is +1 or –1 , as the line makes with 45° and 135°angle with $x - axis.$



Slope of a line joining two points P (x_1, y_1) and Q (x_2, y_2):

Let P (x_1, y_1) and Q (x_2, y_2) be two points on a line making an angle θ with positive direction of x-axis.

m=tanθ = $\frac{\text{Perpendicular}}{\text{Base}} = \frac{y_2 - y_1}{x_2 - x_1}$ (fig 3.11)

Slope of the line PQ is given by, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Conditions of parallelism and perpendicularity:

1. Two lines L_1 and L_2 are parallel:

Let θ_1 and θ_2 be the angle of inclination of the parallel lines L_1 and L_2.

From fig (3.12), we have $\theta_1 = \theta_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

i.e. Two lines are parallel if their slopes are equal.

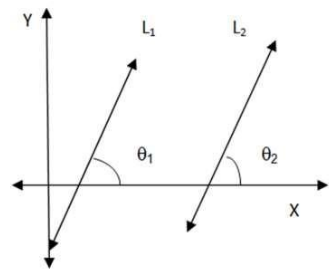


Fig 3.12

2. Two lines are perpendicular to each other

Let θ_1 and θ_2 be the angle of inclination of the perpendicular lines L_1 and L_2.

From Fig. 3.13, we have

$\theta_2 = 90 + \theta_1$

$\Rightarrow \tan\theta_2 = \tan(90 + \theta_1)$

$\Rightarrow m_2 = -\cot\theta_1 = -\frac{1}{\tan\theta_1}$

$\Rightarrow m_2 = \frac{-1}{m_1}$

$\Rightarrow m_1 m_2 = -1$

i.e. two lines are perpendicular if their product is equal to –1.

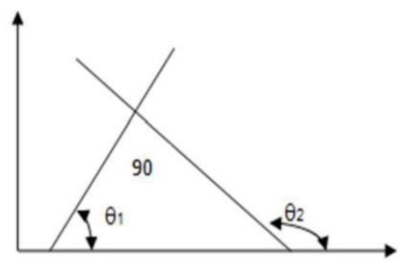


Fig. 3.13

Some Solved Problems

Q- 1: Find Slope of a line joining P (2, 3) and Q (1, 4).

Sol:

Slope of the line joining P (2, 3) and Q (1, 4) = m = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{1-2} = \frac{1}{-1} = -1$

Q- 2: Find slope of the line perpendicular to a line joining P (1, 2) and Q (3, 5).

Sol:

Slope of the line PQ joining P (1, 2) and Q (3, 5)= $\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{3-1} = \frac{3}{2}$

So, slope of the line perpendicular to PQ = $\frac{-1}{3/2} = -2/3$

(Since product of their slopes is -1)

Q- 3: Find slope of the line parallel to the line joining P (1, 4) and Q (2, 6).

Sol:

Slope of the line PQ, joining P (1, 4) and Q (2, 6)= $\frac{6-4}{2-1} = \frac{2}{1} = 2$

So, slope of the line parallel to PQ = 2. (Since slopes of parallel lines are equal)

Angle between two lines

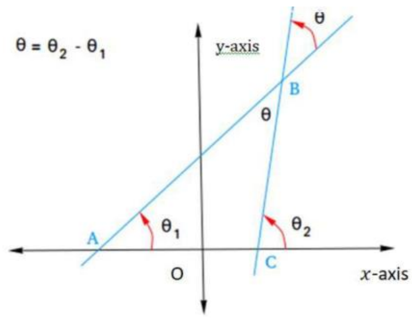


Fig 3.14

Let θ be the angle between two straight lines with slopes m_1 and m_2

i.e $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$,

where θ_1 and θ_2 are the angle of inclinations of two lines.

From Fig. 3.14,

$$\theta + \theta_1 = \theta_2$$

Or, $\tan\theta = \tan(\theta_2 - \theta_1)$

Or, $\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2 \tan\theta_1} = \frac{m_2 - m_1}{1+m_2 m_1}$

The other angle between the lines is given by $\pi - \theta$

So, $\tan(\pi - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_2 m_1}$

Therefore, the angle (θ) between the lines with slopes m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_2 - m_1}{1+m_1 m_2}$$

Note:

The condition of lines to be parallel and perpendicular can also be deduced from the relation

$$\tan\theta = \pm\frac{m_2 - m_1}{1+m_1 m_2}$$

For parallel lines, $\theta = 0^\circ$,

$$\tan\theta = 0$$

Or, $\frac{m_2 - m_1}{1+m_1 m_2} = 0$

Or, $m_1 = m_2$ (Slopes are equal)

and for perpendicular lines, $\theta = 90^\circ$,

Or, $\cot\theta = 0$

Or $\frac{1+m_1 m_2}{m_2 - m_1} = 0$

Or, $1 + m_1 m_2 = 0$

Or, $m_1 m_2 = -1$ (Product of their slopes equal to –1)

Some Solved Problems:

Q-1: If $A(-2,1), B(2,3)$ and $C(-2,-4)$ are three points, find the angle between BA and BC.

Sol: Let m_1 and m_2 be the slopes of BA and BC respectively.

$\therefore \quad m_1 = \frac{3-1}{2+2} = \frac{1}{2}$ and $m_1 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between BA and BC.

$\therefore \quad \tan\theta = \left|\frac{m_2 - m_1}{1+m_1 m_2}\right| = \left|\frac{\frac{7}{4}-\frac{1}{2}}{1+\frac{1}{2}\cdot\frac{7}{4}}\right| = \frac{5/4}{15/8} = \frac{2}{3}$

Therefore, the angle between BA and BC $= \theta = tan^{-1}\left(\frac{2}{3}\right)$

Q-2: Determine x so that the line passing through (3, 4) and (x, 5) makes 135° angle with the positive direction of x-axis**.**

Sol: The slope of the line passing through (3, 4) and (x, 5)= $\frac{5-4}{x-3} = \frac{1}{x-3}$

Again, the line makes 135° angle with the positive direction of x-axis,

So its slope= $\tan 135^\circ = -1$.

Therefore, $\quad \frac{1}{x-3} = -1, \quad$ Or, $x = 2$

Intercepts of a line on the axes

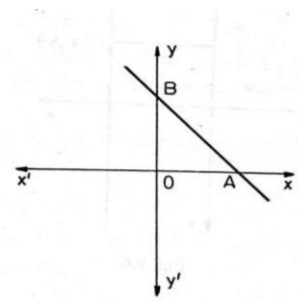


Fig 3.15

If a straight-line cuts x-axis at A and the y–axis at B, then the lengths OA and OB are known as the intercepts of the line x-axis and y-axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of co-ordinate axes.

From Fig. 3.15, OA= x-intercept, OB=y-intercept.

OA is positive or negative according as A lies on OX and OX' respectively. Similarly, OB is positive or negative according as B lies on OY or OY' respectively.

Different forms of equation of a straight line

1. Slope - intercept form:

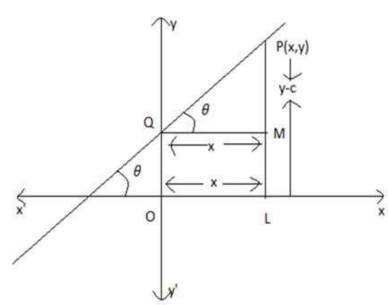


Fig 3.16

Let the given line intersects y-axis at Q and makes an angle θ with x-axis.

Then $m = \tan\theta$.

Let $P(x, y)$ be any point on the line.

From Fig 3.16,.

Clearly $\angle MQP = \theta, QM = OL = x$ and $PM = PL - ML = PL - OQ$

From triangle PMQ, we have

$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$

Or, $\quad m = \frac{y-c}{x}$

Or, $\quad y = mx + c$, is the required equation of the line.

Notes:

1. If the line passes through the origin, then $0 = m.0 + c$ or $c = 0$.
   Therefore, the equation of a line passing through the origin is given by $y = mx$.
2. If the line is parallel to x-axis, then m=0.
   Therefore, the equation of a line parallel to x-axis is y=c.
3. If the line is perpendicular to x-axis, then slope of the line 'm' is not defined But $\frac{1}{m} = 0$.

Therefore, the equation of a line perpendicular to x-axis is $x = \frac{1}{m}y - \frac{c}{m}$. where $\frac{c}{m}$ is the x-intercept. So, $x = 0 - c_1 = c_2$ i.e. $x = constant$ is the equation of line perpendicular to $x-$axis

Some Solved Problems

Q-1: Find equation of the line which has slope 2 and y intercept 3.

Sol:

Given Slope = m =2 and y-intercept , c = 3

Using slope – intercept form,

Equation of straight line with slope $m = 2$ and y-intercept $c = 3$ is given by

$y = mx + c$

Or, $y = 2x + 3$

Or, $2x - y + 3 = 0$

Q-2: Find the equation of a line with slope 1 and cutting off an intercept 2 units on the negative direction of y-axis.

Sol:

Let m be the slope and c be the y-intercept of the required line

Given $m = 1$ and c=–2.

$\therefore$ The equation of the line with slope $m = 1$ and y-intercept $c = -2$ is given by

$y = mx + c$

Or, $y = 1.x + (-2)$

Or, $x - y - 2 = 0$

Q-3: Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of y-axis and makes an angle of 120° with the positive direction of x-axis.

Sol:

Here slope, $m = \tan 120^\circ = \tan(90 + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$

and y-intercept, $c = -5$

Using slope – intercept form $y = mx + c$

Therefore, the equation of the required line is $y = -\sqrt{3}x - 5$ Or, $\sqrt{3}x + y + 5 = 0$

2. One point - slope form:

Let the line with slope m passe through Q(x_1, y_1)

Let P(x,y) be any point on the line .

Then slope of the line is given by m = $\dfrac{y - y_1}{x - x_1}$

Therefore, $y - y_1 = m(x - x_1)$ is the equation of required line.

Some Solved Problems

Q-1: Find equation of the line which passes through (1,2) and slope 2.

Sol:

Using one point-slope form,

Equation of the line passes through (x_1, y_1) =(1,2) and slope m=2 is given by

$y - y_1 = m(x - x_1)$

Or, $y - 2 = 2(x - 1)$

Or, $2x - y = 0$

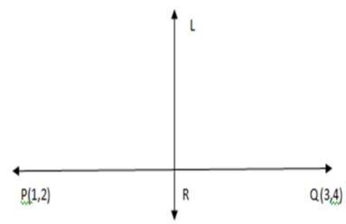
Q-2: Determine the equation of line through the point (4, –5) and parallel to x-axis.
Sol:
Since the line is parallel to x-axis, slope , m = 0.
Using point-slope form,
Equation of the line passes through (x_1, y_1) =(4, –5) and slope m=0 is given by

$y - y_1 = m(x - x_1)$

Or, $y + 5 = 0(x - 4)$

Or, $y + 5 = 0$

Q- 3: Find equation of the line which bisects the line segment joining P (1, 2) and Q (3, 4) at right angle.
Sol:
Let R be the mid-point of the line joining
P (1, 2) and Q (3, 4).



So, co-ordinates of R are R $\left(\frac{1+3}{2}, \frac{2+4}{2}\right)$ = R (2, 3)

Now, slope of PQ = $m_{PQ} = \frac{4-2}{3-1} = \frac{2}{2} = 1$

The line LR passes through R (2, 3) and perpendicular to

So Slope = $m_{LR} = -\frac{1}{m_{PQ}} = \frac{-1}{1} = -1$

Equation of the line LR which passes through the point (2, 3) and slope -1 is

$y - y_1 = m(x - x_1)$

Or, $y - 3 = -1(x - 2)$

Or, $y - 3 = -x + 2$

Or, $x + y - 5 = 0$

3. Two-point form:
Let m be the slope of a line passing through two points (x_1, y_1) and (x_2, y_2).

$\therefore$ Slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

the equation of the required line is

$y - y_1 = m(x - x_1)$ (one point –slope form)

Putting m = $\frac{y_2 - y_1}{x_2 - x_1}$ in above equation, we get

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of required line.

Some Solved Problems
Q-1: Find equation of the line which passes through two points P (1, 2) and Q (3, 4).
Sol:
Let m be the slope of the line PQ joining the points P (1, 2) and Q (3, 4).

$\therefore$ Slope= $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = 1$

Here, $x_1 = 1, y_1 = 2$ and $x_2 = 3, y_2 = 4$
Using two-point form,
Equation of the required line is

$y - y_1 = m(x - x_1)$

Or, $y - 2 = 1(x - 1)$

Or, $x - y + 1 = 0$

Q-2: Prove that the points (5, 1), (1, –1) and (11, 4) are collinear. Find the equation of the line on which these points lie**.**

Proof:

Let the given points be A(5, 1), B(1, –1) and C(11, 4).
Then the equation of the line passing through A(5, 1) and B(1, –1) is

$y - 1 = \frac{-1-1}{1-5}(x - 5)$

Or, $y - 1 = \frac{1}{2}(x - 5)$

Or, $x - 2y - 3 = 0$

Put $x = 11$ and $y = 4$ in the above equation, we get, $11 - 2 \times 4 - 3 = 0$,
Clearly, the point C(11, 4) satisfies the equation $x - 2y - 3 = 0$.
Hence, the given points A, B and C lie on the same straight line and whose equation is $x - 2y - 3 = 0$.

4. Intercept form:

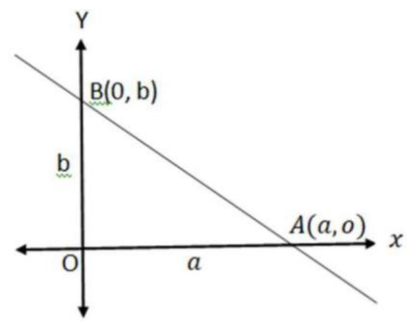


Fig 3.18

Let AB be a straight line cutting the x-axis and y-axis at A($a, 0$) and B(0, b) respectively. (Fig 3.18).
Let x –intercept$= OA = a$ and
Let y-intercept $= OB = b$
Therefore, using two-point form,
The equation of the required straight line passing through A($a, . 0$) and ($0, b$)
By two point form, its equation is given by

$(y - 0) = \frac{b-0}{0-a}(x - a)$

Or, $y = -\frac{b}{a}(x - a)$

Or, $bx + ay = ab$

Dividing both sides by ab

Or, $\frac{x}{a} + \frac{y}{b} = 1$, is the equation of line in **intercept form**.

Some Solved Problems

Q-1: Find equation of the line which has x-intercept is 2 and y – intercept equal to 3 .

Sol:

Given x-intercept $= a = 2$ and y-intercept $= b = 3$
Using intercept form,
Equation of the straight line is

$\frac{x}{a} + \frac{y}{b} = 1$

Or, $\frac{x}{2} + \frac{y}{3} = 1$

Or, $3x + 2y - 6 = 0$

Q-2: Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (2, 3).

Sol:

Let the equation of the line with intercepts a and b is $\frac{x}{a}+\frac{y}{b}=1$.

Since it makes equal intercepts on the co-ordinate axes, then $a=b$

$\therefore$Equation of the line is $\frac{x}{a}+\frac{y}{a}=1$ Or, $x+y=a$

The line $x+y=a$ passes through the point $(2,3)$.

So, $2+3=a$ Or, $a=5$

Hence, the equation of the required line is $\frac{x}{5}+\frac{y}{5}=1$ Or, $x+y=5$.

Q-3: Find the equation of the straight line which passes through the point (3, 4) and the sum of the intercepts on the axes is 14.

Sol:

Let the equation of the line with intercepts a and b be $\frac{x}{a}+\frac{y}{b}=1$ (1)

Given, the line passes through the point (3, 4).

$\therefore \frac{3}{a}+\frac{4}{b}=1$ Or, $3b+4a=ab$ (2)

Also given that, sum of intercepts = 14 i.e. $a+b=14$ (3)

Solving equations (2) and (3), we have,

$3(14-a)+4a=a(14-a)$

Or, $a^2-13a+42=0$

Or, $(a-6)(a-7)=0$

Or, $a=6$ and 7

For $a=6$, the value of $b=8$ and for $a=7$, the value of $b=7$.

Putting the values of a and b in equation (1), we get

$\frac{x}{6}+\frac{y}{8}=1$ and $\frac{x}{7}+\frac{y}{7}=1$

Or, $4x+3y=24$ and $x+y=7,$ are the equations of the required lines.

5. Normal Form / Perpendicular Form:

Let AB be the line whose equation is to be obtained. From O draw OL perpendicular on AB, then $OL=p$, $\angle AOL=\alpha$.

Let $P(x,y)$ be any point on AB. From P draw PM perpendicular on x-axis.

Then, $\angle A=90°-\alpha$

Therefore, $\angle APM=\alpha$. From ΔAOL, we have, $\frac{OL}{OA}=\cos\alpha$.

Therefore $OL=OA\cos\alpha$.

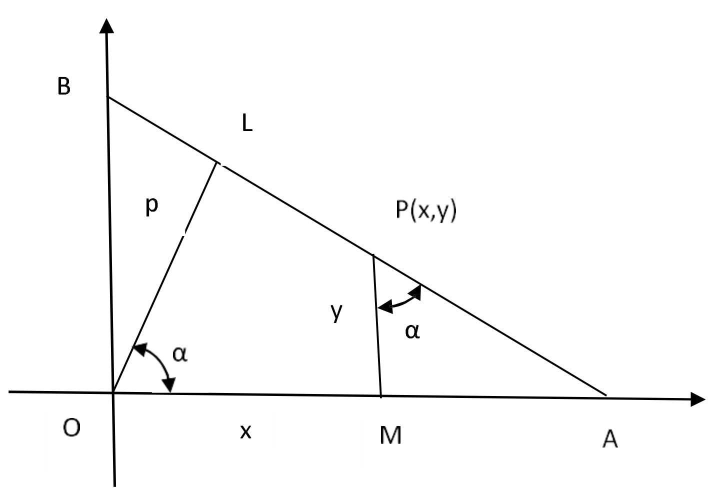


Fig.3.19

Hence, $P = (OM + MA)\cos\alpha = \left(x + \frac{MA}{MP} \times MP\right)\cos\alpha = (x + y.\tan\alpha)\cos\alpha$

$= \left(x + \frac{\sin\alpha}{\cos\alpha} y\right)\cos\alpha.$

Therefore, $p = x\cos\alpha + y\sin\alpha.$

is the equation of required line and known as normal or perpendicular form

Some Solved problems

Q-1: Find equation of the line which is at a distance 2 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of x –axis**.**

Sol:

The required line is 2 unit distance from the origin,

i.e. the perpendicular distance from the origin to the required line, $p = 2$.

Let α be the angle make by the perpendicular from the origin with positive x –axis.

and given that $\alpha = 30^{\circ}$.

Using normal form, the equation of the required line is

$x\cos\alpha + y\sin\alpha = p$

Or, $x\cos 30 + y\sin 30 = 2$

Or, $x\,\frac{\sqrt{3}}{2} + y\,\frac{1}{2} = 2$

Or, $\sqrt{3}x + y - 4 = 0$

Transformation of general equation in different standard forms

The general equation of a straight line is $Ax + By + C = 0$ which can be transformed to various standard forms as discussed below.

To transform $ax + by + c = 0$ in the slope intercept form $(y = mx + c)$

We have $Ax + By + C = 0$.

Or, $By = -Ax - C$

Or, $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$, which is of the form $y = mx + c$, where $m = -\frac{A}{B}$, $c = -\frac{C}{B}$.

Thus, for the straight line $Ax + By + C = 0$,

Slope $= m = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

and $\text{Intercept on } y-\text{axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$

Note. To determine the slope of a line by the formula $= -\frac{coefficient\ of\ x}{coefficient\ of\ y}$, transfer all terms in the equation on one side.

To transform $Ax + By + C = 0$ in intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

We have $Ax + By + C = 0$

Or, $Ax + By = -C$

Or, $\frac{Ax}{-C} + \frac{By}{-C} = 1$

Or, $\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$, which is of the form $\frac{x}{a} + \frac{y}{b} = 1$.

Thus, for the straight line $Ax + By + C = 0$,

$$\text{Intercept on } x\text{-axis} = -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}$$
$$\text{Intercept on } y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

Note. As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We can also use the following method to determine the intercepts on the axes:

For intercepts on x-axis, put $y = 0$ in the equation of the line and find the value of x. Similarly, to find y-intercept, put $x = 0$ in the equation of the line find the value of y.

To transform $Ax + By + C = 0$ in the normal form ($x\cos\alpha + y\sin\alpha = p$).

We have $Ax + By + C = 0$ (1)

Let $x\cos\alpha + y\sin\alpha - p = 0$ (2)

Be the normal form of $Ax + By + C = 0$.

If the equations (1) and (2) represent the same straight line.

Therefore, $\frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$

Or, $\cos\alpha = -\frac{Ap}{C}$ and $\sin\alpha = -\frac{Bp}{C}$ (3)

Or, $\cos^2\alpha + \sin^2\alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$

Or, $1 = \frac{p^2}{C^2}(A^2 + B^2)$

Or, $p = \pm\frac{C}{\sqrt{A^2+B^2}}$

But p denotes the length of the perpendicular from the origin to the line and is always positive. Therefore, $p = \frac{C}{\sqrt{A^2+B^2}}$

Putting the value of p in (3), we get $\cos\alpha = -\frac{A}{\sqrt{A^2+B^2}}, \sin\alpha = -\frac{B}{\sqrt{A^2+B^2}}$.

Therefore, equation (2) takes of the form $-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y - \frac{C}{\sqrt{A^2+B^2}} = 0$

Or, $-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$

which is the required normal form of the line $Ax + By + C = 0$.

Note. To transform the general equation to normal form we perform the following steps:

(i) Shift the constant term on the RHS and make it positive

(ii) Divide both sides by $\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$.

Some Solved Problems

Q-1 : Transform the equation of the line $\sqrt{3}x + y - 8 = 0$ to

(i) slope intercept form and find its slope and y-intercept.

(ii) intercept form and find intercepts on the coordinate axes.

(iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.

Sol:

(i) We have $\sqrt{3}x + y - 8 = 0$ or $y = -\sqrt{3}x + 8$.

This is the slope intercept form of the given line. Therefore, slope$= -\sqrt{3}$ and y-intercept$= 8$.

(ii) We have $\sqrt{3}x + y - 8 = 0$ or $\frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$.

This is the intercept form of the given line. Therefore, x-intercept$= \frac{8}{\sqrt{3}}$, y-intercept$= 8$.

(iii) We have $\sqrt{3}x + y - 8 = 0$

Or, $\sqrt{3}x + y = 8$

Or, $\frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2+1^2}}x + \frac{1}{\sqrt{(\sqrt{3})^2+1^2}}y = \frac{8}{\sqrt{(\sqrt{3})^2+1^2}}$

Or, $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4$.

This is the normal form of the given line. Therefore, $\cos\alpha = \frac{\sqrt{3}}{2}$, $\sin\alpha = \frac{1}{2}$ and $p = 4$.

Since $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant and is equal to $\alpha = \frac{\pi}{6}$.

Hence $\alpha = \frac{\pi}{6}$ and $p = 4$.

Q-2: Reduce the lines $3x - 4y + 4 = 0$ and $4x - 3y + 12 = 0$ to the normal form and hence determine which line is nearer to the origin.

Sol:

The equation of the given line is $3x - 4y + 4 = 0$

Or, $-3x + 4y = 4$

Or, $-\frac{3x}{\sqrt{(-3)^2+4^2}} + \frac{4y}{\sqrt{(-3)^2+4^2}} = \frac{4}{\sqrt{(-3)^2+4^2}}$

Or, $-\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$.

This is the normal form of $3x - 4y + 4 = 0$ and the length of the perpendicular from the origin to it is $p_1 = \frac{4}{5}$.

Again, the equation of second line be $4x - 3y + 12 = 0$

Or, $-4x + 3y = 12$

Or, $-\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$

Or, $-\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$.

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is $p_2 = \frac{12}{5}$.

Clearly, $p_2 > p_1$, therefore the line $3x - 4y + 4 = 0$ is nearer to the origin.

Q-3: Find the equation of a line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$.

Sol:

Let c be the intercept on y-axis.

Then the equation of the line is $y = 2x + c$ (1)

Or, $-2x + y = c$

Or, $-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{1}{\sqrt{(-2)^2+1^2}}y = \frac{c}{\sqrt{(-2)^2+1^2}}$ (Dividing both sides by $\sqrt{(-2)^2 + 1^2}$)

Or, $-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{c}{\sqrt{5}}$, which is the normal form of (1), Therefore RHS denotes the length of the perpendicular from the origin. But the length of the perpendicular from the origin is $\sqrt{5}$.

Therefore, $\frac{c}{\sqrt{5}} = \sqrt{5} \Rightarrow c = 5$.

Putting $c = 5$ in (1), we get $y = 2x + 5$,
which is the required equation of the required line.

Q-4: Find equation of the line which passes through P (1, 2) and parallel to the line $x + 2y + 3 = 0$**.**

Sol:

The given line is $x + 2y + 3 = 0$

So, slope, m $= -\frac{coefficient\ of\ x}{coefficient\ of\ y} = -\frac{1}{2}$

Since the required line is parallel to the given line.

Equation of the required line passes through P (1, 2) and $m = -1/2$ is

$y - y_1 = m\,(x - x_1)$

Taking $m = -1/2$

$y - 2 = \frac{-1}{2}\,(x - 1)$

Or, $2y - 4 = -x + 1$

Or, $x + 2y - 5 = 0$

Q-5: Find equation of the line which passes through (2, 3) and perpendicular to the line $3x + 2y + 5 = 0$**.**

Sol:

The equation of given line is $3x + 2y + 5 = 0$

Slope $= m_{given} = \frac{-3}{2}$

Since requires line is perpendicular to the given line,

therefore $m_{required} = \frac{-1}{m_{given}} = \frac{2}{3}$

So, the equation of the required line which passes through (2, 3) and slope 2/3 is

$y - y_1 = m\,(x - x_1)$

Or, $y - 3 = \frac{2}{3}\,(x - 2)$

Or, $3y - 9 = 2x - 4$

Or, $2x - 3y + 5 = 0$

Equation of a line parallel to a given line

Let m be the slope of the line $ax + by + c = 0$.

Then slope, $m = -\frac{a}{b}$ $\left(\text{using } m = -\frac{coefficient\ of\ x}{coefficient\ of\ y}\right)$.

Let c_1 be the y-intercept of the required line.

Therefore, the equation of the required line is

$y = mx + c_1$ (Using slope-intercept form)

Or, $y = -\frac{a}{b}x + c_1$

Or, $ax + by - bc_1 = 0$

Or, $ax + by + \lambda = 0$, where $\lambda = -bc_1 = \text{constant}$.

Therefore, the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.

Note. To write a line parallel to a given line, we keep the expression containing x and y same and simply replace the given constant by a new constant λ. The value of λ can be determined by some given condition.

Equation of a line perpendicular to a given line

Let m_1 be the slope of the given line and m_2 be the slope of a line perpendicular to the given line.

Then $m_1 = -\frac{a}{b}$ and $m_1 m_2 = -1$. (Using perpendicular condition)

Therefore, $m_2 = -\frac{1}{m_1} = \frac{b}{a}$.

Let c_2 be the y-intercept of the required line. Then its equation is

$y = m_2 x + c_2$

Or, $y = \frac{b}{a}x + c_2$

Or, $bx - ay + ac_2 = 0$

Or, $bx - ay + \lambda = 0$, where $\lambda = ac_2 = \text{constant}$.

Therefore, the equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant

Note. To write a line perpendicular to a given line

(i) Interchange x and y.

(ii) If the coefficients of x and y in the given equation are of the same sign, make them of opposite signs and if the coefficients are of opposite signs, make them of the same sign.

(iii) Replace the given constant by a new constant λ, which is determined by a given condition.

Some Solved problems

Q- 1: Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5, -6)$.

Sol.

The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$3x - 2y + \lambda = 0$ (1)

The line passes through the point $(5, -6)$.

Thus, $3 \times 5 - 2 \times (-6) + \lambda = 0 \Rightarrow \lambda = -27$.

Putting $\lambda = -27$ in (1), we get $3x - 2y - 27 = 0$ which is the required equation of line.

Q-2: Find the equation of the straight line that passes through the point $(3, 4)$ and perpendicular to the line $3x + 2y + 5 = 0$.

Sol.

The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$2x - 3y + \lambda = 0$ (1)

The line passes through the point $(3, 4)$.

Thus, $3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$.

Putting $\lambda = 6$ in (1), we get $2x - 3y + 6 = 0$ which is the required equation of line.

Intersection of two lines:

Let the equations of two lines be

L_1: $A_1x + B_1y + C_1 = 0$ and L_2: $A_2x + B_2y + C_2 = 0$

Slope of $L_1 = -\frac{A_1}{B_1}$ and Slope of $L_2 = -\frac{A_2}{B_2}$

If two line are parallel, i.e. $L_1 \parallel L_2$, then $-\frac{A_1}{B_1} = -\frac{A_2}{B_2}$ Or, $\frac{A_1}{A_2} = \frac{B_1}{B_2}$

If two lines are not parallel to each other they will intersect at a point and solving both the equations, we get the point of intersection.

Concurrency:

Three lines are said to be concurrent if they pass through a common point.

Some Solved Problems

Q-1: Find equation of the line which passes through the point of intersection of two given lines $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$ and parallel to the line $x + y - 2 = 0$**.**

Sol:

To find the point of intersection of two given lines $2x - y - 1 = 0$ and $3x - 4y - 4 = 0$, we solve these equations. We get $x = 2$ and y = 3

The co-ordinate of point of intersection of two given lines is (2, 3)

Now Slope of the given line $x + y - 2 = 0$ is

$m_{given} = \frac{-A}{B} = \frac{-1}{1} = -1$

Since the required line is parallel to the given line $x + y - 2 = 0$.

Therefore, Slope $m_{req} = m_{given}$ (two lines are parallel)

Or, $m_{req} = -1$

So equation of the line passes through the point (2, 3) with slope -1 is

$y - 3 = -1\,(x - 2)$

Or, $x + y - 5 = 0$

Q--2: Find equation of the line which passes through the intersection of the lines $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$ and perpendicular to the line $x + 2y - 1 = 0$.

Sol:

Slope of the given line L_1: $x + 2y - 1 = 0 = m_{\text{given}} = -\frac{1}{2}$ ($m = -\frac{A}{B}$)

Slope of the required line(L_2) perpendicular to the line L_1: $x + 2y - 1 = 0$ is $m_{req} = -\frac{1}{\frac{-1}{2}} = 2$

To find the intersection point of two lines $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$, we solve these equations and we get $x = \frac{8}{5}$, $y = \frac{-6}{5}$,

So, equation of the required line(L_2) passes through the point $\left(\frac{8}{5}, \frac{-6}{5}\right)$ and slope m = 2 is

$$y + \frac{6}{5} = 2\left(x - \frac{8}{5}\right)$$

Or, $2x - y - \frac{22}{5} = 0$

Or, $10x - 5y - 22 = 0$

Perpendicular distance:

(Length of perpendicular from a point $P(x_1, y_1)$ to a line $Ax + By + C = 0$)

PM is the length of perpendicular from the point P(x_1, y_1) to the line AB which has equation Ax + By + C = 0

$$|\overline{PM}| = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

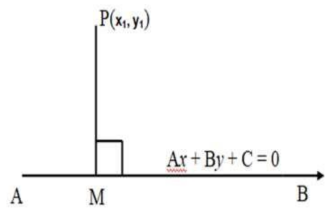


Fig 3.21

Note: The length of the perpendicular from the origin to the line $Ax + By + C = 0$ is $\left|\frac{C}{\sqrt{A^2+B^2}}\right|$.

Some Solved Problems

Q- 1: Find length of perpendicular from a point (2, 3) to a line 3x – y + 4 = 0.

Sol:

The length of perpendicular from a point (2, 3) to a line 3x – y + 4 = 0 is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3\times 2 + (-1)(3) + 4|}{\sqrt{(3)^2 + (-1)^2}} = \frac{|6 - 3 + 4|}{\sqrt{9+1}} = \frac{7}{\sqrt{10}}$$

Q- 2: Find distance between two parallel lines $x + y + 1 = 0$ and $2x + 2y + 3 = 0$.

Sol:

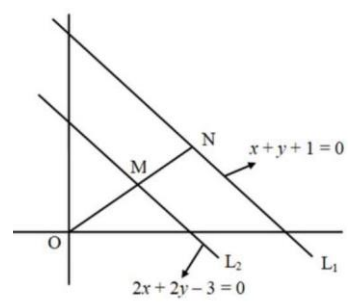


Fig 3.22

From Fig 3.22,

MN =distance between two parallel lines.

=ON – OM

ON= length of perpendicular from (0, 0) to the line $x + y + 1 = 0$

is $\frac{|1(0)+1(0)+1|}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

OM= length of perpendicular from (0, 0) to $2x + 2y - 3 = 0$

is $= \frac{|2(0) + 2(0) - 3|}{\sqrt{(2)^2 + (2)^2}} = \frac{3}{\sqrt{8}} = \frac{3}{2\sqrt{2}}$

Therefore, $MN = ON - OM = \frac{1}{\sqrt{2}} - \frac{3}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$

Alternate method

The distance of the st. line $x + y + 1 = 0$ from the origin is given by

$$p_1 = \frac{1}{\sqrt{2}}$$

The distance of the st. line $2x + 2y + 3 = 0$ from the origin is given by

$$p_2 = \frac{3}{2\sqrt{2}}$$

Since the lines are on the same side of origin

We have,

The distance between the lines = $p_2 - p_1 = \frac{3}{2\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

Note: If the lines are on the opposite sides of the origin, then

The distance between the lines = $p_2 + p_1$

Distance between two parallel lines:
If two lines are parallel, then they have the same distance between them throughout. Therefore, to find the distance between two parallel lines choose an arbitrary point on any one of the line and find the length of the perpendicular on the other line. To choose a point on a line give an arbitrary value to *x* or *y*, and find the value of other variable.

Some Solved Problems

Q-1: Find the distance between the parallel lines $3x - 4y + 9 = 0$ and $6x - 8y - 15 = 0$.
Sol:
Putting $y = 0$ in $3x - 4y + 9 = 0$, we get $x = -3$.
Therefore, $(-3, 0)$ is a point on the line $3x - 4y + 9 = 0$.
Now, the length of the perpendicular from $(-3, 0)$ to $6x - 8y - 15 = 0$ is given by

$$\left|\frac{6(-3) - 8 \times 0 - 15}{\sqrt{6^2 + (-8)^2}}\right| = \left|\frac{-33}{\sqrt{100}}\right| = \frac{33}{10} \text{ units}$$

Q-2: Find the equation of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.
Sol:
Equation of any line parallel to $3x - 4y - 5 = 0$ is

$$3x - 4y + \lambda = 0 \qquad (1)$$

Putting $x = -1$ in $3x - 4y - 5 = 0$, we get $y = -2$.
Therefore, $(-1, -2)$ is a point on $3x - 4y - 5 = 0$.
Length of perpendicular from the point $(-1, -2)$ on the line $3x - 4y + \lambda = 0$ is given by

$$\left|\frac{3(-1) - 4(-2) + \lambda}{\sqrt{3^2 + (-4)^2}}\right| = \left|\frac{5 + \lambda}{5}\right|$$, which is the distance between two lines.

Given that, Distance between two lines $= 1$

Or, $\left|\frac{5+\lambda}{5}\right| = 1$

Or, $5 + \lambda = \pm 5$

Or, $\lambda = 0$ or -10

Putting the values of λ in equation (1), we get $3x - 4y = 0$ or $3x - 4y - 10 = 0$, which are the equations of required lines.

Circle

Definition:
A circle is the locus of a point which moves on a plane in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

In the Fig 3.23, $P(x, y)$ is the moving point, C is the centre and CP is the radius.

1. **Standard form** (Equation of a Circle with given centre and radius)

Let C(α, β) be the centre of the circle and radius of the circle be'r' .
Let $P(x, y)$ be any point on the circumference of the circle.
Then,

$$CP = r$$

By distance formula,

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = r$$

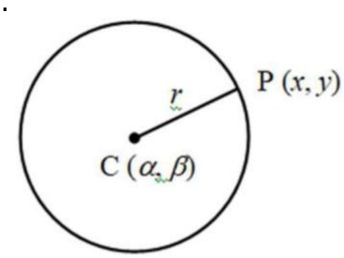


Fig 3.23

Or, $(x-\alpha)^2+(y-\beta)^2=r^2$

Which is the equation of the circle having centre at $(\alpha,\ \beta)$ and radius $'r'$, which is known as standard form of equation of a circle.

Note: If the centre of the circle is at origin, $(0,0)$ and radius is 'r', then the above standard equation of the circle reduces to $x^2+y^2=r^2$.

Some Particular Cases:

The standard equation of the circle with centre at $C(\alpha,\ \beta)$ and radius r, is

$$(x-\alpha)^2+(y-\beta)^2=r^2 \qquad (1)$$

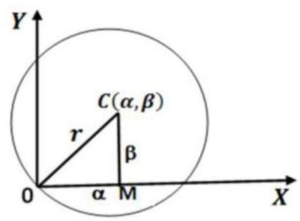


Fig. 3.24

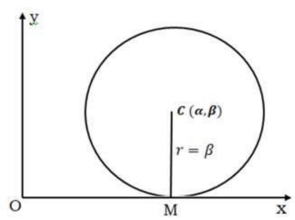


Fig 3.25

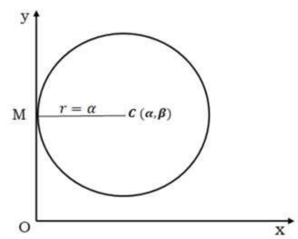


Fig 3.26

(i) **When the circle passes through the origin**

From the Fig 3.24, In right angle triangle ΔOCM,

$OC^2=OM^2+CM^2$ i.e. $r^2=\alpha^2\ +\beta^2$

Then eqn (1) becomes,

$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$

Or, $x^2+y^2-2\alpha x-2\beta y=0$

(ii) **When the circle touches x – axis**

In the Fig 3.25, Here, $r=\beta$

Hence, the eqn (1) of the circle becomes,

$(x-\alpha)^2+(y-\beta)^2=\beta^2$

Or, $x^2+y^2-2\alpha x-2\beta y+\alpha^2=0$

(iii) **When the circle touches y – axis**

In the Fig 3.26, Here, $r=\alpha$

Hence, the eqn (1) of the circle becomes,

$(x-\alpha)^2+(y-\beta)^2=\alpha^2$

Or, $x^2+y^2-2\alpha x-2\beta y+\beta^2=0$

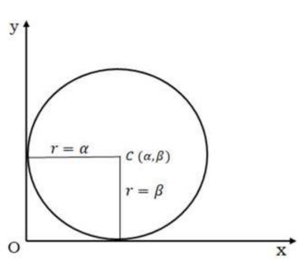


Fig 3.27

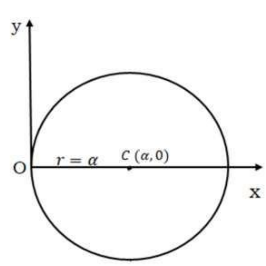


Fig 3.28

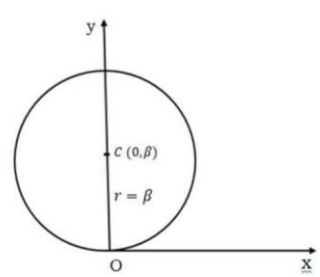


Fig 3.29

(iv) **When the circle touches both the axes**

In the Fig, 3.27, Here, $\alpha = \beta = r$

Hence, the eqn (1) of the circle becomes,

$$(x-r)^2 + (y-r)^2 = r^2$$

Or, $x^2 + y^2 - 2rx - 2ry + r^2 = 0$

(v) **When the circle passes through the origin and centre lies on x- axis**

Here, $\alpha = r$ and $\beta = 0$

Hence, the eqn (1) of the circle becomes,

$$(x-r)^2 + (y-0)^2 = r^2$$

Or, $x^2 + y^2 - 2rx = 0$

(vi) **When the circle passes through the origin and centre lies on y- axis**

Here, $\alpha = 0$ and $\beta = r$

Hence, the eqn (1) of the circle becomes,

$$(x-0)^2 + (y-r)^2 = r^2$$

Or, $x^2 + y^2 - 2ry = 0$

Some Solved Problems

Q- 1: Find equation of the circle which has centre at (2, 3) and radius is 4 .

Sol:

According to the standard form, the equation of circle with centre at (α, β) and radius r is

$$(x - \alpha) + (y - \beta)^2 = r^2$$

$\therefore$ Equation of the circle with centre at (2, 3) and radius 4 is,

$$(x - 2)^2 + (y - 3)^2 = (4)^2$$

Or, $x^2 + y^2 - 4x - 6y + 13 = 16$

Or, $x^2 + y^2 - 4x - 6y - 3 = 0$

Q- 2: Find equation the circle which has centre at (1, 4) and passes through a point (2, 6).

Sol:

Given $C(1,4)$ be the centre and r be the radius of the circle. The circle passes through the point $P(2,6)$

$\therefore$ $PC = r$

Or, $\sqrt{(2-1)^2 + (6-4)^2} = r$ (By using distance formula)

Or, $\sqrt{1+4} = r$

Or, $r = \sqrt{5}$

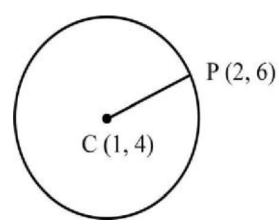


Fig 3.30

By using standard form of the circle,

Equation of the circle with centre at C(1,4) and radius $\sqrt{5}$ is

$$(x - 1)^2 + (y - 4)^2 = \left(\sqrt{5}\right)^2$$

Or, $x^2 + y^2 - 2x - 8y + 1 + 16 = 5$

Or, $x^2 + y^2 - 2x - 8y + 12 = 0$

Q- 3: Find equation of the circle whose centre is at (5, 5) and touches both the axis.

Sol:

The centre of the given circle is at (5, 5).

Since the circle touches both the axes,

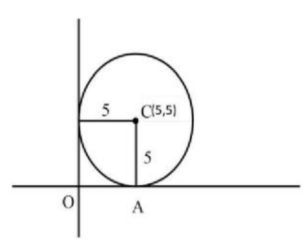


Fig 3.31

$\therefore$ radius, $r = 5$

According to the standard form,

$\therefore$ Equation of the circle with centre at C(5,5)

and radius $r = 5$ is

$$(x-5)^2 + (y-5)^2 = (5)^2$$

Or, $x^2 + y^2 - 10x - 10y + 25 = 0$

Q- 4: If the equation of two diameters of a circle are $x - y = 5$ and $2x + y = 4$, and the radius of the circle is 5, find the equation of the circle.

Sol:

Let the diameters of the circle be AB and LM, whose equations are respectively,

$$x - y = 5 \quad (1)$$

and $2x + y = 4$ (2)

Since, the point of intersection of any two diameters of a circle is its centre and by solving the equations of two diameters we find the co-ordinates of the centre.

$\therefore$ Solving eqns. (1) and (2), we get $x = 3$ and $y = -2$

Therefore, co-ordinates of the centre are $(3, -2)$ and radius is 5.

Hence, equation of required circle is

$$(x-3)^2 + (y+2)^2 = 5^2$$

Or, $x^2 + y^2 - 6x + 4y + 9 + 4 = 25$

Or, $x^2 + y^2 - 6x + 4y - 12 = 0$

Q-5: Find the equation of a circle whose centre lies on positive direction of $y-$ axis at a distance 6 from the origin and whose radius is 4.

Sol:

Given, the centre of the circle lies on positive y-axis at a distance 6 units from origin.

$\therefore$ The centre of the circle lies at the point $C(0, 6)$.

Hence, equation of the circle with centre at C(0, 6) and radius '4' is

$$(x-0)^2 + (y-6)^2 = 4^2$$

Or, $x^2 + y^2 - 12y + 36 = 16$

Or, $x^2 + y^2 - 12y + 20 = 0$

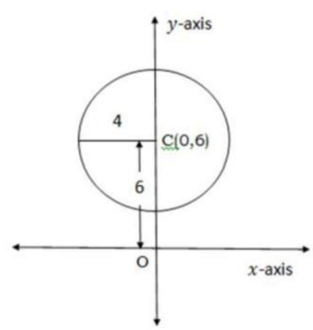


Fig 3.32

2. General form

Theorem: The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is at $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Proof:

The given equation is $x^2 + y^2 + 2gx + 2fy + c = 0$

Or, $(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

Or, $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

Or, $\{x - (-g)\}^2 + \{y - (-f)\}^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

Which is in the standard form (i.e. $(x-\alpha)^2 + (y-\beta)^2 = r^2$) of the circle with centre at (α, β) and radius 'r'.

Hence the given equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle whose centre is $(-g, -f)$ i.e. $\left(-\frac{1}{2} \text{ coeff. of } x, \quad -\frac{1}{2} \text{ coeff. of } y\right)$,

And radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2} \text{ coeff. of } x\right)^2 + \left(-\frac{1}{2} \text{ coeff. of } y\right)^2 - \text{constant term}}$

Notes : Characteristics of the general form of equation of circle

The characteristics of general form $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle are

i. It is quadratic(of second degree) both in x and y.
ii. Coefficient of $x^2 =$ Coefficient of y^2.
iii. It is independent of the term xy, i.e. there is no term containing xy.
iv. Contains three arbitrary constants i.e. g, f and c.

Note : To find the centre and radius of the circle, which is in the form $ax^2 + ay^2 + 2gx + 2fy + c = 0$, where $a \neq 0$,

Divide both sides of the equation by coefficient of x^2 or y^2 (i.e. a) to get

$x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$,

Which is in the general form of the circle

Hence, the co-ordinates of the centre are $\left(-\frac{1}{2} \text{ coeff. of } x, \quad -\frac{1}{2} \text{ coeff. of } y\right)$

$= \left(-\frac{1}{2}\frac{2g}{a}, \quad -\frac{1}{2}\frac{2f}{a}\right) = \left(-\frac{g}{a}, \quad -\frac{f}{a}\right)$

and radius= $\sqrt{\left(-\frac{1}{2} \text{ coeff. of } x\right)^2 + \left(-\frac{1}{2} \text{ coeff. of } y\right)^2 - \text{constant term}}$

$= \sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}}$

Example-1:

Let the equation of a circle be $25x^2 + 25y^2 - 30x - 10y - 6 = 0$

To find the centre and radius of the above circle, divide by coefficient of x^2 i.e. 25, as

$\mathrm{x}^2 + \mathrm{y}^2 - \frac{30}{25}\mathrm{x} - \frac{10}{25}\mathrm{y} - \frac{6}{25} = 0$

Or, $x^2 + y^2 - \frac{6}{5}x - \frac{2}{5}y - \frac{6}{25} = 0$

Or, $x^2 + y^2 - \frac{6}{5}x - \frac{2}{5}y - \frac{6}{25} = 0$

Or, $x^2 + y^2 + 2\left(-\frac{3}{5}\right)x + 2\left(-\frac{1}{5}\right)y + \left(-\frac{6}{25}\right) = 0$,

which is the general form of circle with centre at $(-g, -f) = \left(\frac{3}{5}, \frac{1}{5}\right)$ and radius

$= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{1}{5}\right)^2 - \left(-\frac{6}{25}\right)} = \frac{4}{5}$

Example-2:

Consider the equation of a circle $x(x + y - 6) = y(x - y + 8)$

Or, $x^2 + xy - 6x = xy - y^2 + 8y$

Or, $x^2 + y^2 - 6x - 8y = 0$,

Which, is in the general form of circle.

$\therefore$ Centre $= (-g, -f) = \left(-\frac{1}{2} \text{ coeff. of } x, \quad -\frac{1}{2} \text{ coeff. of } y\right) = \left(-\frac{1}{2}(-6), -\frac{1}{2}(-8)\right) = (3, 4)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{1}{2}\text{ coeff. of }x\right)^2 + \left(-\frac{1}{2}\text{ coeff. of }y\right)^2 - \text{constant term}}$

$$= \sqrt{\left(-\frac{1}{2}(-6)\right)^2 + \left(-\frac{1}{2}(-8)\right)^2 - 0} = \sqrt{9 + 16 - 0} = 5$$

Example-3:
Let the equation of the circle be $x^2 + y^2 + 4x + 6y + 2 = 0$.
Compare this given equation with the general equation of the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$.
Here, $2gx = 4x$, $2fy = 6y$, and $c = 2$
So, $g = 2$, $f = 3$ and $c = 2$
Now, Centre is at (-g, -f) = (-2, -3) and $r = \sqrt{g^2 + f^2 - C} = \sqrt{4 + 9 - 2} = \sqrt{11}$

Some Solved Problems*:*
Q-1 : Determine which of the circles $x^2 + y^2 - 3x + 4y = 0$ and $x^2 + y^2 - 6x + 8y = 0$ is greater.
Sol:
The equations of two given circles are

$C_1 : x^2 + y^2 - 3x + 4y = 0$

and $C_2 : x^2 + y^2 - 6x + 8y = 0$

In 1st circle C_1, $g = \frac{3}{2}$, $f = 2$, $c = 0$

radius $= r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 - 0} = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$

Similarly, In 2nd circle C_2, $g = 3$, $f = 4$, $c = 0$

radius $= r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + 4^2 - 0} = \sqrt{9 + 16} = 5$

Since, $r_1 < r_2$, So, the 2nd circle $C_2 : x^2 + y^2 - 6x + 8y = 0$ is greater.

Q-2: Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ and having radius 10 units.
Sol:
The coordinates of the centre of the given circle, $x^2 + y^2 - 4x + 6y + 10 = 0$,are
$\left(-\frac{1}{2}\text{ coeff. of }x, \quad -\frac{1}{2}\text{ coeff. of }y\right) = (2, -3)$.
Since the required circle is concentric with the above circle , the centre of the required circle and above given circle are same.
$\therefore$ Centre of the required circle is at $(2, -3)$.
Hence, the equation of the required circle with centre at $(2, -3)$ and radius 10 is

$$(x - 2)^2 + (y + 3)^2 = (10)^2$$

Or, $x^2 + y^2 - 4x + 6y - 87 = 0$

Q-3: Find the equation of the circle whose centre is at the point (4, 5) and passes through the centre of the circle: $x^2 + y^2 - 6x + 4y - 12 = 0$.
Sol:
The co-ordinates of the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are

$$C_1\left(-\frac{1}{2}\text{ coeff. of }x, \quad -\frac{1}{2}\text{ coeff. of }y\right) = C_1(3, -2).$$

Therefore, the required circle passes through the point $C_1(3, -2)$.
Given, the centre of the required circle is at $C(4, 5)$

$\therefore$ radius of the required circle $= CC_1 = \sqrt{(4-3)^2+(5+2)^2} = \sqrt{1+49} = \sqrt{50}$

Hence, the equation of the required circle with centre at $C(4,5)$ and radius '$\sqrt{50}$' is

$$(x-4)^2+(y-5)^2 = \left(\sqrt{50}\right)^2$$

Or, $x^2+y^2-8x-10y-9=0$

Q-4: Find the equation of the circle concentric with the circle $4x^2 + 4y^2 - 24x + 16y - 9 = 0$ and having its area equal to 9π sq. units.

Sol:

The equation of given circle is $4x^2 + 4y^2 - 24x + 16y - 9 = 0$

Or, $x^2+y^2-6x+4y-\frac{9}{4}=0$

$\therefore$ Centre $\left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right) = (3,-2)$.

Since the required circle is concentric with the above circle , the centre of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is $(3,-2)$ and let its radius be 'r'

Again, Given Area of the required circle $= 9\pi$

Or, $\pi r^2 = 9\pi$

Or, $r = 3$ units

Therefore, the equation of the required circle with centre at $(3,-2)$ and radius '3' is

$$(x-3)^2+(y+2)^2 = (3)^2$$

Or, $x^2+y^2-6x+4y+4=0$

Q-5: Find the equation of the circle concentric with the circle $2x^2+2y^2+8x+12y-25=0$ and having its circumference equal to 6π sq. units.

Sol:

The equation of given circle be $2x^2+2y^2+8x+12y-25=0$

Or, $x^2+y^2+4x+6y-\frac{25}{2}=0$

$\therefore$ centre $= \left(-\frac{1}{2}\text{ coeff. of } x, \quad -\frac{1}{2}\text{ coeff. of } y\right) = (-2,-3)$.

Since the required circle is concentric with the above circle, the centre of the required circle and above given circle are same.

$\therefore$ Centre of the required circle is $(-2,-3)$ and let its radius be 'r'

Again, Given, circumference of the required circle $= 6\pi$

Or, $2\pi r = 6\pi$

Or, $r = 3$ units

Therefore, the equation of the required circle with centre at $(-2,-3)$ and radius '3' is

$$(x+2)^2+(y+3)^2 = 3^2$$

Or, $x^2+y^2+4x+6y+4=0$

Equation of a Circle satisfying certain given conditions

The general equation of a circle $x^2+y^2+2gx+2fy+c=0$ involves three unknown quantities $g,\ f$ and c called the arbitrary constants. These three constants can be determined from three equations involving $g,\ f$ and c. These three equations can be obtained from three independent given conditions. We find the values of g, f and c by solving these three equations, and putting these values in the equation of circle, we get the required equation of circle.

Some Solved Problems:

Q-1: Find equation of the circle passes through the points (0, 0), (1, 0) and (0, 1).

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle (1) passes through the points (0, 0), (1, 0) and (0, 1). We have,

$0 + 0 + 0 + 0 + c = 0$, Or. $c = 0$ (2)

$1 + 0 + 2g + 0 + 0 = 0$, Or, $g = -1/2$ (3)

and, $0 + 1 + 0 + 2f + 0 = 0$, Or, $f = -1/2$ (4)

Putting the values of g, f and c in equation (1), we get

$$x^2 + y^2 + 2\left(\frac{-1}{2}\right)x + 2\left(\frac{-1}{2}\right)y + 0 = 0$$

Or, $x^2 + y^2 - x - y = 0$, is the equation of required circle.

Q-2: Find the equation of the circle passes through the points $(0, 2)$, $(3, 0)$ and $(3, 2)$. Also, Find the centre and radius.

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Since, the circle (1) passes through the points (0, 2), (3, 0) and (3, 2) i.e these points lie on the circle (1), we have,

$\therefore$ $0 + 4 + 0 + 4f + c = 0$,

Or, $4f + c = -4$ (2)

$9 + 0 + 6g + 0 + c = 0$

Or, $6g + c = -9$ (3)

and, $9 + 4 + +6g + 4f + c = 0$

Or, $6g + 4f + c = -13$ (4)

On solving equations (2), (3) and (4),

Eqns (2)+(3) : $6g + 4f + 2c = -13$ (5)

Eqns (5)-(4) : $c = 0$

Putting the value of c in (2) and (3), we get

$4f = -4$ Or, $f = -1$

$6g = -9$ Or, $g = -\frac{3}{2}$

Putting the values of $g,\ f$ and c in the general eqn of circle (1), we get

$$x^2 + y^2 + 2\left(-\frac{3}{2}\right)x + 2(-1)y + 0 = 0$$

Or, $x^2 + y^2 - 3x - 2y = 0$, is the equation of required circle.

Now, the centre of the circle$= (-g, -f) = \left(\frac{3}{2},\ 1\right)$

and radius$= \sqrt{g^2 + f^2 - c}\ = \sqrt{\frac{9}{4} + 1 - 0} = \frac{\sqrt{13}}{2}$

Q-3: Find the equation of the circle which passes through the origin and cuts off intercepts a and b from the positive parts of the axes**.**

Sol:

Let the equation of the circle be

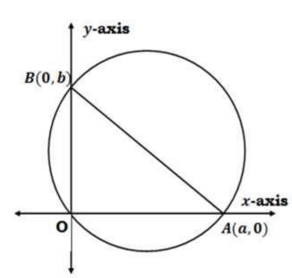


Fig 3.33

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Since, the circle passes through the origin
and cuts off the intercepts a and b from the positive axes.
So, the circle passes through the points
$O(0,0)$, $A(a,0)$ and $B(0,b)$. We have,

$0 + 0 + 0 + 0 + c = 0$, Or, $c = 0$ (2)

$a^2 + 0 + 2ag + 0 + 0 = 0$, Or, $g = -a/2$ (3)

and $0 + b^2 + 0 + 2bf = 0 = 0$, Or, $f = -b/2$ (4)

Putting the values of $g,\ f$ and c in the equation of circle (1), we get

$x^2 + y^2 - ax - by = 0$ is the equation of required circle.

Q-4: Prove that the points (2, –4), (3, –1), (3, –3) and (0, 0) are concyclic.

Sol:

Note : To prove that four given points are concyclic (i.e. four points lie on the circle), we find the equation of the circle passing through any three given points and show that the fourth point lies on it.

Let the equation of the circle passing through the points (0, 0), (2, –4) and (3, –1) be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Since the point (0, 0) lies on circle (1),we have,

$0 + 0 + 0 + 0 + c = 0$, Or, $c = 0$ (2)

Again, since the point (2, –4) lies on circle (1),we have,

$4 + 16 + 4g - 8f + 0 = 0$,

Or, $4g - 8f = -20$,

Or, $g - 2f = -5$ (3)

Also, since the point (3, –1) lies on circle (1),we have,

$9 + 1 + 6g - 2f + 0 = 0$,

Or, $6g - 2f = -10$,

Or, $3g - f = -5$ (4)

Now, solving equations (3) and (4), we get

$g = -1$ and $f = 2$

Putting the values of $g,\ f$ and c in the equation of circle (1), we get

$x^2 + y^2 - 2x + 4y = 0$, (5)

is the equation of circle. Now, to check the 4th point (3, –3) lies on the circle (5),we put $x = 3$ and $y = -3$ in eqn (5),

$9 + 9 - 6 - 12 = 0$,

Therefore, the point (3, –3) satisfies the equation of circle (5) and lies on the circle.
Hence, the given points are concyclic.

Q-5: Find the equation of circle which passes through (3, –2), (–2, 0) and has its centre on the line $2x - y = 3$.

Sol:

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Since, the circle (1) passes through the points (3, –2) and (–2, 0) i.e these points lie on the circle (1),we have

$9 + 4 + 6g - 4f + c = 0$

Or, $6g - 4f + c = -13$ (2)

and $4 + 0 + 4g + 0 + c = 0$

Or, $4g + c = -4$ (3)

Again, the centre $(-g, -f)$ of circle (1) lies on the line $2x - y = 3$

$\therefore$ $-2g + f = 3$ (4)

Now, solving equations (2), (3) and (4), we get

eqn(2)-eqn(3) : $2g - 4f = -9$ (5)

eqn(4)+eqn(5): $-3f = -6$, Or, $f = 2$

i.e $-2g + 2 = 3$ Or, $g = -1/2$

i.e $4(-1/2) + c = -4$, Or, $c = -2$

Putting the values of g, f and c in the equation of circle (1), we get

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\,(2)y + (-2) = 0$$

Or, $x^2 + y^2 - x + 4y - 2 = 0$ is the equation of required circle.

Q-6: Find the equation of the circle circumscribing the triangle △ABC whose vertices are A(1, –5), B(5, 7) and C(–5, 1).

Sol:

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Since, the circle circumscribing the triangle △ABC with vertices A(1, –5), B(5, 7) and C(–5, 1), So, the circle (1) passes through the points A(1, –5), B(5, 7) and C(–5, 1).

Therefore,

$1 + 25 + 2g - 10f + c = 0$,

Or, $2g - 10f + c = -26$ (2)

$25 + 49 + 10g + 14f + c = 0$

Or, $10g + 14f + c = -74$ (3)

and $25 + 1 - 10g + 2f + c = 0$

Or, $-10g + 2f + c = -26$ (4)

On solving equations (2), (3) and (4), we get

eqn(3)-eqn(2): $8g + 24f = -48$, Or, $g + 3f = -6$ (5)

eqn(3)-eqn(4): $20g + 12f = -48$, Or, $5g + 3f = -12$ (6)

eqn(5)-eqn(6): $-4g = 6$, Or, $g = -\frac{3}{2}$

i.e $-\frac{3}{2} + 3f = -6$, Or, $f = -\frac{3}{2}$

i.e $2\left(-\frac{3}{2}\right) - 10\left(-\frac{3}{2}\right) + c = -26$, Or, $c = -38$

Putting the values of g, f and c in the equation of circle (1),

$$x^2 + y^2 + 2\left(-\frac{3}{2}\right)x + 2\left(-\frac{3}{2}\right)y + (-38) = 0$$

Or, $x^2 + y^2 - 3x - 3y - 38 = 0$ is the equation of required circle.

3. Diameter form (Equation of a circle with given end points of a diameter)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two end points of diameter AB of a circle. Let $P(x.y)$ be any point on the circle.

Join AP and BP. $\therefore \angle APB = 90^\circ$

($\because$ An angle on a semi-circle is right angle)

Now, slope of $AP = \frac{y - y_1}{x - x_1}$ and slope of $BP = \frac{y - y_2}{x - x_2}$

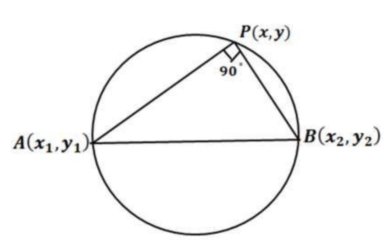


Fig 3.34

Since $AP \perp BP$

By condition of perpendicularity, the product of their slopes $= -1$

Or, $\frac{y-y_1}{x-x_1} \frac{y-y_2}{x-x_2} = -1$

Or. $(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

Or. $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

is the equation of circle with end points of a diameter (x_1, y_1) and (x_2, y_2), which is known as the diameter form of equation of circle.

Some Solved Problems:

Q- 1 : Find equation of a circle whose end points of a diameter are (1, 2) and (–3, –4).

Sol:

We know that, equation of the circle with end points (x_1, y_1) and (x_2, y_2) of a diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Given, the end points of diameter are (1, 2) and (–3, –4).

Therefore, the equation of the circle is

$$(x - 1)(x + 3) + (y - 2)(y + 4) = 0$$

Or, $x^2 + 2x - 3 + y^2 + 2y - 8 = 0$

Or, $x^2 + y^2 + 2x + 2y - 11 = 0$

Q-2: Find the equation of the circle passing through the origin and making intercepts 4 and 5 on the axes of co-ordinates.

Sol:

Let the intercepts be $OA = 4$ and $OB = 5$.

$\therefore$ The co-ordinates of *A* and *B* are $(4, 0)$ and $(0, 5)$ respectively.

Since $\angle AOB = \frac{\pi}{2}$, therefore AB is the diameter.

According to the diameter form,

Equation of the circle with end points $(4, 0)$ and $(0, 5)$ of diameter *AB* is

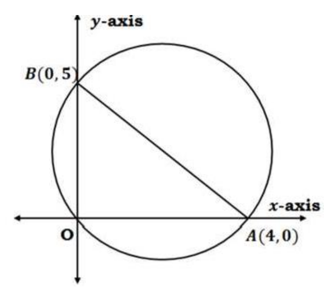


Fig 3.35

$$(x-4)(x-0) + (y-0)(y-5) = 0$$

Or, $x^2 + y^2 - 4x - 5y = 0$

EXERCISE

1. 02 Mark Questions

i. Find the distance between two points (3, –4) and (3, 5).

ii. Find slope, *x* intercept and *y* intercept of the line. $x - 2y + 4 = 0$.

iii. Determine the area of the triangle with the vertices at (0,0), (4,0) and (4,10).

iv. Find the slope of the line which makes an angle of 45° with *x*-axis.

v. Find the slope of a line which passes through the points (3, 2) and (–1, 5).

vi. Determine x so that the line passing through (3, 4) and (x, 5) makes 135° angle with the positive direction of x –axis.

vii. If A(–2, 1), B(2, 3) and C(–2, –4) are three points, find the angle between BA and BC.

viii. Find slope of the line whose equation is y+2=0.

ix. Find slope of the line joining the point (-k,-k) and the origin.

x. Find the slope a line perpendicular to the line joining the points (6, 4) and (2, 12).

xi. Without using Pythagoras theorem, show that the points A(0, 4), B(1, 2) and C(3, 3) are the vertices of a right angled triangle.

xii. Find the equation of a line with slope 2 and y-intercept is 3.

xiii. Find the equation of a straight line cutting off an intercept of -1 units on negative direction of y –axis and being equally inclined to the axis.

xiv. Determine the equation of a line through the point(–4, –3) and parallel to x –axis.

xv. Find the point of intersection of the lines whose equations are $x + 3 = 0$ and $y - 4 = 0$.

xvi. Determine the x-intercept and y- intercept of the line $y = 2x + 3$.

xvii. Find the equation of the line passing through the origin and parallel to the line $y = 3x + 4$.

xviii. Find the equation of the line passing through the origin and perpendicular to the line $y = -2x + 4$.

xix. Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of $x -$ axis.

xx. Reduce the equation $3x - 2y + 6 = 0$ to the intercept form and find the x- and y-intercepts.

xxi. Find the distance of the point (4, 5) from the straight line $3x - 5y + 7 = 0$.

xxii. Find centre and radius of the circle $2x^2 + 2y^2 - 5x + 6y + 2 = 0$.

xxiii. Find the equation of the circle whose two end points of a diameter are $(-1, 2)$ and $(4, -3)$.

2. 05 Mark Questions

i. Find the co-ordinates of a point whose distance from (3, 5) is 5 units and from (0, 1) is 10 units.

ii. A line AB is of length 5. A is the point (2, –3). If the abscissa of the point B is 5, prove that the ordinate of the B is 1 or 7.

iii. Show that the points (–2, 3), (1, 2) and (7, 0) are collinear.

iv. If the points (*a*, 0), (0, *b*) and (*x*, *y*) are collinear, prove that $\frac{x}{a}+\frac{y}{b}=1$

v. Show that the points (7, 3), (3, 0), (0, –4) and (4, –1) are the vertices of a rhombus.

vi. Find the ratio in which the line segment joining (2, 3) and (-3, -4) divided by x- axis and hence find the co-ordinates of the point.

vii. Find the ratio in which the line $x-y-2=0$ cuts the line segment joining (3,-1) and (8, 9).

viii. If the vertices of a right angled ΔABC are (0, 0) and (3,0), then find the third vertex.

ix. Determine the ratio in which the line joining the points (3,4) and (-3,-4) divided by the origin.

x. Show that the points (a, b+c), (b, c+a) and (c, a+b) are collinear.

xi. For what value of k, the points (k, 1), (5, 5) and (10, 7) are collinear.

xii. Find the equation of the perpendicular bisector of the line segment joining the points A(2, 3) and B(6, –5).

xiii. Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes 14.

xiv. Find the equation of the perpendicular bisector of the line joining the points (1, 3) and (3, 1).

xv. Find equation of the circle whose centre is on x-axis and the circle passes through (4, 2) and (0, 0).

xvi. Find Co-ordinates of the point where the circle $x^2 + y^2 - 7x - 8y + 12 = 0$ meets the co-ordinates axes and hence find the intercepts on the axes.

xvii. Find equation of the circle which passes through the points (0, 0), (1,2) and (2,-1).

xviii. Find the equation of the circle which touches the lines $x=0$, $y=0$ and $x=a$.

xix. Find the equation of a circle passing through the point $(2,-1)$ and which is concentric with the circle $5x^2+5y^2-12x+15y-420=0$.

xx. Show that the points $(9,1),(7,9),(-2,12)$ and $(6,10)$ are concyclic.

3. 10 Mark Questions

i. If the point (x, y) be equidistant from the points $(a+b,\ b-a)$ and $(a-b.a+b)$, prove that $bx=ay$.

ii. Find the area of a quadrilateral whose vertices are (1, 1), (7, –3), (12, 2) and (7, 21).

iii. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y=x+3$. Find the third vertex.

iv. Find equation of the circle whose centre is on the line 8x + 5y = 0 and the circle passes through (2, 1) and (3, 5).

v. ABCD is a square whose side is 'a', taking AB and AD as axes, prove that the equation to the circle circumscribing the square is $x^2+y^2=a(x+y)$.

VECTORS

INTRODUCTION:-

In our real life situation we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively.

We have also come across physical quantities such as displacement, velocity, acceleration, momentum etc, which are of different type in comparison to above.

Consider the figure-1, where A, B, C are at a distance 4k.m. from P. If we start from P, then covering 4k.m. distance is not sufficient to describe the destination where we reach after the travel, So here the end point plays an important role giving rise the need of direction. So we need to study about direction of a quantity, along with magnitude.

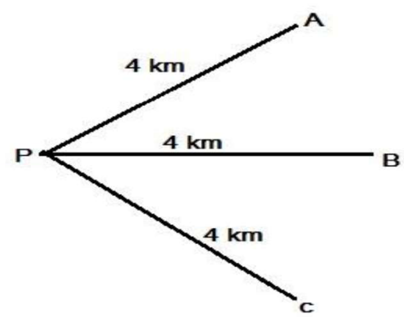


Fig - 1

OBJECTIVE

After completion of the topic you are able to :-

i) Define and distinguish between scalars and vectors.
ii) Represent a vector as directed line segment.
iii) Classify vectors in to different types.
iv) Resolve vector along two or three mutually perpendicular axes.
v) Define dot product of two vectors and explain its geometrical meaning.
vi) Define cross product of two vectors and apply it to find area of triangle and parallelogram.

Expected background knowledge

i) Knowledge of plane and co-ordinate geometry
ii) Trigonometry.

Scalars and vectors

All the physical quantities can be divided into two types.
i) Scalar quantity or Scalar.
ii) Vector quantity or Vector.

Scalar quantity: - The physical quantities which requires only magnitude for its complete specification is called as scalar quantities.
Examples: - Speed, mass, distance, velocity, volume etc.

Vector: - A directed line segment is called as vector.

Vector quantities:- A physical quantity which requires both magnitude & direction for its complete specification and satisfies the law of vector addition is called as vector quantities.
Examples: - Displacement, force, acceleration, velocity, momentum etc.

Representation of vector:- A vector is a directed line segment $\overrightarrow{AB}$where A is the initial point and B is the terminal point and direction is from A to B. (see fig-2).

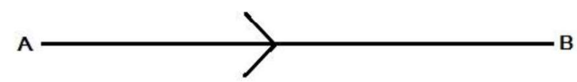


Fig - 2

Similarly $\overrightarrow{BA}$ is a directed line which represents a vector having initial point B and terminal point A.

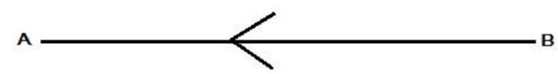


Fig - 3

Notation: - A vector quantity is always represented by an arrow (→) mark over it or by bar (−) over it. For example $\overrightarrow{AB}$. It is also represented by a single small letter with an arrow or bar mark over it. For example $\vec{a}$.

Magnitude of a vector: - Magnitude or modulus of a vector is the length of the vector. It is a scalar quantity.

Magnitude of $\overrightarrow{AB}$ = $|\overrightarrow{AB}|$ = Length AB. = AB

<u>Types of Vector: -</u> Vectors are of following types.

1) **Null vector or zero vector or void vector: -** A vector having zero magnitude and arbitrary direction is called as a null vector and is denoted by $\vec{0}$.

Clearly, a null vector has no definite direction. If $\vec{a}$ = $\overrightarrow{AB}$, then $\vec{a}$ is a null (or zero) vector iff $|\vec{a}|$ = 0 i.e. if $|\overrightarrow{AB}|$ = 0

For a null vector initial and terminal points are same.

2) Proper vector: - Any non zero vector is called as a proper vector. If $|\vec{a}|$ ≠ 0 then $\vec{a}$ is a proper vector.

3) Unit vector : - A vector whose magnitude is unity is called a unit vector. Unit vectors are denoted by a small letter with ^ over it. For example $\hat{a}$. $|\hat{a}|$ = 1

Note: - The unit vector along the direction of a vector $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

4) Co-initial vectors:- Vectors having the same initial point are called co-initial vector.

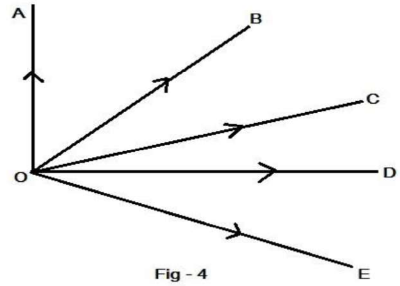


Fig - 4

In figure-4, $\overrightarrow{OA}$,$\overrightarrow{OB}$,$\overrightarrow{OC}$,$\overrightarrow{OD}$ and $\overrightarrow{OE}$ are Co-initial vectors.

5) Like and unlike vectors: - Vectors are said to be like if they have same direction and unlike if they have opposite direction.

6) Co-Linear vectors:- Vectors are said to be co-linear or parallal if they have the same line of action. In f figure-5 $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.


Fig - 5

7) Parallel vectors: - Vectors are said to be parallel if they have same line of action or have line of action parallel to one another. In fig-6 the vectors are parallel to each other.

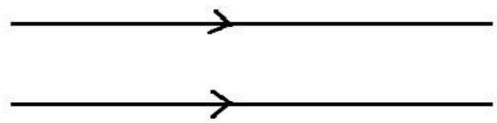


Fig - 6

8) Co-planner Vectors: - Vectors are said to be co-planner if they lies on the same plane. In fig-7 vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanner.

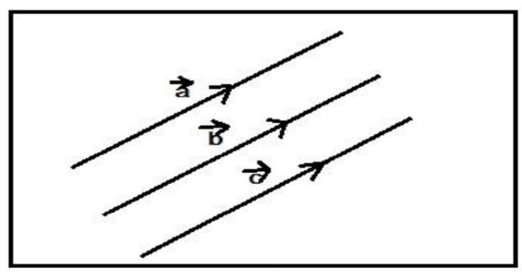


Fig - 7

9) Negative of a vector: - A vector having same magnitude but opposite in direction to that of a given vector is called negative of that vector. If $\overrightarrow{a}$ is any vector then negative vector of it is written as -$\overrightarrow{a}$ and $|\vec{a}| = |-\vec{a}|$ but both have direction opposite to each other as shown in fig-8.

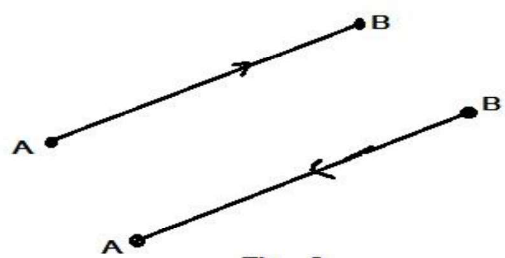


Fig - 8

10) Equal Vectors: - Two vectors are said to be equal if they have same magnitude as well as same direction.

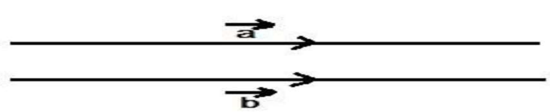


Fig - 9

Thus $\vec{a} = \vec{b}$

Remarks:- Two vectors can not be equal

i) If they have different magnitude .

ii) If they have inclined supports.

iii) If they have different sense.

Vector operations

Addition of vectors: -

Triangle law of vector addition: - The law states that If two vectors are represented by the two sides of a triangle taken in same order their sum or resultant is represented by the 3rd side of the triangle with direction in reverse order.

As shown in figure-10 $\vec{a}$ and $\vec{b}$ are two vectors represented by two sides OA and AB of a triangle ABC in same order. Then the sum $\vec{a}+\vec{b}$ is represented by the third side OB taken in reverse order i.e. the vector $\vec{a}$ is represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$, so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then $\overrightarrow{OB}$ represents the sum (or resultant) $(\vec{a}+\vec{b})$. Thus $\overrightarrow{OB} = \vec{a}+\vec{b}$

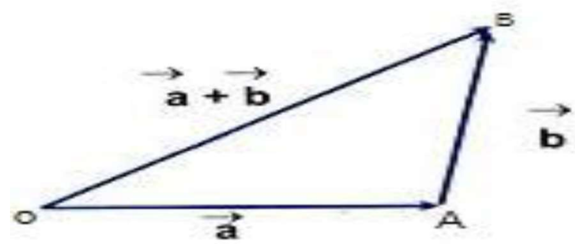


Fig – 10

Note-1 – The method of drawing a triangle in order to define the vector sum $(\vec{a}+\vec{b})$ is called triangle law of addition of the vectors.

Note-2 – Since any side of a triangle is less than the sum of the other two sides

$|\overrightarrow{OB}| \neq |\overrightarrow{OA}| + |\overrightarrow{AB}|$

Parallelogram law of vector addition: - If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogram in magnitude and direction, then their sum (resultant) is represented in magnitude and direction by the diagonal which is passing through the common initial point of the two vectors. As shown in fig-II if OA is $\vec{a}$ and AB is $\vec{b}$ then OB diagonal represent $\vec{a}+\vec{b}$.

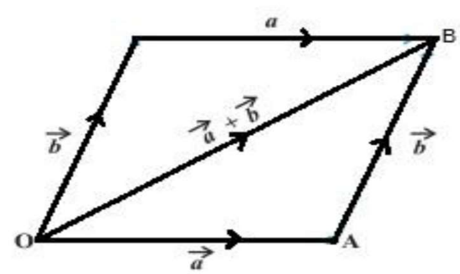


Fig - 11

i.e. $\vec{a}+\vec{b} = \overrightarrow{OA}+\overrightarrow{AB}$

Polygon law of vector addition: - If $\vec{a}$, $\vec{b}, \vec{c}$ and $\vec{d}$ are the four sides of a polygon in same order then their sum is represented by the last side of the polygon taken in opposite order as shown in figure-12.

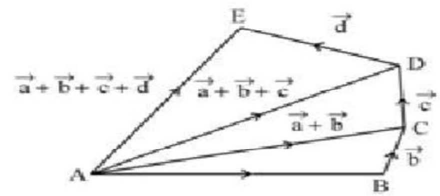


Fig - 12

Subtraction of two vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the subtraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a}$ - $\vec{b}$ is defined as addition of -$\vec{b}$ with $\vec{a}$. i.e. $\vec{a}$ - $\vec{b}$ =$\vec{a}$+ (-$\vec{b}$).

Properties of vector addition:- i) Vector addition is commutative i.e. if $\vec{a}$ & $\vec{b}$ are any two vectors then:-

$\vec{a}+\vec{b}$ = $\vec{b}+\vec{a}$

ii) Vector addition is associative i.e. if $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

iii) Existence of additive identity i.e. for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e. $\vec{0}+\vec{a}$ = $\vec{a}+\vec{0}$ = $\vec{a}$ where $\vec{0}$ is a null vector.

iv) Existence of additive Inverse :- If $\vec{a}$ is any non zero vector then -$\vec{a}$ is the additive inverse of $\vec{a}$, so that $\vec{a}+(-\vec{a})$ = $(-\vec{a})+\vec{a}$ = $\vec{0}$

Multiplication of a vector by a scalar : -

If $\vec{a}$ is a vector and k is a nonzero scalar then the multiplication of the vector $\vec{a}$ by the scalar k is a vector denoted by k$\vec{a}$ or $\vec{a}$k whose magnitude $|k|$ times that of $\vec{a}$.

i.e k$\vec{a}$ = $|k|$ x $|\vec{a}|$

= k x $|\vec{a}|$ if $k \geq 0$.

= (-k) x $|\vec{a}|$ if k<0 .

The direction of k$\vec{a}$ is same as that of $\vec{a}$ if k is positive and opposite as that of $\vec{a}$ if k is negative.

k$\vec{a}$ and $\vec{a}$ are always parallel to each other.

Properties of scalar multiplication of vectors :-

If h and k are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

i) k $(\vec{a}+\vec{b})$ = k$\vec{a}$ + k$\vec{b}$

ii) (h+k)$\vec{a}$ = h$\vec{a}$ + k$\vec{a}$, (Distributive law)

iii) (hk)$\vec{a}$ = h(k$\vec{a}$), (Associative law)

iv) 1.$\vec{a}$= $\vec{a}$

v) 0. $\vec{a}$ = $\vec{0}$

Position Vector of a point

Let O be a fixed point called origin, let P be any other point, then the vector $\overrightarrow{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{p}$.

As shown in figure-13, let AB be any vector , then applying triangle law of addition we have

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ where $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

=> $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

= (Position vector of B)–(Position vector of A)

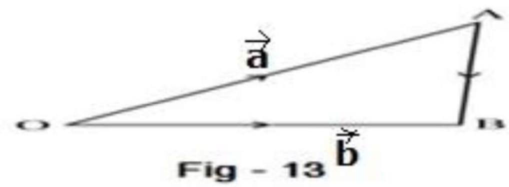


Fig - 13

Section Formula:- Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB , dividing it in the ration m:n. internally. Then the position vector of P i.e.$\vec{r}$ is given by the formula: $\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}$

If P divides AB externally in the ratio m:n then $\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}$

If P is the midpoint of AB then $\vec{r} = \frac{\vec{a}+\vec{b}}{2}$

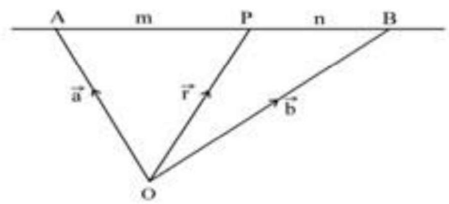


Fig - 14

Example-1 :- Prove that by vector method the medians of a triangle are concurrent.

Solution:- Let ABC be a triangle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A,B and C respectively. We have to show that the medians of this triangle are concurrent.

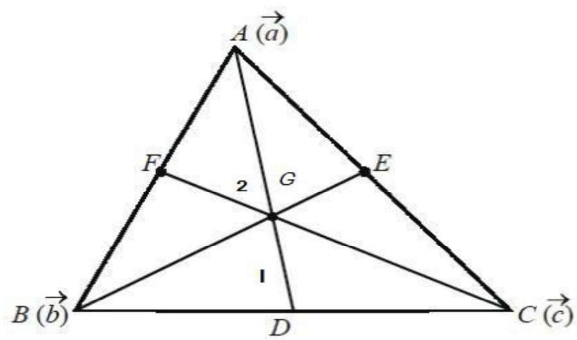


Fig - 15

Let AD , BE and CF are the three medians of the triangle.

Now as D be the midpoint of BC, so position vector of D i.e. $\vec{d} = \frac{\vec{b}+\vec{c}}{2}$.

Let G be any point of the median AD which divides AD in the ratio 2:1 . Then position vector of G is given

by $\vec{g} = \frac{2\vec{d}+\vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1\vec{a}}{3}$ (by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Let G' be a point which divides BE in the ratio 2:1 ,

Position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$.

Then position vector of G' is given by $\overrightarrow{g'} = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{2\left(\frac{\vec{a}+\vec{c}}{2}\right)+1\vec{b}}{3}$ (by applying section formula)

$$\Rightarrow \overrightarrow{g'} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

As position vector of a point is unique , so G = G' .

Similarly if we take G'' be a point on CF dividing it in 2:1 ratio then the position vector of G'' will be same as that of G.

Hence G is the one point where three median meet.

∴ The three medians of a triangle are concurrent. (proved)

Example2: - Prove that i) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (It is known as Triangle Inequality).

ii) $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$

iii) $|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Proof:- Let O,A and B be three points, which are not collinear and then draw a triangle OAB.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, then by triangle law of addition we have $\overrightarrow{OB} = \vec{a} + \vec{b}$

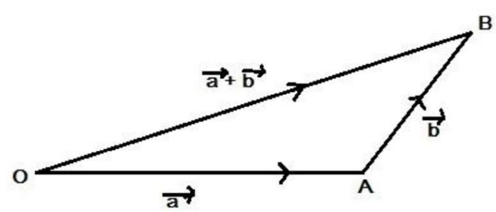


Fig - 16

From properties of triangle we know that the sum of any two sides of a triangle is greater than the third side.

- ⇨ OB < OA + AB
- ⇨ $|\overrightarrow{OB}| < |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ -------------------(1)

When O,A, B are collinear then

From figure-17 it is clear that

OB = OA + AB

- ⇨ $|\overrightarrow{OB}| = |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ ------------------(2)

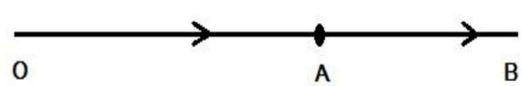


Fig-17

From (1) and (2) we have,

$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$ --------------(1)

But $|(\vec{a}-\vec{b})+\vec{b}\,| \leq |\vec{a}-\vec{b}| + |\vec{b}|$ (From triangle inequality)--------(2)

From (1) and (2) we get $|\vec{a}| \leq |\vec{a}-\vec{b}| + |\vec{b}|$

⇨ $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$(proved)

iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \leq |\vec{a}| + |-\vec{b}|$ (From triangle inequality)

$= |\vec{a}| + |\vec{b}|$ (as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

Components of vector in 2D

Let XOY be the co-ordinate plane and P(x,y) be any point in this plane.

The unit vector along direction of X axis i.e. $\overrightarrow{OX}$ is denoted by $\hat{\imath}$.

The unit vector along direction of Y axis i.e. $\overrightarrow{OY}$ is denoted by $\hat{\jmath}$.

Then from figure-18 it is clear that $\overrightarrow{OM}$ = x$\hat{\imath}$ and $\overrightarrow{ON}$ = y$\hat{\jmath}$.

So, the position vector of P is given by

$$\overrightarrow{OP} = \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$$

And OP =$|\overrightarrow{OP}|$ =r = $\sqrt{x^2+y^2}$

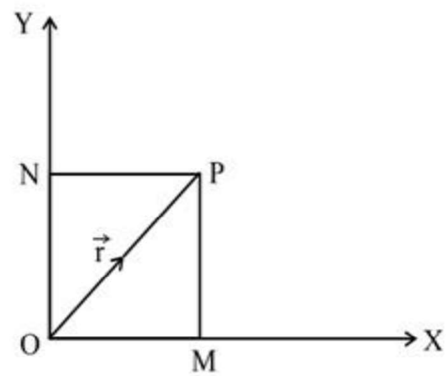


Fig-18

Representation of vector in component form in 2D

If $\overrightarrow{AB}$is any vector having end points A(x_1, y_1) and B (x_2, y_2) , then it can be represented by

$\overrightarrow{AB}$ = ($x_2 - x_1$)$\hat{\imath}$ + ($y_2 - y_1$)$\hat{\jmath}$

Components of vector in 3D

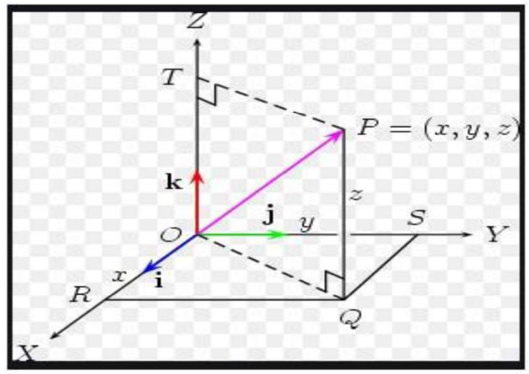


Fig-19

Let P(x,y,z) be a point in space and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ be the unit vectors along X axis , Y axis and Z axis respectively. (as shown in fig-19)

Then the position vector of P is given by

$\overrightarrow{OP} = x\,\hat{\imath} + y\,\hat{\jmath} + \hat{k}$, The vectors $x\hat{\imath}$, $y\,\hat{\jmath}$, $z\hat{k}$ are called the components of $\overrightarrow{OP}$ along x–axis, y-axis and z-axis respectively.

And OP $= |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Addition and scalar multiplication in terms of component form of vectors: -

For any vector $\vec{a} = a_1\hat{\imath} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{j} + b_3\hat{k}$

i) $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{\imath} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

ii) $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{\imath} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

iii) k $\vec{a} = ka_1\hat{\imath} + ka_2\hat{j} + ka_3\hat{k}$, where K is a scalar.

iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{\imath} + a_2\hat{j} + a_3\hat{k} = b_1\hat{\imath} + b_2\hat{j} + b_3\hat{k}$

$\Leftrightarrow a_1=b_1,\ a_2=b_2,\ a_3=b_3$

Representation of vector in component form in 3-D & Distance between two points:

If $\overrightarrow{AB}$ is any vector having end points A(x_1, y_1, z_1) and B (x_2, y_2, z_2) , then it can be represented by

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$=(x_2\hat{\imath} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3:-

Show that the points A(2,6,3) , B(1,2,7) and c(3,10,-1) are collinear.

Solution:- From given data Position vector of A, $\overrightarrow{OA} = 2\,\hat{\imath} + 6\hat{\jmath} + 3\,\hat{k}$.

Position vector of B, $\overrightarrow{OB} = \hat{\imath} + 2\hat{\jmath} + 7\hat{k}$

Position vector of C, $\overrightarrow{OC} = 3\,\hat{\imath} + 10\hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1-2)\,\hat{\imath} + (2-6)\,\hat{\jmath} + (7-3)\,\hat{k} = -\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3-2)\,\hat{\imath} + (10 - 6)\,\hat{\jmath} + (-1-3)\,\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$.

$= -(-\hat{\imath} - 4\hat{\jmath} + 4\hat{k}) = -\overrightarrow{AB}$

⇨ $\overrightarrow{AB} \parallel \overrightarrow{AC}$ or collinear.

∴ They have same support and common point A.
As 'A' is common to both vector , that proves A,B and C are collinear.

Example-4: - Prove that the points having position vector given by $2\,\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\,\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ form a right angled triangle. [2009(w)]

S0lution :- Let A,B and C be the vertices of a triangle with position vectors $2\,\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\,\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ respectively

Then . $\overrightarrow{AB}$ = Position vector of B – Position vector of A.

$= (1-2)\,\hat{\imath} + (-3 - (-1))\,\hat{\jmath} + (-5-1)\,\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$.

$\overrightarrow{BC}$ = Position vector of C – Position vector of B.

$= (3-1)\,\hat{\imath} + (-4 - (-3))\,\hat{\jmath} + (-4-(-5))\,\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$.

$\overrightarrow{AC}$ = Position vector of C – Position vector of A.

$= (3-2)\,\hat{\imath} + (-4 - (-1))\,\hat{\jmath} + (-4-1)\,\hat{k} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$.

Now AB = $|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

BC = $|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

AC = $|\overrightarrow{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

From above $BC^2 + AC^2$ = 6+35 = 41 = AB^2.

Hence ABC is a right angled triangle.

Example-5 :- Find the unit vector in the direction of the vector $\vec{a} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k}$. (2017-W)

Ans:- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{3^2+(-4)^2+1^2}} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}\hat{\imath} - \frac{4}{\sqrt{26}}\hat{\jmath} + \frac{1}{\sqrt{26}}\hat{k}.$$

Example-6 :- Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = = \hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans:- Let $\vec{r} = \vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + (\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 2\,\hat{\imath} + 2\hat{k}$.

Unit vector along direction of $\vec{a} + \vec{b}$ is given by $= \frac{\vec{r}}{|\vec{r}|} = \frac{2\,\hat{\imath} + 2\hat{k}}{\sqrt{2^2+2^2}} = \frac{2\,\hat{\imath} + 2\hat{k}}{\sqrt{8}} = \frac{2}{\sqrt{8}}\hat{\imath} + \frac{2}{\sqrt{8}}\hat{k}$

$$= \frac{2}{2\sqrt{2}}\hat{k}\hat{\imath} + \frac{2}{2\sqrt{2}}\hat{\jmath} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}.$$

Angle between the vectors:-

As shown in figure-20 angle between two vectors $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ can be determined as follows.

Let $\overrightarrow{OB}$ be a vector parallel to $\overrightarrow{RS}$ and $\overrightarrow{OA}$ is a vector parallel to $\overrightarrow{PQ}$ such that $\overrightarrow{OB}$ and $\overrightarrow{OA}$ intersect each other.

Then θ = ∟AOB = angle between $\overrightarrow{RS}$ and $\overrightarrow{PQ}$.

If θ = 0 then vectors are said to be parallel.

If θ = $\frac{\pi}{2}$ then vectors are said to be orthogonal or perpendicular.

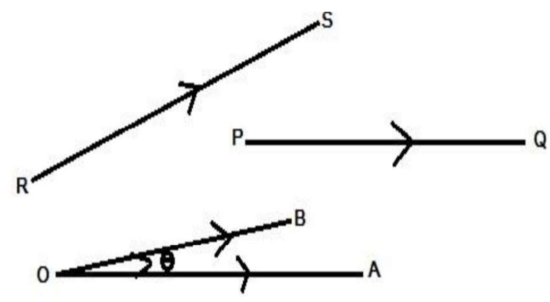


Fig-20

Dot Product or Scalar product of vectors

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are, a and b respectively denoted by $\vec{a}.\vec{b}$ is defined as the scalar abcosθ , where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$.

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\, b \cos\theta$$

Geometrical meaning of dot product

In figure21(a), $\vec{a}$ and $\vec{b}$ are two vectors having θ angle between them. Let M be the foot of the perpendicular drawn from B to OA.

Then OM is the Projection of $\vec{b}$ on $\vec{a}$ and from figure-21(a) it is clear that ,

$|OM| = |OB| \cos\theta = |\vec{b}| \cos\theta$.

Now $\vec{a}.\vec{b} = |\vec{a}| (|\vec{b}| \cos\theta) = |\vec{a}|$ x projection of $\vec{b}$ on $\vec{a}$

which gives projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Similarly we can write $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$= |\vec{b}| (|\vec{a}| \cos\theta) = |\vec{b}|$ projection of $\vec{a}$ on $\vec{b}$.

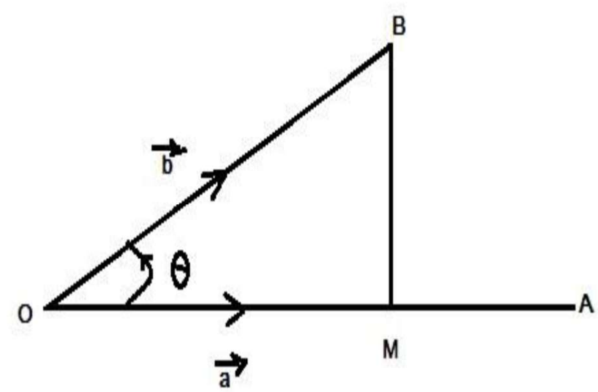


Fig-21

(a)

Similarly, let us draw a perpendicular from A on OB and let N be the foot of the perpendicular in fig-21(b).

Then ON = Projection of $\vec{a}$ on $\vec{b}$

and ON = OAcosθ =$|\vec{a}|$ cosθ.

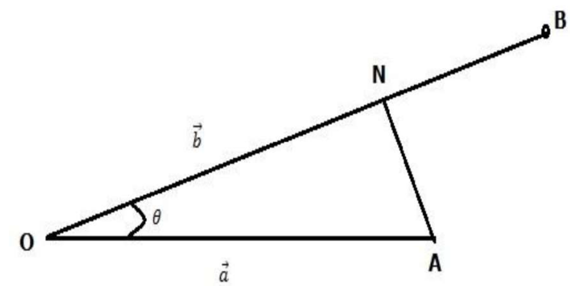


Fig-21(b)

Properties of Dot product

i) $\vec{a}.\vec{b}$ = $\vec{b}.\vec{a}$ (commutative)

ii) $\vec{a}.(\vec{b}+\vec{c})$ = $\vec{a}.\vec{b}$ + $\vec{a}.\vec{c}$ (Distributive)

iii) If $\vec{a}$ II $\vec{b}$, then $\vec{a}.\vec{b}$ = ab { as θ = 0 in this case cos 0 = 1 }

In particular $(\vec{a})^2$ = $\vec{a}\,.\vec{a}$ = $|\vec{a}|^2$

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

iv) If $\vec{a} \perp \vec{b}$, then $\vec{a}.\vec{b}$ = 0. { as θ = 90° in this case cos90º= 0 }

In particular $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0 = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k}$

v) $\vec{a}.\vec{0}$ = $\vec{0}.\vec{a}$ = 0

vi) $(\vec{a}+\vec{b})\,.\,(\vec{a}-\vec{b})$ = $|\vec{a}|^2$- $|\vec{b}|^2$= a^2 - b^2 {Where $|\vec{a}|$ = a and $|\vec{b}|$ = b }

viii) Work done by a Force:- The work done by a force $\vec{F}$ acting on a body causing displacement $\vec{d}$ is given by W = $\vec{F}.\vec{d}$

Dot product in terms of rectangular components

For any vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ we have,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (by applying distributive(ii) , (iii) and (iv) successively)

Angle between two non zero vectors

For any two non zero vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, having θ is the angle between them we have,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \hat{a}.\hat{b} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}} \quad \textbf{(In terms of components.)}$$

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Condition of Perpendicularity: -

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are perpendicular to each other $\Leftrightarrow \boldsymbol{a_1 b_1 + a_2 b_2 + a_3 b_3 = 0}$

Condition of Parallelism :-

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are parallel to each other $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$)

Scalar & vector projections of two vectors (Important formulae)

Scalar Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\hat{a} = [\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}]\vec{a}$

Scalar Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$

Examples: -

Q.- 7. Find the value of p for which the vectors $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are perpendicular to each other.

Solution:- Let $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$.

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 9$

$b_1 = 1$, $b_2 = p$ & $b_3 = 3$

Given $\vec{a} \perp \vec{b}$ => $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

=> 3.1+2.p+9.3 = 0

=> 3 +2p +27 = 0

=> 2p = -30

=> p = -15 (Ans)

Q-8 Find the value of p for which the vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are parallel to each other. (2014-W)

Solution:- Given $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Leftrightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$ { Taking 1st two terms}

$\Leftrightarrow 3 = \frac{2}{p} \Leftrightarrow$ p $= \frac{2}{3}$ (Ans) {Note:- any two expression may be taken for finding p.}

Q-9 Find the scalar product of $3\hat{\imath} - 4\hat{\jmath}$ and $-2\hat{\imath} + \hat{\jmath}$. (2015-S)

Solution:- $(3\hat{\imath} - 4\hat{\jmath}) . (-2\hat{\imath} + \hat{\jmath}) = (3\times(-2)) + ((-4)\times 1) = (-6)+(-4) = -10$

Q-10 Find the angle between the vectors $5\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $6\hat{\imath} - 8\hat{\jmath} - \hat{k}$. (2015-W)

Solution:- Let $\vec{a} = 5\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = 6\hat{\imath} - 8\hat{\jmath} - \hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\theta = cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}}\right)$

$= cos^{-1}\left(\frac{5.6+3.(-8)+4.(-1)}{\sqrt{5^2+3^2+4^2}\sqrt{6^2+(-8)^2+(-1)^2}}\right) = cos^{-1}\left(\frac{30-24-4}{\sqrt{50}\sqrt{101}}\right) = cos^{-1}\left(\frac{2}{\sqrt{50}\sqrt{101}}\right)$

Q-11 Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where,

$\vec{a} = \hat{\imath} - \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 3\hat{k}$. { 2013-W ,2017-W, 2017-S}

Solution:- Scalar Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{I\vec{b}I} = \frac{1.3+(-1).1+(-1).3}{\sqrt{3^2+1^2+3^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$

Vector Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{I\vec{b}I^2}\vec{b} = \frac{1.3+(-1).1+(-1).3}{(\sqrt{3^2+1^2+3^2})^2}$ $(3\hat{\imath} + \hat{\jmath} + 3\hat{k})$

$= \frac{3-1-3}{19}(3\hat{\imath} + \hat{\jmath} + 3\hat{k}) = \frac{-1}{19}(3\hat{\imath} + \hat{\jmath} + 3\hat{k})$

Q-12 Find the scalar and vector projection of $\vec{b}$ on $\vec{a}$ where,

$\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$. { 2015-S}

Solution: - Scalar Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}.\vec{b}}{I\vec{a}I} = \frac{3.2+1.3+(-2).(-4)}{\sqrt{3^2+1^2+(-2)^2}} = \frac{6+3+8}{\sqrt{14}} = \frac{17}{\sqrt{14}}$

Vector Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}.\vec{b}}{I\vec{a}I^2}\vec{a} = \frac{3.2+1.3+(-2).(-4)}{(\sqrt{3^2+1^2+(-2)^2})^2}$ $(3\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$= \frac{17}{14}(3\hat{\imath} + \hat{\jmath} - 2\hat{k})$.

Q-13 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Proof:- Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

⇨ $(\vec{a}.\vec{b}) - (\vec{a}.\vec{c}) = \vec{0}$ => $\vec{a}.(\vec{b} - \vec{c}) = \vec{0}$ { applying distributive property}

Dot product of above two vector is zero indicates the following conditions

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{o}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

⇨ $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ (proved)

Example:-14 Find the work done by the force $\vec{F} = \hat{\imath} + \hat{\jmath} - \hat{k}$. acting on a particle if the particle is displace A from A(3,3,3) to B(4,4,4).

Ans:- Let O be the origin, then

Position vector of A $\overrightarrow{OA} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$

Position vector of B $\overrightarrow{OB} = 4\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

Then displacement is given by, $\vec{d} = \overrightarrow{AB}$ = ($\overrightarrow{OB}$ - $\overrightarrow{OA}$) = (4$\hat{\imath}$+4$\hat{\jmath}$+ 4$\hat{k}$) - (3$\hat{\imath}$+3$\hat{\jmath}$+3$\hat{k}$) = $\hat{\imath}$+$\hat{\jmath}$+ $\hat{k}$.

So work done by the force W = $\vec{F}.\vec{d}$ = $\vec{F}.\overrightarrow{AB}$ = ($\hat{\imath}$+$\hat{\jmath}$- $\hat{k}$). ($\hat{\imath}$+$\hat{\jmath}$+ $\hat{k}$)

= 1.1+1.1+(-1).1 = 1 units

Example:-15 If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$$

Proof: - $(|\hat{a} - \hat{b}|)^2 = (\hat{a} - \hat{b}).(\hat{a} - \hat{b}) = (\hat{a}.\hat{a}) - (\hat{a}.\hat{b}) - (\hat{b}.\hat{a}) + (\hat{b}.\hat{b})$ { Distributive property}

$= (|\hat{a}|)^2 - (\hat{a}.\hat{b}) - (\hat{a}.\hat{b}) + (|\hat{b}|)^2$ {commutative property}

$= 1^2 - 2\,\hat{a}.\hat{b} + 1^2$ { as $\hat{a}$ and $\hat{b}$ are unit vectors so their magnitudes are 1}

$= 2 - 2\,\hat{a}.\hat{b} = 2(1 - \hat{a}.\hat{b})$

$= 2(1 - |\hat{a}|.|\hat{b}|\cos\theta)$ { as θ is the angle between $\hat{a}$ and $\hat{b}$ }

$= 2(1 - 1.1.\cos\theta)$

$= 2(1 - \cos\theta) = 2.2\,sin^2\frac{\theta}{2}$

Taking square root of both sides we have $|\hat{a} - \hat{b}| = 2\sin\frac{\theta}{2}$

⇨ $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$ (proved)

Example:-16 If the sum of two unit vectors is a unit vector. Then show that the magnitude of their difference is $\sqrt{3}$.

Proof:-$\hat{a}$,$\hat{b}$ and $\hat{c}$ are three unit vectors such that $\hat{a} + \hat{b} = \hat{c}$

Squaring both sides we have,

=> $(|\hat{a} + \hat{b}|)^2 = (|\hat{c}|)^2$

=> $(|\hat{a}|)^2 + (|\hat{b}|)^2 + 2\,\hat{a}.\hat{b} = 1^2$

=> $1^2 + 1^2 + 2|\hat{a}||\hat{b}|\cos\theta = 1$ { where θ is the angle between $\hat{a}$ and $\hat{b}$ }

=> $1 + 1 + 2\cos\theta = 1$

=> $2\cos\theta = -1$

=> $\cos\theta = \frac{-1}{2}$

Now we have to find the magnitude of their difference i.e $\hat{a} - \hat{b}$.

So $(|\hat{a} - \hat{b}|)^2 = (|\hat{a}|)^2 + (|\hat{b}|)^2 - 2\,\hat{a}.\hat{b} = 1^2 + 1^2 - 2|\hat{a}||\hat{b}|\cos\theta$

$= 2 - 2\cos\theta = 2 - 2(\frac{-1}{2}) = 2 - (-1) = 3$

∴ $|\hat{a} - \hat{b}| = \sqrt{3}$ (Proved)

Vector Product or Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them , then the vector product of these two vectors denoted by $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}| \sin\theta \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

As shown in figure-21 the direction of $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$.

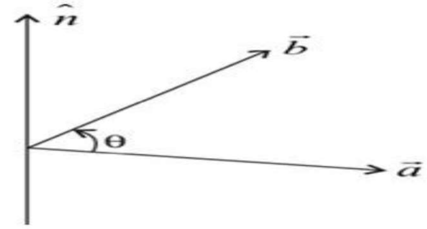


Fig-22

Properties of cross product

i) Vector product is not commutative $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

ii) For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

iii) For any scalar m, $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

iii) Distributive $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

iv) Vector product of two parallel or collinear vectors is zero.

$\vec{a} \times \vec{a} = \vec{0}$ and if $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$ { as θ = 0 or 180° => $\sin\theta = 0$ }

Using this property we have,

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

v) Vector product of orthonormal unit vectors form a right handed system.

As shown in figure- 23 the three mutually perpendicular unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ form a right handed system , i.e. $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$

(as θ =90 , then $\sin\theta = 1$)

$\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$

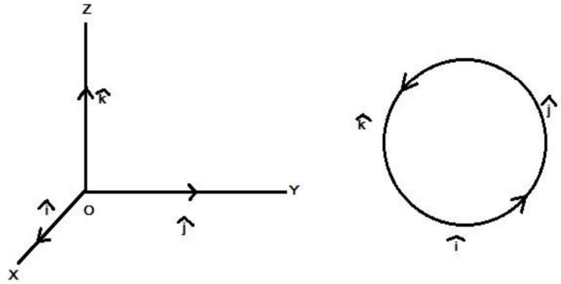


Fig-23

Unit vector perpendicular to two vectors:- Unit vector perpendicular to two given vectors $\vec{a}$and $\vec{b}$ is given by $\hat{\boldsymbol{n}} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Angle between two vectors

Let θ be the angle between $\vec{a}$and $\vec{b}$. Then $\vec{a} \times \vec{b}$ = (I $\vec{a}$ I. I $\vec{b}$ I sin θ) $\hat{n}$.

Taking modulus of both sides we have,

I $\vec{a} \times \vec{b}$I = I $\vec{a}$ I. I $\vec{b}$ I sin θ

⇨ $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}$

Hence $\boldsymbol{\theta} = \boldsymbol{sin^{-1}} \{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}\}$

Geometrical Interpretation of vector product or cross product

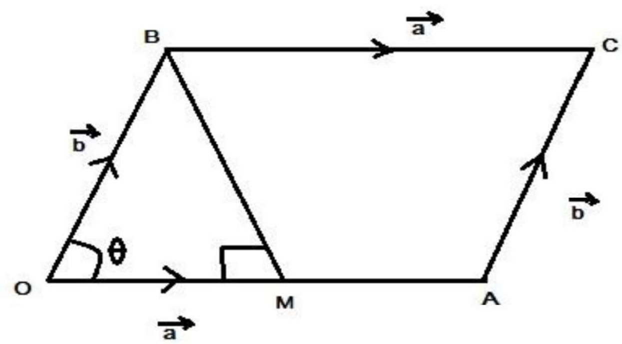


Fig-24

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Then $\vec{a} \times \vec{b}$ = (I $\vec{a}$ I. I $\vec{b}$ I sin θ) $\hat{n}$

= (I $\vec{a}$ I).(I $\vec{b}$ I sin θ)$\hat{n}$

From fig-24 below it is clear that

BM = OB sin θ = I $\vec{b}$ I sin θ = I $\vec{a}$ I IBMI $\hat{n}$

{ as sin θ = BM/OB &$\overrightarrow{OB} = \vec{b}$}

Now I $\vec{a} \times \vec{b}$I = I $\vec{a}$ I IBMI I $\hat{n}$ I = OA.

BM = Area of the parallelogram with side $\vec{a}$ and $\vec{b}$.

Therefore the magnitude of cross product of two vectors is equal to area of the parallelogram formed by these vectors as two adjacent sides.

From this it can be concluded that area of △ ABC = ½ I$\overrightarrow{AB}$. $\overrightarrow{AC}$I

Application of cross product

1. Moment of a force about a point ($\vec{M}$) :-Let O be any point and Let $\vec{r}$ be the position vector w.r.t. O of any point 'P' on the line of action of the force $\vec{F}$, then the moment or torque of the force F about origin 'O' is given by

$\vec{M} = \vec{r} \times \vec{F}$

2. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a triangle then the area of the triangle is given by

$\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

3. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram then area of the parallelogram is given by

$\Delta = |\vec{a} \times \vec{b}|$ Sq. unit

4. If $\vec{a}$ and $\vec{b}$ represent two diagonals of a parallelogram then area of the parallelogram is given by

$\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

Vector product in component form : -

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$

$= a_1b_1(\hat{\imath} \times \hat{\imath}) + a_1b_2(\hat{\imath} \times \hat{\jmath}) + a_1b_3(\hat{\imath} \times \hat{k}) + a_2b_1(\hat{\jmath} \times \hat{\imath}) + a_2b_2(\hat{\jmath} \times \hat{\jmath}) + a_2b_3(\hat{\jmath} \times \hat{k})$

$+ a_3b_1(\hat{k} \times \hat{\imath}) + a_3b_2(\hat{k} \times \hat{\jmath}) + a_3b_3(\hat{k} \times \hat{k})$

{ using properties $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$, $\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$ and

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$ }

$= (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ i.e. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Condition of Co-planarity

If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ lies on the same plane then the perpendicular to $\vec{a}$ *and* $\vec{b}$ must be perpendicular to $\vec{c}$.

In particular $(\vec{a} \times \vec{b}) \perp \vec{c}$ => $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

In component form if $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

Then $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

⇨ $(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 = 0$

⇨ $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$ (interchanging rows two times R_1 *and* R_2, *then* R_2 *and* R_3)

⇨ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Example:- 17

If $\vec{a} = \hat{\imath} + 3\,\hat{\jmath} - 2\hat{k}$ and $\vec{b} = -\,\hat{\imath} + 3\hat{k}$ then find $|\vec{a} \times \vec{b}|$

Ans: - We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \{(3\times3)-(0\times(-2))\}\,\hat{\imath} - \{(1\times3)-((-1)\times(-2))\}\,\hat{\jmath} + \{(1\times0)-((-1)\times3)\}\,\hat{k}$

$= 9\,\hat{\imath} - \hat{\jmath} + 3\,\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81+1+9} = \sqrt{91}$ (Ans)

Example:-18 Determine the area of the parallelogram whose adjacent sides are the vectors

$\vec{a} = \widehat{2\imath}$ and $\vec{b} = 3\,\hat{\jmath}$. (2013-W)

Ans:- Area of the parallelogram with adjacent sides given by $\vec{a}$ and $\vec{b}$ is given by

area = $|\vec{a} \times \vec{b}| = |\widehat{2\imath} \times 3\,\hat{\jmath}| = |6\hat{k}| = 6$ sq units (Ans)

Example:-19 Find a unit vector perpendicular to both the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\,\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans: - (2015-W and 2017-S)

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$……………….(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix}$

$= (3-1)\,\hat{\imath} - (6+3)\,\hat{\jmath} + (-2-3)\,\hat{k}$

$= 2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}$-----------------(2)

From (1) and (2) we have,

$\hat{n} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{|2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}|} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{\sqrt{2^2+(-9)^2+(-5)^2}} = \frac{2\,\hat{\imath} - 9\,\hat{\jmath} - 5\,\hat{k}}{\sqrt{110}}$

$= \frac{2}{\sqrt{110}}\hat{\imath} - \frac{9}{\sqrt{110}}\hat{\jmath} - \frac{5}{\sqrt{110}}\hat{k}$ (ans)

Example:-20 If $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = 3\,\hat{\imath} + 4\hat{\jmath} - \hat{k}$, then find the sine of the angle between these vectors. (2016-w)

Ans :- We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}$……………..(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\,\hat{\imath} - (-2-3)\,\hat{\jmath} + (8+3)\,\hat{k} = -3\,\hat{\imath} + 5\,\hat{\jmath} + 11\,\hat{k}$

Hence $\left|\vec{a} \times \vec{b}\right| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9 + 25 + 121} = \sqrt{155}$ …………………(2)

Again $\left|\vec{a}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$ ………………………(3)

and $\left|\vec{b}\right| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$……………………….(4)

From equation (1),(2),(3) and (4) we have,

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$ (Ans)

Q-21 Calculate the area of the triangle ABC (by vector method) where A(1,1,2), B(2,2,3) and C(3,-1,-1). (2013-W)

Solution: - Let the position vector of the vertices A,B and C is given by $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ respectively.

Then $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{c} = 3\hat{\imath} - \hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = Position\ vector\ of\ B - Position\ vector\ of\ A$

$= 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (2-1)\hat{\imath} + (2-1)\hat{\jmath} + (3-2)\hat{k}$

$= \hat{\imath} + \hat{\jmath} + \hat{k}$

Similarly $\overrightarrow{AC} = Position\ vector\ of\ C - Position\ vector\ of\ A$

$= 3\hat{\imath} - \hat{\jmath} - \hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (3-1)\hat{\imath} + (-1-1)\hat{\jmath} + (-1-2)\hat{k}$

$= 2\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= (-3 + 2)\hat{\imath} - (-3 - 2)\hat{\jmath} + (-2 - 2)\hat{k} = -\hat{\imath} + 5\hat{\jmath} - 4\hat{k}$

Hence area of the triangle is given by

$\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{(-1)^2 + 5^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{1 + 25 + 16} = \frac{1}{2}\sqrt{42}$ sq units. (Ans)

Example:-22 Find the area of a parallelogram whose diagonals are determined by the vectors $\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$. (2014-W, 2017-W)

Ans: - Area of the parallelogram with diagonals $\vec{a}$ *and* $\vec{b}$ are given by

$$\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= (4 - 6)\,\hat{\imath} - (12 + 2)\,\hat{\jmath} + (-9 - 1)\,\hat{k} = -2\,\hat{\imath} - 14\,\hat{\jmath} - 10\,\hat{k}$

Now area $\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$

$= \frac{1}{2}\sqrt{4 + 196 + 100} = \frac{\sqrt{300}}{2} = \frac{10\sqrt{3}}{2} = 5\sqrt{3}$ sq unit. (ans)

Example:-23 For any vector $\vec{a}$ *and* $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}\,.\vec{b})^2$ where a and b are magnitude of $\vec{a}$ *and* $\vec{b}$ respectively.

Proof: - $(\vec{a} \times \vec{b})^2 = (\mathrm{I}\,\vec{a}\,\mathrm{I}.\mathrm{I}\,\vec{b}\,\mathrm{I}\sin\theta\hat{n})^2$

$= (ab\sin\theta\hat{n})^2 = a^2b^2sin^2\theta$ (As $(\hat{n})^2 = (\left|\hat{n}\right|)^2 = 1^2 = 1$)

$= a^2b^2(1 - cos^2\theta) = a^2b^2 - a^2b^2cos^2\theta$

$= a^2b^2 - (abcos\theta)^2 = a^2b^2 - (\vec{a}\,.\vec{b})^2$ (Proved)

Example:-24 In a ΔABC , prove by vector method that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$,

where BC = a, CA = b and AB =c. (2017-S)

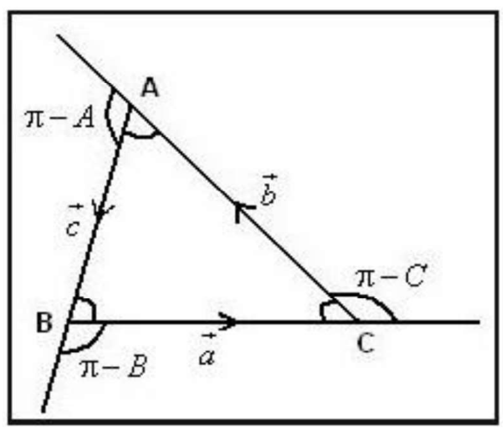


Fig-25

Proof:- As shown in figure- 25 ABC is a triangle having, $\vec{a} = \overrightarrow{BC}, \vec{b} = \overrightarrow{CA}$ and $\vec{c} = \overrightarrow{AB}$.

From triangle law of vector we know that ,

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

⇨ $\vec{a} + \vec{b} = -\vec{c}$

⇨ $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ………..(1)

(taking cross product of both sides with $\vec{a}$ we get)

⇨ $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

⇨ $(\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $\vec{0} + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $(\vec{a} \times \vec{b}) = -(\vec{a} \times \vec{c})$

⇨ $(\vec{a} \times \vec{b}) = (\vec{c} \times \vec{a})$ ---------------------(2)

Similarly taking cross product with $\vec{b}$ both sides of (1) we have,

⇨ $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c})$ ------------------(3)

From (2) and (3) , $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$

⇨ $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

⇨ $ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

As from fig-25 it is clear that angle between $\vec{a}$ *and* $\vec{b}$ is $\pi - C$, $\vec{b}$ *and* $\vec{c}$ is $\pi - A$and $\vec{c}$ *and* $\vec{a}$ is $\pi - B$.

Dividing above equation by abc we have,

⇨ $\frac{ab\sin(\pi-C)}{abc} = \frac{bc\sin(\pi-A)}{abc} = \frac{ca\sin(\pi-B)}{abc}$

⇨ $\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Hence $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ (Proved).

Example:-25 What inference can you draw when $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

Ans: - Given $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

⇨ { Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ II $\vec{b}$ } and { $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$ }

⇨ As $\vec{a}$ II $\vec{b}$ and $\vec{a} \perp \vec{b}$ cannot be hold simultaneously so $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

Hence either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

Example:-26 If $|\vec{a}| = 2$ and $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then find $\vec{a}.\vec{b}$.

Ans: - Given $|\vec{a} \times \vec{b}| = 8$

⇨ $|\vec{a}||\vec{b}|\sin\theta = 8$

⇨ $2 \times 5 \sin\theta = 8$

⇨ $\sin\theta = \frac{8}{10} = \frac{4}{5}$

∴ $\cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1 - (\frac{4}{5})^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Hence $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 2 \times 5 \times \frac{3}{5} = 6$ (Ans)

Example:-27 Show that the vectors $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$, and $4\hat{\imath} - 7\hat{\jmath} + 10\hat{k}$ are co-planar.(2017-S)

Ans: - Now let us find the following determinant ,

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) - (-3)(20-8) + 4(-14+4) = 4 + 36 - 40 = 0$$

Hence the three given vectors are co-planar.

Exercise

1. Show that the points (3,4) ,(1,7) and (-5,16) are collinear. (2 Marks)
2. If $\vec{a} = 3\hat{\imath} - 5\hat{\jmath}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath}$, then find the unit vector parallel to $\vec{a} + 2\vec{b}$. (2 Marks)
3. Show that the vectors $\vec{a} = 3\sqrt{3}\hat{\imath} - 3\hat{\jmath}$, $\vec{b} = 6\hat{\jmath}$ and $\vec{c} = 3\sqrt{3}\hat{\imath} + 3\hat{\jmath}$ form the sides of an equilateral triangle. (5 Marks)
4. Find the unit vector parallel to the sum $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$. (2014-W,2017-W).(2 Marks)
5. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$,where $\vec{a} = \hat{\imath} + \hat{\jmath}$ and $\vec{b} = \hat{\jmath} + \hat{k} - 2\hat{\imath}$.(2015-W) (5 Marks)
6. The position vector of A,B and C are $2\hat{\imath} + \hat{\jmath} - \hat{k}, 3\hat{\imath} - 2\hat{\jmath} + \hat{k}, \hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ respectively . Show that A, B and C are collinear. (2 Marks)
7. Find the value of 'a' such that the vectors $\hat{\imath} - \hat{\jmath} + \hat{k},\ 2\hat{\imath} + \hat{\jmath} + \hat{k}$ and $a\hat{\imath} - \hat{\jmath} + a\hat{k}$ are coplanar. (2 Marks)
8. Find the value of 'k' so that the vectors $\vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{b} = k\hat{\imath} + \hat{\jmath} + 5\hat{k}$ are perpendicular to each other. (2015-W) (2 Marks)
9. Find the unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} - \hat{k}$. (5 Marks)
10. Find the angle between the vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$ and $\vec{b} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$. (5 Marks)
11. Calculate the area of the triangle ABC by vector method where A(1,2,4), B(3,1,-2) and C(4,3,1). (5 Marks)
12. Obtain the area of the parallelogram whose adjacent sides are given by vectors $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $-3\hat{\imath} - 2\hat{\jmath} + \hat{k}$. (5 Marks)
13. Determine the sine of the angle between $\vec{a} = \hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k}$. (5 Marks)
14. Find the unit vector along the direction of vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$. (2015-S) (2 Marks)
15. Find the area of the parallelogram having adjacent sides $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $2\hat{\imath} + \hat{\jmath}$. (5 Marks)
16. Find the unit vector perpendicular to both $3\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\hat{\imath} + \hat{\jmath} - \hat{k}$. (5 Marks)
17. Find the area of the parallelogram having vertices A(5,-1,1), B(-1,-3,4), C(1,-6,10) and D(7,-4,7). (5 Marks)
18. Find the vector joining the points (2,-3) and (-1,1).Find its magnitude and the unit vector along the same direction. Also determine the component vectors along the co-ordinate axes. (5 Marks)
19. Prove by vector method , that in a triangle ABC,

$cosA = \frac{b^2+c^2-a^2}{2bc}$ where BC = a, CA = b and AB =c. (5 Marks)

20. Find the work done by the force $4\hat{\imath} - 3\hat{k}$ on a particle to displace it from (1,2,0) to (0,2,3) (2 Marks)
21. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then show that $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$. (2 Marks)

22. If $\vec{a}$,$\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $(\vec{a}+\vec{b}+\overrightarrow{c})$ is equally inclined with the vectors $\vec{a}$,$\vec{b}$ and $\vec{c}$. (10 Marks)

23. Find the area of the parallelogram whose diagonals are $\vec{a} = 2\hat{\imath} - 3\,\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -3\hat{\imath} + 4\,\hat{\jmath} - \hat{k}$. (5 Marks)

Answers

2) $\frac{7}{5\sqrt{2}}\hat{\imath} + \frac{1}{5\sqrt{2}}\hat{\jmath}$

4) $\frac{3\hat{\imath}+6\,\hat{\jmath}-2\hat{k}}{7}$

5) $\frac{-1}{\sqrt{6}}$, $\frac{-1}{6}(\hat{\jmath} + \hat{k} - 2\hat{\imath})$

7) 1

8) 3

9) $\frac{11}{\sqrt{122}}\hat{\imath} - \frac{1}{\sqrt{122}}\hat{k}$

10) $\frac{\pi}{2}$

11) $\frac{5\sqrt{10}}{2}$ sq units

12) $6\sqrt{5}$ sq units

13) $\frac{4\sqrt{2}}{\sqrt{33}}$

14) $\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$

15) $\sqrt{45}$ sq units

16) $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{k}$

17) $\sqrt{2257}$ sq units

18) $-3\hat{\imath} + 4\,\hat{\jmath}$, 5 , $\frac{-3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$, $\frac{-3}{5}\hat{\imath}$ and $\frac{4}{5}\hat{\jmath}$.

20) -13 units

23) $\frac{3\sqrt{30}}{2}$ sq units.