LEARNING MATERIAL
OF
MATHEMATICS - II

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Derivatives

Introduction

The study of differential calculus originated in the process of solving the following three problems

1. From the astronomical consideration particularly involving an attempt to have a better approximation of π as developed by Bhaskaracharya, Madhava and Nilakantha.
2. Finding the tangent to any arbitrary curve as developed by Fermat and Leibnitz.
3. Finding rate of change as developed by Fermat and Newton.

In this chapter we define derivative of a function, give its geometrical and physical interpretation and discuss various laws of derivatives etc.

Objectives

After studying this lesson, you will be able to:

(1) Define and Interpret geometrically the derivative of a function y = f(x) at x = a.
(2) State derivative of some standard function.
(3) Find the derivative of different functions like composite function, implicit function using different techniques.
(4) Find higher order derivatives of a particular function by successive differentiation method.
(5) Determine rate of change and tangent to a curve.
(6) Find partial derivative of a function with more than one variable with respect to variables.
(7) Define Euler's theorem and apply it solve different problems based on partial differentiation.

Expected background knowledge

1. Function
2. Limit and continuity of a function at a point.

Derivative of a function

Consider a function y = x^2

Table-1

x	5	5.1	5.01	5.001	5.0001
y	25	26.01	25.1001	25.010001	25.00100001

Let x = 5 and y = 25 be a reference point

We denote the small changes in the value of x as 'δx' ,

δx = small change in x

δy = change in y, when there is a change of δx in x.

Now, $\frac{\delta y}{\delta x}$ is called Increment ratio or Newton quotient or average rate of change of y.

Now, let us write table -1 in terms of δx, δy as

Table-2

δx	0.1	0.01	0.001	0.0001
δy	1.01	0.1001	0.010001	0.00100001
$\frac{\delta y}{\delta x}$	10.1	10.01	10.001	10.0001

From table-2 δy varies as δx varies

It is clear from the table when $\delta x \to 0$
⇨ $\delta y \to 0$ and $\frac{\delta y}{\delta x} \to 10$

This $\frac{\delta y}{\delta x}$ when $\delta x \to 0$ is the instantaneous rate of change of y at the value of x.

In above case x = 5, so $\frac{dy}{dx}$ at x = 5 is 10

Definition of derivative of a function (Differentiation)

If y = f(x) is a function. Then derivative of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$\frac{dy}{dx}$ is also denoted by f '(x)

$\frac{dy}{dx} = \frac{df(x)}{dx}$ = f ' (x) = f '
are same notations

Process of finding derivatives of dependent variable w.r.t. independent varibale is called differentiation.

Derivative of a function at a point 'a'

Derivative of y = f(x) at a point 'a' in the domain D_f is given by

$$\frac{dy}{dx}\Big]_{x=a} = f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Example -1

Find the derivative of f(x) = x^2 at x = 5

Ans. $\frac{dy}{dx}\Big]_{x=5}$ = f ' (5) = $\lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$

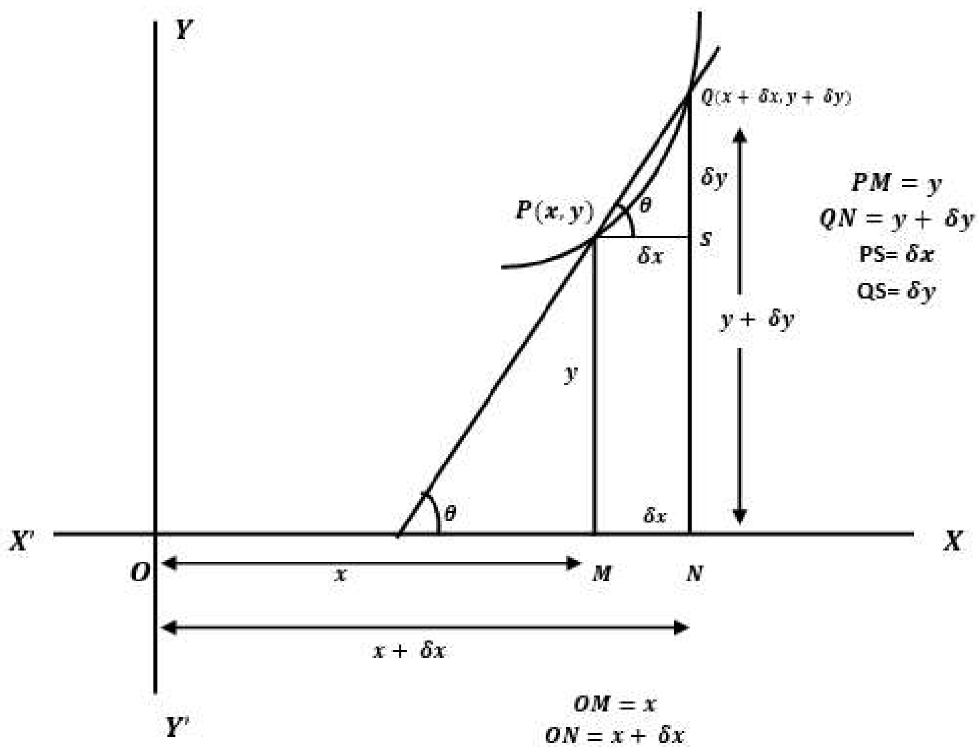
$= \lim_{h\to 0}\frac{(5+h)^2-5^2}{h}$

$= \lim_{h\to 0}\frac{(5+h+5)(5+h-5)}{h}$

$= \lim_{h\to 0}\frac{(10+h)h}{h} = \lim_{h\to 0}(10+h) = 10$

Geometrical Interpretation of $\frac{dy}{dx}$

(Fig.-1)



Let f(x) is represented by the curve in fig-1 given above.

Let Q(x+$\delta x, y + \delta y$) be the neighbourhood of P(x,y). PM and QN are drawn perpendicular to X-axis.

PS⊥ QN

Let QP Secant meets x-axis, (by extending it) and $\overleftrightarrow{QP}$ make angle θ with x-axis then angle QPS = θ

In ΔQPS, tan θ = $\frac{QS}{PS}$ = $\frac{\delta y}{\delta x}$

As QN = y + δy, NS = PM = y

=> QS = QN – NS = δy.

Similarly, ON = x+δx and OM = x => PS = MN = ON – OM = δx

When δx →0 then Q→P and QP secant becomes tangent at P.

In Δ PQS $\tan\theta = \frac{\delta y}{\delta x}$ { $\tan\theta$ gives slope of PQ line}

We know

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \tan\theta$$

Now when δx → 0 the line PQ becomes tangent at P

So,

$$\frac{dy}{dx} = \tan\theta = \text{slope of the tangent to the curve at P.}$$

So derivative of a function at a point represents the slope or gradient of the tangent at that point.

Example 2

Q. Find the slope of the tangent to the curve y = x^2 at x = 5.

Ans. As we have done it in example – 1.

$\frac{dy}{dx}\Big]_{x=5}$ = 10

Therefore, slope of the tangent at x = 5 is 10.

Derivative of some standard functions

1. $\frac{d}{dx}$ (c) = 0
2. $\frac{d}{dx}(x^n)$ = n.x^{n-1}
3. $\frac{d}{dx}(a^x) = a^x \log_e a$ In particular $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ In particular $\frac{d}{dx}(\log_e x) = \frac{d}{dx}$ (lnx) $= \frac{1}{x}$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x)$ = - $\sin x$
7. $\frac{d}{dx}$ (tan x) = $sec^2 x$
8. $\frac{d}{dx}$ (cot x) = - $cosec^2 x$
9. $\frac{d}{dx}$ (sec x) = sec x.tan x
10. $\frac{d}{dx}$ (cosec x) = - cosec x. cot x
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
12. $\frac{d}{dx}(\cos^{-1} x)$ = - $\frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
14. $\frac{d}{dx}(\cot^{-1} x)$ = - $\frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
16. $\frac{d}{dx}(\text{cosec}^{-1} x)$ = - $\frac{1}{|x|\sqrt{x^2-1}}$

Algebra of derivatives or fundamental theorems of derivatives

If f(x) and g(x) are both derivable functions i.e. their derivative exists then,

(i) $\frac{d}{dx}$ {cf(x)} = c f ' (x)

(ii) $\frac{d}{dx}$ (f + g) = f ' + g'

(iii) $\frac{d}{dx}$ (f - g) = f ' – g'

(iv) $\frac{d}{dx}$ {fg} = fg' + f ' g

(v) $\frac{d}{dx}\{\frac{f}{g}\} = \frac{f'g - fg'}{g^2}$

Example-3

Find the derivative of the following:

(i)$3x^3$ (ii) $6\sqrt{x}$ (iii) $9\ 3^x$ (iv) 5 cot x

Ans.

(i) $\frac{dy}{dx} = \frac{d}{dx}(3x^3) = 3\frac{d(x^3)}{dx} = 3 \text{ X } 3\ x^{3-1} = 9x^2$

(ii) $\frac{dy}{dx} = \frac{d(6\sqrt{x})}{dx} = 6\frac{d(\sqrt{x})}{dx} = 6\frac{d}{dx}(x^{\frac{1}{2}}) = 6\ \frac{1}{2}x^{\frac{1}{2}-1} = 6\ \frac{1}{2}x^{\frac{-1}{2}} = \frac{3}{\sqrt{x}}$

(iii) $\frac{dy}{dx} = \frac{d(9\ 3^x)}{dx} = 9\frac{d(3^x)}{dx} = 9\ 3^x \ln 3$

(iv) $\frac{d(5\cot x)}{dx} = 5\frac{d(\cot x)}{dx} = 5\ (-\ cosec^2 x) = -\ 5\ cosec^2 x$

Example 4

Find $\frac{dy}{dx}$ (i) y = $x^3 - x^2 + 6$

(ii) y = $\frac{1}{\sqrt{x}} + x^2(1-x) + \sin^{-1} x$

(iii) y = cosec x - $\sec^{-1} x$.cot x

Ans.

(i) $\frac{dy}{dx} = \frac{d}{dx}(x^3 - x^2 + 6)$

$= \frac{d(x^3)}{dx} - \frac{d(x^2)}{dx} + \frac{d(6)}{dx}$

$= 3x^2 - 2x + 0$

$= 3x^2 - 2x$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(\frac{1}{\sqrt{x}} + x^2(1-x) + \sin^{-1} x)$

$= \frac{d(\frac{1}{\sqrt{x}})}{dx} + \frac{d}{dx}\{x^2\ (1 - x\ )\} + \frac{d}{dx}(\sin^{-1} x)$

$= \frac{d(x^{\frac{-1}{2}})}{dx} + \{x^2\frac{d}{dx}\ (1 - x) + \frac{d}{dx}(x^2).\ (1 - x)\} + \frac{d}{dx}(\sin^{-1} x)$

{ $as\ \frac{d}{dx}$(fg) = fg ' + f ' g }

$= (-\frac{1}{2})\ x^{-\frac{1}{2}-1} + \{\ x^2\ (0 - 1) + 2x.\ (1 - x)\} + \frac{1}{\sqrt{1-x^2}}$

$= -\frac{1}{2x^{\frac{3}{2}}} - x^2 + 2x - 2x^2 + \frac{1}{\sqrt{1-x^2}}$

$= -\frac{1}{2x^{\frac{3}{2}}} + 2x - 3x^2 + \frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{dy}{dx} = \frac{d(\text{cosec x} - \sec^{-1}x.\text{cot x})}{dx}$

$= \frac{d(\text{cosec x})}{dx} - \frac{d(\sec^{-1}x.\text{cot x})}{dx}$

=(- cosecx.cotx) – $\{\sec^{-1}x$. (- $cosec^2$ x) + $\frac{1}{x\sqrt{x^2-1}}$ cot x}

= $\sec^{-1}x\, cosec^2 x$ - cosec x cot x - $\frac{1}{x\sqrt{x^2-1}}$ cot x

Example-5

Find the derivative of following functions w.r.t x.

(i) $\frac{3x^2+2x+5}{\sqrt{x}}$ (ii) $\frac{a^x-b^x}{x}$ (iii) $\frac{\tan x}{\cos^{-1}x}$ (iv) $(\frac{x^{\frac{3}{5}}-2e^{2lnx}+\ln x^{\frac{2}{3}}}{x+1})$ (v) x sin x - $\frac{e^x}{1+x^2}$

Ans.

(i) $\frac{dy}{dx} = \frac{\{3(2x)+2+0\}\sqrt{x} - (3x^2+2x+5)\frac{1}{2}x^{\frac{1}{2}-1}}{(\sqrt{x})^2}$

{ As $\frac{d}{dx}(\frac{f}{g}) = \frac{f'g-fg'}{g^2}$}

$= \frac{(6x+2)\sqrt{x}-(3x^2+2x+5)\frac{1}{2\sqrt{x}}}{x}$

$= \frac{6x^{\frac{3}{2}}+2\sqrt{x}-\frac{3}{2}x^{\frac{3}{2}}-\sqrt{x}-\frac{5}{2\sqrt{x}}}{x}$

$= \frac{\frac{9}{2}x^{\frac{3}{2}}+\sqrt{x}-\frac{5}{2\sqrt{x}}}{x}$

$= \frac{9}{2}\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{5}{2x^{\frac{3}{2}}}$

(ii) $y = \frac{a^x-b^x}{x}$

$\frac{dy}{dx} = \frac{(a^x\ln a - b^x \ln b)x - 1(a^x-b^x)}{x^2}$ $\{\frac{d}{dx}(\frac{f}{g}) = \frac{f'g-fg'}{g^2}\}$

$= \frac{xa^x\ln a - xb^x\ln b - a^x + b^x}{x^2}$

$= \frac{a^x(xln\, a-1)+b^x(1-x\ln b)}{x^2}$

(iii) y = $\frac{\tan x}{\cos^{-1} x}$

$$\frac{dy}{dx} = \frac{\cos^{-1} x\, sec^2 x - \tan x\, (\frac{-1}{\sqrt{1-x^2}})}{(\cos^{-1} x)^2}$$

$$= \frac{\cos^{-1} x\, sec^2 x + \frac{\tan x}{\sqrt{1-x^2}})}{(\cos^{-1} x)^2}$$

(iv) y = $\frac{x^{\frac{3}{5}} - 2e^{2lnx} + \ln x^{\frac{2}{3}}}{x+1}$

$$= \frac{x^{\frac{3}{5}} - 2e^{lnx^2} + \frac{2}{3}\ln x}{x+1}$$ {As $e^{\ln x}$ = x and $lne^x = x$}

$$= \frac{x^{\frac{3}{5}} - 2x^2 + \frac{2}{3}\ln x}{x+1}$$

$$\frac{dy}{dx} = \frac{\left(\frac{3}{5}x^{\frac{3}{5}-1} - 4x + \frac{2}{3}\frac{1}{x}\right)(x+1) - \left(x^{\frac{3}{5}} - 2x^2 + \frac{2}{3}\ln x\right)(1+0)}{(x+1)^2}$$

$$= \frac{\left(\frac{3}{5}x^{\frac{-2}{5}} - 4x + \frac{2}{3x}\right)(x+1) - x^{\frac{3}{5}} + 2x^2 - \frac{2}{3}\ln x}{(x+1)^2}$$

$$= \frac{\frac{3}{5}x^{\frac{3}{5}} - 4x^2 + \frac{2}{3} + \frac{3}{5}x^{\frac{-2}{5}} - 4x + \frac{2}{3x} - x^{\frac{3}{5}} + 2x^2 - \frac{2}{3}\ln x}{(x+1)^2}$$

$$= \frac{\frac{2}{3} - 4x - 2x^2 - \frac{2}{5}x^{\frac{3}{5}} + \frac{3}{5x^{\frac{2}{5}}} + \frac{2}{3x} - \frac{2}{3}\ln x}{(x+1)^2}$$

(v) $\frac{dy}{dx} = \frac{d}{dx}(xsinx) - \frac{d}{dx}(\frac{e^x}{1+x^2})$

= {*x* cos *x*+ *1.* sin *x*} – $\{\frac{e^x.(1+x^2) - e^x(0+2x)}{(1+x^2)^2}\}$

= *x* cos *x* + sin *x* – $\{\frac{e^x + x^2e^x - 2xe^x}{(1+x^2)^2}\}$

= *x* cos *x* + sin *x* - $e^x\{\frac{1+x^2-2x}{(1+x^2)^2}\}$

= *x* cos *x* + sin *x* - $e^x\left(\frac{1-x}{1+x^2}\right)^2$

Example 6

Find the slope of the tangent to the curve y = $\ln x$ at x = $\frac{1}{2}$ **[2017-w]**

Ans.

Slope of tangent to the curve y = $\ln x$ at x = $\frac{1}{2}$ is $\frac{dy}{dx}\Big]_{x=\frac{1}{2}}$

Now, $\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$

Now $\frac{dy}{dx}\Big]_{x=\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

Example – 7

Find f ' ($\sqrt{3}$) if f(x) = x $\tan^{-1} x$**[2017-w]**

Ans. f(x) = $x \tan^{-1} x$

$$f'(x) = \frac{d(x \tan^{-1} x)}{dx} = x\frac{1}{1+x^2} + 1.\ \tan^{-1} x$$

$$= x\frac{1}{1+x^2} + \tan^{-1} x$$

$$f'(\sqrt{3}) = \frac{\sqrt{3}}{1+(\sqrt{3})^2} + \tan^{-1}\sqrt{3}$$

$$= \frac{\sqrt{3}}{1+3} + \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

Example-8

Find the gradient of the tangent to the curve 2x^2-3x-1 at (1,-2).

Ans.

$\frac{dy}{dx}$ = 4x – 3

$\frac{dy}{dx}\Big]_{at\ (1,-2)}$ = 4X1-3 = 1

Derivative of a composite function (Chain Rule)

Composite function

A function formed by composition of more than one function is called composite function.

Example of composite functions

1)$\sin x^2$ is form by composition of two functions, one is $\sin x$ function and other is x^2.

y= $\sin x^2$ = $\sin u$ where u = x^2

2) Similarly y = $\sqrt{x^2+3x+1}$ is written as

y= $\sqrt{u}$ where u = x^2+3x+1

3) y= $\sqrt{\sin(x^2+1)}$ is form by composition of three functions.

y = $\sqrt{u}$ where u = $\sin v$ and v = $(x^2$+1)

Chain Rule

If y = f(u) and u is a function of x defined by u = g(x), then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Generalized chain rule

If y is a differentiable function of u, u is a differentiable function v, ….. and finally t is a differentiable function of x. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \;\; \ldots\ldots \;\; \frac{dt}{dx}$$

Example – 9

Find $\frac{dy}{dx}$

(i). y = $(x^2+2x-1)^5$ (ii) y = $cot^3 x$

(iii) $\sqrt{\sin\sqrt{x}}$ (2016-S) (iv) $a^{\ln x}$ (v) $5^{\sin x^2}$

Ans.

(i) y = $(x^2 + 2x - 1)^5$

Here, y = u^5 and u = $x^2 + 2x - 1$

$\frac{du}{dx}$ = 2x + 2 = 2 (x+1) and $\frac{dy}{dx}$ = 5 u^4

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

= 5 u^4 2(x+1)

= 10 $(x^2 + 2x - 1)^4$ (x+1)

(ii) y = cot^3x can be written as y = u^3

where u = cot x

$\frac{du}{dx}$ = - $cosec^2x$, $\frac{dy}{du}$ = 3 u^2

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ = 3 $u^2(-cosec^2x)$

= - 3 $cot^2xcosec^2x$

(iii) y = $\sqrt{\sin\sqrt{x}}$

Here y = $\sqrt{u}$, u = $\sin v$, v = $\sqrt{x}$

So, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$, $\frac{du}{dv}$ = cos v, $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

Therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

= $\frac{1}{2\sqrt{u}}$. cos v. $\frac{1}{2\sqrt{x}}$

= $\frac{1}{2\sqrt{\sin v}}$. cos $\sqrt{x}$. $\frac{1}{2\sqrt{x}}$

= $\frac{cos\sqrt{x}}{4\sqrt{sin\sqrt{x}}\sqrt{x}}$

(iv) y = $a^{\ln x}$

Here y = a^u where u = $\ln x$

$\frac{dy}{du}$ = $a^u \ln a$ and $\frac{du}{dx} = \frac{1}{x}$

Hence $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ = $a^u \ln a . \frac{1}{x}$

= $a^{lnx} \ln a \ \frac{1}{x}$

= $\frac{1}{x} \ln a \ a^{\ln x}$

(v) y = $5^{\sin x^2}$

Here y = 5^u, u = $\sin v$, v = x^2

$\frac{dy}{du}$ = $5^u \ln 5$, $\frac{du}{dv}$ = $\cos v$, $\frac{dv}{dx}$ = 2x

Therefore, $\frac{dy}{dx}$ = $\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$ = $5^u \ln 5 . \cos v$. 2x

= $5^{\sin v} \ln 5 \cos x^2 \, 2x$

= $2x \ ln 5 \ 5^{\sin x^2} \cos x^2$

Example – 10

Differentiate the following functions w.r.t. x.

(i) $\sqrt{\cot^{-1}\sqrt{x}}$ (ii) $\frac{1}{f(x)}$ (2016-S) (iii) $\frac{1}{f(ax+b)}$ (2014-S)

(iv) $\tan^{-1}(\sec x + \tan x)$ (2017-S) (v) $\cos^{-1}(\frac{\cos x + \sin x}{\sqrt{2}})$

Ans.

(i) $\frac{d\sqrt{\cot^{-1}\sqrt{x}}}{dx} = \frac{d(\cot^{-1}\sqrt{x})}{dx}$

{Here y = $\sqrt{u}$, Then $\frac{dy}{du} = \frac{d\sqrt{u}}{du} = \frac{1}{2\sqrt{u}}$, u = $\cot^{-1}\sqrt{x}$ = $\cot^{-1} v$, $\frac{du}{dv} = -\frac{1}{1+v^2}$}

= $\frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}}$ {- $\frac{1}{1+(\sqrt{x})^2}$} $\frac{d\sqrt{x}}{dx}$ { v = $\sqrt{x}$, then by chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$}

= - $\frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}\,(1+x)} \frac{1}{2\sqrt{x}}$

= - $\frac{1}{4\sqrt{x}\sqrt{\cot^{-1}\sqrt{x}}\,(1+x)}$

= - $\frac{1}{4\sqrt{x}(1+x)\sqrt{\cot^{-1}\sqrt{x}}}$

(ii) $\frac{d}{dx}\{\frac{1}{f(x)}\}$ = - $\frac{1}{\{f(x)\}^2}$ f ' (x)

= - $\frac{f'(x)}{\{f(x)\}^2}$

(iii) $\frac{d}{dx}\{\frac{1}{f(ax+b)}\}$ = - $\frac{1}{\{f(ax+b)\}^2}$ f '(ax+b) $\frac{d}{dx}$ (ax+b)

= - $\frac{a f'(ax+b)}{f(ax+b)^2}$

(iv) y = $\tan^{-1}(\sec x + \tan x)$

= $\tan^{-1}(\frac{1}{\cos x} + \frac{\sin x}{\cos x})$

= $\tan^{-1}(\frac{1+\sin x}{\cos x})$

= $\tan^{-1}\{\frac{sin^2\frac{x}{2}+ cos^2\frac{x}{2}+2sin\frac{x}{2}cos\frac{x}{2}}{cos^2\frac{x}{2}-sin^2\frac{x}{2}}\}$

= $\tan^{-1}\{\frac{(sin\frac{x}{2}+cos\frac{x}{2})^2}{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)(cos\frac{x}{2}+sin\frac{x}{2})}\}$

= $\tan^{-1}(\frac{cos\frac{x}{2}+sin\frac{x}{2}}{cos\frac{x}{2}-sin\frac{x}{2}})$ {dividing numerator and denominator by $\frac{x}{2}$ }

= $\tan^{-1}(\frac{\frac{cos\frac{x}{2}}{cos\frac{x}{2}}+\frac{sin\frac{x}{2}}{cos\frac{x}{2}}}{\frac{cos\frac{x}{2}}{cos\frac{x}{2}}-\frac{sin\frac{x}{2}}{cos\frac{x}{2}}})$ = $\tan^{-1}(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}})$

= $\tan^{-1}(\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{x}{2}})$ = $\tan^{-1}\{\tan(\frac{\pi}{4} + \frac{x}{2})\}$ = $\frac{\pi}{4} + \frac{x}{2}$

Hence $\frac{dy}{dx}$ = $\frac{d}{dx}$ { $\tan^{-1}(\sec x + \tan x)$ } = $\frac{d}{dx}$ $(\frac{\pi}{4} + \frac{x}{2})$ = $\frac{1}{2}$

(v) y = $\cos^{-1}(\frac{\cos x+\sin x}{\sqrt{2}})$

= $\cos^{-1}(\cos x\ \frac{1}{\sqrt{2}} + \sin x.\frac{1}{\sqrt{2}})$

= $\cos^{-1}(\cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4})$

= $\cos^{-1}(\cos(x - \frac{\pi}{4}))$ = x - $\frac{\pi}{4}$

Hence $\frac{dy}{dx}$ = $\frac{d}{dx}$ $(\cos^{-1}(\frac{\cos x+\sin x}{\sqrt{2}}))$

= $\frac{d}{dx}$ (x - $\frac{\pi}{4}$) = 1

Example –11

If y = sin 5x cos 7x then find $\frac{dy}{dx}$

Ans.

$\frac{dy}{dx}$ = sin 5x $\frac{\mathrm{d}}{\mathrm{dx}}$(cos 7x) + $\frac{\mathrm{d}}{\mathrm{dx}}$(sin 5x). cos 7x

= sin 5x. (-7 sin 7x) + 5 cos 5x cos 7x

= 5 cos 5x cos 7x – 7 sin 5x sin 7x

Example –12

Find $\frac{dy}{dx}$ if y = $cosec^2(2x^2 + \log_7 x)$

Ans.

$\frac{dy}{dx}$ = 2 cosec $(2x^2 + \log_7 x)$\{(-cosec$(2x^2 + \log_7 x)).\cot(2x^2 + \log_7 x)$\} (4x + $\frac{1}{x\log_e 7}$)

= - 2 $cosec^2(2x^2 + \log_7 x)$. Cot $(2x^2 + \log_7 x)$*[4x* + $\frac{1}{x\log_e 7}$]

Methods of differentiation

We use following two methods for differentiation of some functions.

(i) Substitution

(ii) Use of logarithms

Substitution

Sometimes with proper substitution we can transform the given function to a simpler function in the new variable so that the differentiation w.r.t to new variable becomes easier. After differentiation we again re-substitute the old variable. This can be better understood by following examples.

Example – 13

y = $\tan^{-1}(\frac{\sqrt{x}-x}{1+x^{\frac{3}{2}}})$

Ans.

y = $\tan^{-1}(\frac{\sqrt{x}-x}{1+x^{\frac{3}{2}}})$

(If we differentiate directly by applying chain rule , it will be very complicated. So, we have to adopt substitution technique here.)

Now y = $\tan^{-1}(\frac{\sqrt{x}-x}{1+x^{\frac{3}{2}}})$ = $\tan^{-1}(\frac{\sqrt{x}-x}{1+\sqrt{x}.x})$

Now Put $\sqrt{x}$ = tan ∝ , x = tan β

Then, y = $\tan^{-1}(\frac{\tan\propto-\tan\beta}{1+tan\alpha.tan\beta})$

= $\tan^{-1}(\tan(\propto-\beta))$

= ∝ - β = $\tan^{-1}\sqrt{x}$ - $\tan^{-1}x$

(As $\sqrt{x}$ = tan ∝ ⇒ ∝= $\tan^{-1}\sqrt{x}$ and x = tan β ⇒ β = $\tan^{-1}\sqrt{x}$)

Now $\frac{dy}{dx}$ = $\frac{d}{dx}$ ($\tan^{-1}\sqrt{x}$ - tan x)

$= \frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x}) - \frac{1}{1+x^2}$

$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{1+x^2}$

$= \frac{1}{(1+x)2\sqrt{x}} - \frac{1}{1+x^2}$ (Ans)

Example –14

Find $\frac{d}{dt}$ if y = $\cos^{-1}(\frac{1-t^2}{1+t^2})$ **(2015-S)**

Ans.

y = $\cos^{-1}(\frac{1-t^2}{1+t^2})$ {Put tan θ = t ⇒ $\theta = \tan^{-1} t$}

$= \cos^{-1}(\frac{1-tan^2\theta}{1+tan^2\theta})$

$= \cos^{-1}(\cos 2\theta)$

= 2 θ

= 2 $\tan^{-1} t$

$\frac{dy}{dt} = \frac{d}{dt}(2\tan^{-1} t) = \frac{2}{1+t^2}$

Note

When we apply substitution method, then we must have proper knowledge about trigonometric formulae. Because it makes the choice of new variable easy. If proper substitution is not made, then problem will be more complicated than original.

Example –15

If y = $\sec^{-1}(\frac{\sqrt{a^2+x^2}}{a})$ then find $\frac{dy}{dx}$

Ans.

y = $\sec^{-1}(\frac{\sqrt{a^2+x^2}}{a})$ Put x = a tan θ

$= \sec^{-1}(\frac{\sqrt{a^2+a^2tan^2\theta}}{a})$ $= \sec^{-1}(\frac{\sqrt{a^2(1+tan^2\theta)}}{a})$

$= \sec^{-1}(\frac{\sqrt{a^2sec^2\theta}}{a})$ $\quad$ $= \sec^{-1}(\frac{a\sec\theta}{a})$

$= \sec^{-1}(\sec\theta) = \theta$ $\quad$ $= \tan^{-1}(\frac{x}{a})$

Now $\frac{dy}{dx} = \frac{d}{dx}\{\tan^{-1}\left(\frac{x}{a}\right)\} = \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$

$= \frac{1}{(1+\frac{x^2}{a^2})}\left(\frac{1}{a}\right) = \frac{1}{a}\frac{1}{\frac{a^2+x^2}{a^2}}$

$= \frac{a^2}{a(x^2+a^2)} = \frac{a}{x^2+a^2}$

Example – 16

Differentiate $sin^2(\cot^{-1}\sqrt{\frac{1+x}{1-x}})$ w.r.t x. **[2018-S]**

Ans.

y $= sin^2(\cot^{-1}\sqrt{\frac{1+x}{1-x}})$ $\quad$ {Put x = cos 2θ ⇒ $\theta = \frac{\cos^{-1}x}{2}$}

$= sin^2(\cot^{-1}\sqrt{(\frac{1+\cos 2\theta}{1-\cos 2\theta})}) = sin^2(\cot^{-1}\sqrt{\frac{2cos^2\theta}{2sin^2\theta}})$

$= sin^2(\cot^{-1}\sqrt{cot^2\theta}) = sin^2\cot^{-1}(\cot\theta) = sin^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}(sin^2\theta)\frac{d}{dx}(\frac{\cos^{-1}x}{2})$

$= 2\sin\theta\cos\theta\frac{1}{2}(\frac{-1}{\sqrt{1-x^2}})$

$= \sin 2\theta(\frac{-1}{2\sqrt{1-x^2}}) = -\frac{\sqrt{1-cos^2 2\theta}}{2\sqrt{1-x^2}}$

$= -\frac{\sqrt{1-x^2}}{2\sqrt{1-x^2}} = -\frac{1}{2}$ (Ans)

Example – 17

Find the derivative of $\cot^{-1}(\sqrt{1+x^2}+x)$ w.r.t x

Ans.

y $= \cot^{-1}(\sqrt{1+x^2}+x)$ $\quad$ { Put x = cot θ =>$\theta = \cot^{-1}x$}

$= \cot^{-1}(\sqrt{1+cot^2\theta}+\cot\theta)$

$= \cot^{-1}(\sqrt{cosec^2\theta}+\cot\theta)$

$= \cot^{-1}(\operatorname{cosec}\theta + \cot\theta)$

$= \cot^{-1}(\frac{1}{sin\theta} + \frac{cos\theta}{sin\theta})$

$= \cot^{-1}(\frac{1+cos\theta}{sin\theta})$

$= \cot^{-1}(\frac{2cos^2\frac{\theta}{2}}{2sin\frac{\theta}{2}.cos\frac{\theta}{2}})$

$= \cot^{-1}(\frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}})$

$= \cot^{-1}(\cot(\frac{\theta}{2}))$ $= \frac{\theta}{2} = \frac{\cot^{-1}x}{2}$ Type equation here.

∴ $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cot^{-1}x}{2}\right) = \frac{1}{2}\left(\frac{-1}{1+x^2}\right) = -\frac{1}{2(1+x^2)}$

Differentiation using logarithm

When a function appears as an exponent of another function we make use of logarithms.

Example – 18

Differentiate $(\sin x)^{\tan x}$

Ans.

y = $(\sin x)^{\tan x}$

Taking logarithms of both sides we have,

$\ln y = \ln(\sin x)^{\tan x}$

$\Rightarrow \ln y = \tan x . \ln\sin x$

Differentiating both sides w.r.t x, we have

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x \frac{1}{\sin x} . \cos x + sec^2x . \ln\sin x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x . \cot x + sec^2x . \ln\sin x = 1 + sec^2x . \ln\sin x$

$\Rightarrow \frac{dy}{dx} = y\,(1+sec^2x . \ln\sin x)$

Hence $\frac{dy}{dx} = (\sin x)^{\tan x}\,(1+sec^2x . \ln\sin x)$

Example – 19

Differentiate y = $\frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}$

Ans.

y = $\frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}$

Taking logarithm of both sides

⇒ $\ln y = \ln(x-1)^2 + \ln\sqrt{3x-1} - \ln x^7 - \ln(6-7x^2)^{\frac{3}{2}}$ {as $\log ab = \log a + \log b$ & $\log\frac{a}{b} = \log a - \log b$}

⇒ $\ln y = 2\ln(x-1) + \frac{1}{2}\ln(3x-1) - 7\ln x - \frac{3}{2}\ln(6-7x^2)$ { as $\ln x^n$ = n lnx}

Differentiating both sides w.r.t, we have

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-1}\frac{d\,(x-1)}{dx} + \frac{1}{2}\frac{1}{(3x-1)}\frac{d(3x-1)}{dx} - \frac{7}{x} - \frac{3}{2}\frac{1}{6-7x^2}\frac{d(6-7x^2)}{dx}$

$= \frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{3(14x)}{2(6-7x^2)}$

$= \frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}$

⇒ $\frac{dy}{dx} = y\left[\frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}\right]$

⇒ $\frac{dy}{dx} = \frac{(x-1)^2\sqrt{3x-1}}{x^7(6-7x^2)^{\frac{3}{2}}}\left[\frac{2}{x-1} + \frac{3}{2(3x-1)} - \frac{7}{x} + \frac{21x}{6-7x^2}\right]$ (Ans)

Example – 20

Find the derivative of y = $(\log x)^{\tan x}$ **(2017-W, 2015-S)**

Ans: - y = $(\log x)^{\tan x}$

Taking logarithm of both sides,

$\log y = \log(\log x)^{\tan x}$

⇒ logy = tanx log(logx)

⇒ $\frac{1}{y}\frac{dy}{dx}$ = tan x $\frac{1}{\log x}\cdot\frac{1}{x} + sec^2x\log(\log x)$

⇒ $\frac{dy}{dx} = y\left(\frac{\tan x}{x\log x} + sec^2x\log(\log x)\right)$

⇒ $\frac{dy}{dx} = (\log x)^{\tan x}\left(\frac{\tan x}{x\log x} + sec^2x\log(\log x)\right)$

Example – 21

Differentiate $(\sin x)^{\ln x}$ w.r.t x **(2017-S)**

Ans.

y = $(\sin x)^{\ln x}$

Then log y = log $(\sin x)^{\ln x}$ = ln x log (sin x)

Differentiating w.r.t x ,

$\frac{1}{y}\frac{dy}{dx}$ = $lnx \frac{1}{\sin x}\ cosx$ + $\frac{1}{x}$ log(sin x)

$\Rightarrow \frac{dy}{dx}$= *y [lnx cot x* +$\frac{\log(\sin x)}{x}$]

$\therefore \frac{dy}{dx}$ = $(\sin x)^{\ln x}$ [*ln cot x* + $\frac{\log(\sin x)}{x}$]

Example – 22

Find $\frac{dy}{dx}$ if y = x^x

Ans. y = x^x

Taking logarithm of both sides,

$\Rightarrow$ log y = $\log x^x$ = x log x

Differentiating w.r.t x ,

$\Rightarrow \frac{1}{y}\frac{dy}{dx}$ = $x.\frac{1}{x}$ + 1 . log x

$\Rightarrow \frac{dy}{dx}$ = y (1+log x) = $x^x(1 + \log x)$.

Example – 23

Differentiate $(\ln x)^x$ + $(\sin^{-1} x)^x$ w.r.t. x.

Ans:- y= $(\ln x)^x$ + $(\sin^{-1} x)^x$ = u +v

u = $(\ln x)^x$ and v = $(\sin^{-1} x)^x$

Taking logarithm of both sides,

$\Rightarrow$ $\log u = \log (\ln x)^x$ and $\log v = \log (\sin^{-1} x)^x$

$\Rightarrow$ log u= x log (ln x) and log v = x log $(\sin^{-1} x)$

Differentiating w.r.t x ,

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\text{x log (ln x)})$ ()) and $\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(xlog(sin^{-1}x))$

$\Rightarrow \frac{1}{u}\frac{du}{dx}$ = [$x\frac{1}{\ln x} \cdot \frac{1}{x}$ + 1 . log (*ln x)*] and $\frac{1}{v}\frac{dv}{dx}$= $x\frac{1}{sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$ + 1. log($sin^{-1}x$)

$\Rightarrow \frac{du}{dx} = u[\frac{1}{\ln x}$ + log (ln x)] and $\frac{dv}{dx} = v[\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)]

$\Rightarrow \frac{du}{dx} = (\ln x)^x$ [$\frac{1}{\ln x}$ + log (ln x)] and $\frac{dv}{dx} = (\sin^{-1}x)^x$ [$\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)]

Now, $\frac{dy}{dx} = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

= $(\ln x)^x$ [$\frac{1}{\ln x}$ + log (ln x)] + $(\sin^{-1}x)^x$ [$\frac{x}{sin^{-1}x\sqrt{1-x^2}}$ + log($sin^{-1}x$)] (Ans)

Differentiation of parametric function

Sometimes the variables x and y of a function is represent by function of another variable 't', which is called as a parameter. Such type of representation of a fnction is called parametric form. For example equation of circle can be given by x = r cos t, y = r sin t.

Here x, y both are functions of parameter 't'.

So, this form of the function is called parametric form.

Derivative of function given in parametric form

If y = f(t) , x = g(t) , Then

$$\frac{dy}{dx} = \frac{f'(t)}{g'(t)} = \frac{\frac{df(t)}{dt}}{\frac{dg(t)}{dt}} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}$$

Example – 24

Find $\frac{dy}{dx}$ if x = at^2 and y = 2 bt

Ans.

$\frac{dx}{dt}$ = 2at , $\frac{dy}{dt}$ = 2b

Now, $\frac{dy}{dx} = (\frac{dy}{dt}) \Big/ (\frac{dx}{dt}) = \frac{2b}{2at} = \frac{b}{at}$

Example -25

Find $\frac{dy}{dx}$ if x = a(1 + $\cos\theta$) and y =b (1 - $\sin\theta$)   (2018-S)

Ans. $\frac{dx}{d\theta}$ = a (- $\sin\theta$) = -a sin θ , $\frac{dy}{d\theta}$ = b (- $\cos\theta$) = -b cosθ

Hence $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-b\cos\theta}{-a\sin\theta} = \frac{b}{a}$ cot θ

Example – 26

Find $\frac{dy}{dx}$ when x = a (cos t +t sin t) and y = a (sin t – t cos t)   **(2017-S, 2017-W)**

Ans.

$\frac{dx}{dt}$ = a (-sin t + t cos t + 1. sin t)

= a (t cos t) = a t cos t

$\frac{dy}{dt}$ = a (cos t – t (-sint) – 1. cos t)

= a t sin t ,

∴ $\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{a\,t\sin t}{a\,t\cos t}$ = tan t.

Example – 27

If sin x = $\frac{2t}{1+t^2}$ and tan y = $\frac{2t}{1-t^2}$ then find $\frac{dy}{dx}$.

Ans.

Put t = tan θ { In this case by substitution we can convert both x and y into functions of another parameter θ, which are easily differentiable w.r.t to θ .}

Then sin x = $\frac{2\tan\theta}{1+tan^2\theta}$ = sin 2θ

⇒ x = $\sin^{-1}(\sin 2\theta)$ = 2θ

tan y = $\frac{2t}{1-t^2} = \frac{2\tan\theta}{1-tan^2\theta}$ = tan 2θ

⇒ y = 2θ

Now, $\frac{dy}{d\theta}$ = 2 and $\frac{dx}{d\theta}$ = 2

Hence $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2}{2} = 1$

Differentiation of a function w.r.t another function

Suppose we have two differentiable functions given by y = f(x) and z = g(x). Then to find the derivative of y w.r.t. z we have to follow the following formula.

$$\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})}$$

Example-28

Find the derivative of tan x w.r.t cot x **(2017-w)**

Ans.

Let y = tan x and z = cot x

$\frac{dy}{dx} = sec^2x$, $\frac{dz}{dx}$ = - $cosec^2x$

Now, $\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})} = \frac{sec^2x}{-cosec^2x}$ = - $sec^2x sin^2x$,Hence $\frac{d(\tan x)}{d\ (\cot x)}$ = - $sec^2x sin^2x$

Example – 29

Find the derivative of $e^{2\log x}$ w.r.t $2x^2$ **[2018-S, 2017-w]**

Ans.

y = $e^{2\log x}$ and z = $2x^2$

$\frac{dy}{dx} = \frac{d}{dx}(e^{2\log x}) = \frac{d}{dx}(e^{log x^2}) = \frac{d}{dx}(x^2)$ = 2x

$\frac{dz}{dx} = \frac{d}{dx}(2x^2)$ = 4x

Hence $\frac{dy}{dz} = (\frac{dy}{dx})/(\frac{dz}{dx}) = \frac{2x}{4x} = \frac{1}{2}$

Example – 30

Differentiate a^x w.r.t x^a **[2014-S]**

Ans. y = a^x and z = x^a

Now, $\frac{dy}{dx} = a^x$log a and $\frac{dz}{dx}$ = ax^{a-1}

Hence $\frac{dy}{dz} = \frac{(\frac{dy}{dx})}{(\frac{dz}{dx})} = \frac{a^x \log a}{\mathrm{a}x^{a-1}} = \frac{a^{x-1}}{x^{a-1}}$ log a (Ans)

Example – 31

Differentiate $\sin^{-1}(\frac{2x}{1+x^2})$ w.r.t $\cos^{-1}(\frac{1-x^2}{1+x^2})$ **(2016-S)**

Ans. Let y = $\sin^{-1}(\frac{2x}{1+x^2})$& z = $\cos^{-1}(\frac{1-x^2}{1+x^2})$

Put x = tan t

y = $\sin^{-1}(\frac{2x}{1+x^2})$ = $\sin^{-1}(\frac{2\tan t}{1+tan^2 t})$ = $\sin^{-1}(\sin 2t)$ = 2 t

$\Rightarrow \frac{dy}{dt}$ = 2

Similarly, z = $\cos^{-1}(\frac{1-x^2}{1+x^2})$ = $\cos^{-1}(\frac{1-tan^2 t}{1+\ tan^2 t})$ = $\cos^{-1}(\cos 2t)$ = 2t

$\Rightarrow \frac{dz}{dt}$ = 2

$\frac{dy}{dz}$ = $\frac{(\frac{dy}{dt})}{(\frac{dz}{dt})}$ = $\frac{2}{2}$ = 1

Hence $\frac{d\ (\sin^{-1}(\frac{2x}{1+x^2}))}{d\ (\cos^{-1}(\frac{1-x^2}{1+x^2}))}$ = 1

Example – 32

Find derivative of log x w.r.t $\sqrt{x}$ **[2017-W]**

Ans.

y = log x and z = $\sqrt{x}$

Now, $\frac{dy}{dx} = \frac{1}{x}$ and $\frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

Hence $\frac{dy}{dz}$ = $\frac{(\frac{dy}{dx})}{(\frac{dz}{dx})}$ = $\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$ = $\frac{2\sqrt{x}}{x}$ = $\frac{2}{\sqrt{x}}$ (Ans)

Example –33

Differentiate $\frac{1-\cos x}{1+\cos x}$ w.r.t $\frac{1-\sin x}{1+\sin x}$

Ans.

y = $\frac{1-\cos x}{1+\cos x}$ and z = $\frac{1-\sin x}{1+\sin x}$

$\frac{dy}{dx} = \frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$ { As $\frac{d}{dx}\{\frac{f}{g}\} = \frac{f'g - fg'}{g^2}\}$

$= \frac{\sin x+\sin x cos x+\sin x-sin x cos x}{(1+\cos x)^2}$

$= \frac{2\sin x}{(1+\cos x)^2}$

$\frac{dz}{dx} = \frac{(1+\sin x)(-\cos x)-(1-\sin x)\cos x}{(1+\sin x)^2}$

$= \frac{-cos x-\cos x\sin x-\cos x+\cos x\sin x}{(1+\sin x)^2}$

$= \frac{-2\cos x}{(1+\sin x)^2}$

$\frac{dy}{dz} = \frac{(^{dy}/_{dx})}{(\frac{dz}{dx})} = \frac{\frac{2\sin x}{(1+\cos x)^2}}{\frac{-2\cos x}{(1+\sin x)^2}} = \frac{-\tan x(1+\sin x)^2}{(1+\cos x)^2}$

Differentiation of implicit function

Functions of the form F(x,y) = 0 where x and y cannot be separated or in other words y cannot be expressed in terms of x is called Implicit function.

e.g. x^2 + y^2 - 25 = 0

$x^y = y^x$

x^2y + y^2x + xy = 25 etc

Derivative of Implicit functions can be found without expressing y explicitly in terms of x. Simply we differentiate both side w.r.t x and express $\frac{dy}{dx}$ in terms of both x and y.

Example – 34

Find $\frac{dy}{dx}$ when x^3 + y^3- 3xy = 0 **[2015-S]**

Ans.

Given x^3 + y^3- 3xy = 0

Differentiating both sides w.r.t x We have,

3x^2 + 3$y^2\frac{dy}{dx}$ - 3x $\frac{dy}{dx}$ - 3.1.y = 0

$\Rightarrow \frac{dy}{dx}(3y^2 - 3x)$ = 3y - 3x^2

$\Rightarrow \frac{dy}{dx} = \frac{3y-3x^2}{3y^2-3x} = \frac{y-x^2}{y^2-x}$ (Ans)

Example – 35

Find $\frac{dy}{dx}$ if $\ln\sqrt{x^2+y^2}$ = $\tan^{-1}(\frac{y}{x})$ **[2017-w]**

Ans.

Given $\ln\sqrt{x^2+y^2}$ = $\tan^{-1}(\frac{y}{x})$

Differentiating both sides w.r.t x,

$$\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}(2x+2y\frac{dy}{dx}) = \frac{1}{1+(\frac{y}{x})^2}(\frac{x\frac{dy}{dx}-y.1}{x^2})$$

$$\Rightarrow\frac{2(x+y\frac{dy}{dx})}{2(x^2+y^2)} = \frac{1}{\frac{x^2+y^2}{x^2}}(\frac{x\frac{dy}{dx}-y}{x^2})$$

$$\Rightarrow\frac{x+y\frac{dy}{dx}}{(x^2+y^2)} = \frac{x^2(x\frac{dy}{dx}-y)}{(x^2+y^2)x^2}$$

$\Rightarrow$ x + y $\frac{dy}{dx}$ = x $\frac{dy}{dx}$ - y

$\Rightarrow$ (x-y)$\frac{dy}{dx}$ = x + y

$\Rightarrow$ $\boxed{\frac{dy}{dx} = \frac{x+y}{x-y}}$

Example – 36

Find $\frac{dy}{dx}$ if $y^x = x^y$ **[2014-S, 2016-S, 2017-w]**

Ans. Given $y^x = x^y$

Taking logarithm of both sides

$\Rightarrow\ln y^x = \ln x^y$

$\Rightarrow$ x ln y = y ln x

Differentiating both sides w.r.t x, we have

$\Rightarrow$ x $\frac{1}{y}\frac{dy}{dx}$ + 1. ln y = $\frac{dy}{dx}$. ln x + y $\frac{1}{x}$

$\Rightarrow\frac{x}{y}\frac{dy}{dx}$ + ln y = ln x $\frac{dy}{dx}$ + $\frac{y}{x}$

$\Rightarrow(\frac{x}{y}-\ln x)\frac{dy}{dx} = (\frac{y}{x}-\ln y)$

$\Rightarrow (\frac{x-y\ln x}{y})\frac{dy}{dx} = \frac{y-x\ln y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(y-x\ln y)}{x(x-y\ln x)}$

$\therefore \frac{dy}{dx} = \frac{y(y-x\ln y)}{x(x-y\ln x)}$

Example – 37

Find $\frac{dy}{dx}$ if y^2 cot x = x^2 cot y

Ans. y^2 cot x = x^2 cot y

Differentiating both sides w.r.t x,

$\Rightarrow$ 2y $\frac{dy}{dx}$ cot x + y^2(- $cosec^2x$) = 2x cot y + x^2(-$cosec^2y\frac{dy}{dx}$)

$\Rightarrow$ 2y cot x $\frac{dy}{dx}$ - y^2cosec^2x = 2x cot y - $x^2cosec^2y\frac{dy}{dx}$

$\Rightarrow$ (2y cot x + x^2cosec^2y)$\frac{dy}{dx}$ = 2x cot y + y^2cosec^2x

$\Rightarrow \frac{dy}{dx} = \frac{2x\cot y + y^2cosec^2x}{2y\cot x + x^2cosec^2y}$

Example – 38

Find $\frac{dy}{dx}$ if $y^x = x^{siny}$

Ans. $y^x = x^{siny}$

Taking logarithm of both sides

log y^x = log x^{siny}

$\Rightarrow$ *x log y = sin y log x*

Differentiating both sides w.r.t x,

$\Rightarrow$ 1. log y + x $\frac{1}{y}\frac{dy}{dx}$ = cos y $\frac{dy}{dx}$. $\log x + \sin y.\frac{1}{x}$

$\Rightarrow (\frac{x}{y} - \log x\cos y)\frac{dy}{dx} = \frac{\sin y}{x} - logy$

$\Rightarrow (\frac{x-y\log x\cos y}{y})\frac{dy}{dx} = \frac{\sin y - x\log y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(\sin y - x\log y)}{x(x-y\log x\cos y)}$

Differentiation of Infinite series

Example – 39

If y = $x^{x^{x^{\cdots\cdots}}}$, find $\frac{dy}{dx}$

Ans.

y = $x^{x^{x^{\cdots\cdots}}}$

⇒ y = $x^{(x^{x^{\cdots\cdots}})}$ = x^y

Taking logarithm of both sides

⇒ log y = log x^y = y log x

Differentiating both sides

⇒$\frac{1}{y}\frac{dy}{dx}$ = $\frac{dy}{dx}$ log x + y $\frac{1}{x}$

⇒ $(\frac{1}{y} - \log x)\frac{dy}{dx} = \frac{y}{x}$

⇒$\frac{dy}{dx} = \frac{y}{x}\frac{1}{(\frac{1}{y} - \log x)} = \frac{y^2}{x(1 - y\log x)}$

Example – 2

If y = $\sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$

⇒ y = $\sqrt{\sin x + (\sqrt{\sin x + \sqrt{\sin x + \cdots}})}$

⇒ y = $\sqrt{\sin x + y}$

Squaring both sides

⇒y^2 = sin x + y

Differentiating both sides w.r.t x,

⇒ 2y $\frac{dy}{dx}$ = cos x + $\frac{dy}{dx}$

⇒ (2y-1)$\frac{dy}{dx}$ = cos x

⇒ $\boxed{\frac{dy}{dx} = \frac{\cos x}{2y - 1}}$

Miscellaneous examples

Example – 1

Differentiate the following functions w.r.t x

(i) $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

(ii) $|x|$ for $x \neq 0$

(iii) $\tan^{-1} e^{2x}$

(iv) $e^{\tan^{-1} x^2}$

(v) $\tan^{-1} \frac{7\,ax}{a^2-12x^2}$

(vi) $x^{\sqrt{x}}$

(vii) $\log_{10} \sin x + \log_x 10$ x > 0

(viii) $(x^e)^{e^x} + (e^x)^{x^e}$

(ix) x^{x^x}

(x) $\tan^{-1}(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}})$

Ans.

(i) y = $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

$$= \frac{(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$$

$$= \frac{(\sqrt{a+x}+\sqrt{a-x})^2}{(\sqrt{a+x})^2-(\sqrt{a-x})^2}$$

$$= \frac{(a+x)+(a-x)+2\sqrt{a+x}\sqrt{a-x}}{(a+x)-(a-x)}$$

$$= \frac{2a+2\sqrt{(a+x)(a-x)}}{2x}$$

$$= \frac{a+\sqrt{a^2-x^2}}{x}$$

Now $\frac{dy}{dx} = \frac{d}{dx}(\frac{a+\sqrt{a^2-x^2}}{x})$

$$= \frac{x\{0+\frac{1}{2\sqrt{a^2-x^2}}(-2x)\}-(a+\sqrt{a^2-x^2}).1}{x^2}$$

$$= \frac{-\frac{x^2}{\sqrt{a^2-x^2}}-(a+\sqrt{a^2-x^2})}{x^2}$$

$= \frac{-x^2 - a\sqrt{a^2-x^2} - a^2 + x^2}{x^2\sqrt{a^2-x^2}}$

$= \frac{-x^2 - a\sqrt{a^2-x^2} - a^2 + x^2}{x^2\sqrt{a^2-x^2}}$

$= -\frac{(a^2 + a\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$

(ii) y = |x|

When x < 0 , y = |x| = -x

When x > 0 , y = |x| = x

So $\frac{d|x|}{dx} = \frac{d(-x)}{dx}$ = -1 when x < 0

$\frac{d|x|}{dx} = \frac{d(x)}{dx}$ = 1 when x > 0

(iii) y = $\tan^{-1} e^{2x}$

$\frac{dy}{dx} = \frac{1}{1+(e^{2x})^2}\frac{d}{dx}(e^{2x}) = \frac{2\,e^{2x}}{1+e^{4x}}$

(iv) y = $e^{\tan^{-1} x^2}$

$\frac{dy}{dx} = e^{\tan^{-1} x^2}\frac{d}{dx}(\tan^{-1} x^2) = e^{\tan^{-1} x^2}\frac{1}{1+(x^2)^2}\frac{d}{dx}(x^2)$

$= \frac{2\,x\,e^{\tan^{-1} x^2}}{1+x^4}$

(v) y = $\tan^{-1}\frac{7\,ax}{a^2-12x^2} = \tan^{-1}\left(\frac{\frac{7\,ax}{a^2}}{\frac{a^2-12x^2}{a^2}}\right)$

$= \tan^{-1}\left(\frac{\frac{7x}{a}}{1-\frac{12x^2}{a^2}}\right) = \tan^{-1}\left(\frac{\frac{3x}{a}+\frac{4x}{a}}{1-\frac{3x}{a}\cdot\frac{4x}{a}}\right)$

(Putting $\frac{3x}{a} = \tan\theta_1$ & $\frac{4x}{a} = \tan\theta_2$)

$= \tan^{-1}\left(\frac{\tan\theta 1+\tan\theta 2}{1-\tan\theta 1\tan\theta 2}\right) = \tan^{-1}\{\tan(\theta_1+\theta_2)\}$

$= \theta_1 + \theta_2 = \tan^{-1}\frac{3x}{a} + \tan^{-1}\frac{4x}{a}$

Now $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\frac{3x}{a}) + \frac{d}{dx}(\tan^{-1}\frac{4x}{a})$

$= \frac{1}{1+(\frac{3x}{a})^2}\left(\frac{3}{a}\right) + \frac{1}{1+(\frac{4x}{a})^2}\left(\frac{4}{a}\right) = \frac{1}{1+\frac{9x^2}{a^2}}\left(\frac{3}{a}\right) + \frac{1}{1+\frac{16x^2}{a^2}}\left(\frac{4}{a}\right)$

$= \frac{3a^2}{(a^2+9x^2)a} + \frac{4a^2}{(a^2+16x^2)a}$

$= \frac{3a}{a^2+9x^2} + \frac{4a}{a^2+16x^2}$ (Ans)

(vi) y = $x^{\sqrt{x}}$

Taking logarithm of both sides,

ln y = ln $x^{\sqrt{x}}$

⇒ ln y = $\sqrt{x}\ lnx$

Differentiating both sides w.r.t x,

⇒ $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}lnx + \sqrt{x}\frac{1}{x}$

⇒ $\frac{dy}{dx} = y(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}})$

⇒ $\frac{dy}{dx} = x^{\sqrt{x}}\ (\frac{\ln x+2}{2\sqrt{x}})$ (ans)

(vii) y = $\log_{10}\sin x + \log_x 10$

= $\log_{10}\sin x + \frac{1}{\log_{10}x}$ {as $\log_b a = \frac{1}{\log_a b}$}

$\frac{dy}{dx} = \frac{1}{\sin x\ \log_e 10}cosx + \{-\frac{1}{(\log_{10}x)^2}\cdot\frac{1}{x\ \log_e 10}\}$

= $\frac{\cot x}{\log_e 10} - \frac{1}{x(\log_{10}x)^2\log_e 10}$

= $cotx\ \log_{10}e - \frac{\log_{10}e}{x(\log_{10}x)^2}$ (ans)

(viii) y = $(x^e)^{e^x} + (e^x)^{x^e}$

Let y = $y_1 + y_2$ where $y_1 = (x^e)^{e^x}$, $y_2 = (e^x)^{x^e}$

Now, $y_1 = (x^e)^{e^x}$

Taking logarithm of both sides

log y_1 = log $(x^e)^{e^x} = e^x$ log x^e

⇒ log $y_1 = e^x$ e log x = $e^{x+1}logx$

Differentiating w.r.t x we have,

⇒ $\frac{1}{y_1}\frac{dy_1}{dx} = e^{x+1}logx + e^{x+1}\frac{1}{x}$

⇒ $\frac{dy_1}{dx}$ = $y_1\,(e^{x+1}logx + \frac{e^{x+1}}{x})$

= $(x^e)^{e^x}e^{x+1}\ (\ logx + \frac{1}{x})$ -------------- (1)

Again $y_2 = (e^x)^{x^e}$

Taking log of both sides

⇒ ln $y_2 = x^e$ln $e^x = x^e$x = x^{e+1}

Differentiating w.r.t x we have,

⇒$\frac{1}{y_2}\frac{dy_2}{dx}$ = (e+1) x^{e+1-1} = (e+1)x^e

⇒$\frac{dy_2}{dx}$ = y_2(e+1)x^e = $(e^x)^{x^e}$ (e+1)x^e ------------------------ (2)

From (1) and (2)

$\frac{dy}{dx} = \frac{dy1}{dx} + \frac{dy2}{dx}$

$= (x^e)^{e^x} e^{x+1} (\log x + \frac{1}{x}) + (e^x)^{x^e}$ (e+1)x^e (ans)

(ix) y = (x^{x^x})

Taking logarithm of both sides,

ln y = ln x^{x^x} = x^xln x

Differentiating w.r.t x we have,

$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{d(x^x)}{dx} + x^x \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y(lnx \frac{d(x^x)}{dx} + x^{x-1})$

$= x^{x^x}$(ln x $\frac{d(x^x)}{dx}$ + x^{x-1}) --------------- (1)

Now let z = x^x

Taking logarithm both sides,

⇒ log z = ln x^x

$\Rightarrow logz = x\ lnx$

Differentiating w.r.t x we have,

$\Rightarrow \frac{1}{z}\frac{dz}{dx}$ = 1. *ln* x + $x \frac{1}{x}$

$\Rightarrow \frac{dz}{dx}$= *z (ln x + 1)*

⇒ $\boxed{\frac{d(x^x)}{dx} = x^x (\ln x + 1)}$ ---------------(2)

From (1) and (2)

$\frac{dy}{dx}$ = x^{x^x}(ln x x^x (ln x +1)+ x^{x-1})

= x^{x^x}($x^x(\ln x)^2$ + x^x ln x + x^{x-1})

= $x^{x^x} x^{x-1}(x (\ln x)^2 + x \ln x + 1)$ (Ans)

x) y = $\tan^{-1}(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}})$

Put = x^2 = cos θ

Then y = $\tan^{-1}(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}})$

$= \tan^{-1}(\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}})$

$= \tan^{-1}(\frac{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}})$

$$= \tan^{-1}\left(\frac{\frac{cos\frac{\theta}{2}}{cos\frac{\theta}{2}} + \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}}{\frac{cos\frac{\theta}{2}}{cos\frac{\theta}{2}} - \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}}\right) \quad \{\text{dividing numerator and denominator by } \frac{\theta}{2}\}$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2}$$

$$= \frac{\pi}{4}+\frac{\cos^{-1}x^2}{2}$$

Now $\frac{dy}{dx} = 0 + \frac{1}{2}\left(\frac{-1}{\sqrt{1-(x^2)^2}}\right)2x$

$$= -\frac{x}{\sqrt{1-x^4}}$$

Example – 2

If *cos y = x cos (a+y)* then show that $\frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$.

Ans. Given *cos y = x cos (a+y)*

$\Rightarrow x = \frac{\cos y}{\cos(a+y)}$

Differentiate both sides w.r.t *x* we have,

$$\Rightarrow 1 = \frac{\cos(a+y)(-\sin y)\frac{dy}{dx} - \cos y\,(-\sin(a+y))\frac{dy}{dx}}{cos^2(a+y)}$$

$$\Rightarrow 1 = \frac{dy}{dx}\left\{\frac{\sin(a+y)cosy - \cos(a+y)siny}{cos^2(a+y)}\right\}$$

$$\Rightarrow 1 = \frac{dy}{dx}\left\{\frac{\sin(a+y-y)}{cos^2(a+y)}\right\}$$

$\Rightarrow \frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$ (Proved)

Example – 3

Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1-x^2}$

Ans.

Here y = $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$, z = $\sqrt{1-x^2}$

Let x = cos θ

Then y = $\sec^{-1}\left(\frac{1}{2cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x$

$\frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1} x) = \frac{-2}{\sqrt{1-x^2}}$

$\frac{dz}{dx} = \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

Now $\frac{dy}{dz} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \frac{-2}{-x} = \frac{2}{x}$

Example – 4

If y = $10^{\log sinx}$ find $\frac{dy}{dx}$.

Ans.

$\frac{dy}{dx} = \frac{d}{dlogsinx}\left(10^{logsinx}\right).\frac{d}{dx}(logsinx)$

$= 10^{\log sinx} \log_e 10 \frac{d}{dx}(logsinx)$ (As $\frac{d}{dx}((a^x) = a^x \log_e a)$

$= 10^{\log sinx}$ln 10 $\frac{1}{\sin x}\, cosx = ln10\, cotx\, 10^{\log sinx}$

Example – 5

If $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$ then find $\frac{dy}{dx}$

$x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ (Put $t = tan\theta$)

$= \cos^{-1}\left(\frac{1}{\sqrt{1+tan^2\theta}}\right) = \cos^{-1}\left(\frac{1}{\sec\theta}\right)$

$= \cos^{-1}(\cos\theta) = \theta = \tan^{-1} t$

$\Rightarrow \frac{dx}{dt} = \frac{1}{1+t^2}$

Similarly $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}} = \sin^{-1}\left(\frac{tan\theta}{\sqrt{1+tan^2\theta}}\right) = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$

$= \sin^{-1}\left(\frac{\sin\theta}{\cos\theta\sec\theta}\right) = \sin^{-1}(\sin\theta) = \theta = \tan^{-1} t$

$\therefore \frac{dy}{dt} = \frac{1}{1+t^2}$

Hence, $\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{1/_{1+t^2}}{1/_{1+t^2}} = 1$

Exercise

Short Questions (2 marks)

1) Find the slope of the tangent to the curve y = x^2 at x = - $\frac{1}{2}$**[2014-S]**

2) Find the derivative of sin x w.r.t cos x. **[2017-S]**

3) Find the derivative of cos x w.r.t $\log_e x$. **[2015-S]**

4) Find $\frac{dy}{dx}$ if x = at^2 , y = 2at. **[2017-w]**

5) Differentiate $\tan^{-1} x$ w.r.t $\cos^{-1} x$.

6) Differentiate y = $x^{\sin^{-1} x}$, w.r.t. x.

7) Differentiate cosec($\cot^{-1} x$) w.r.t sec ($\tan^{-1} x$)

8) Differentiate $sec^2(\tan^{-1} x)$ w.r.t (1-x^2)

9) Differentiate $\tan^{-1} \sqrt{\frac{1}{x} - 1}$ w.r.t. x.

10) Differentiate$\cot^{-1} x$ w.r.t. $\operatorname{cosec}^{-1} x$

11) Find $\frac{dy}{dx}$ of each of the following

i) $(\tan^{-1} 5x)^2$

ii) $(\sin^{-1} x^4)^4$

iii) $\tan^{-1}(cos\sqrt{x})$

iv) $\log_7(\log_7 x)$

v) $\sin(e^{x^2})$

Long Questions (5 marks)

12) Differentiate $\tan^{-1}(\frac{\cos x - \sin x}{\cos x + \sin x})$ w.r.t x. **[2017-w]**

13) If y = $\tan^{-1} \sqrt{\frac{1+\sin x}{1-\sin x}}$ then find $\frac{dy}{dx}$.

14)Find $\frac{dy}{dx}$ if x = y ln (xy) (2016-S)

15) Find $\frac{dy}{dx}$ if y = $(\tan x)^{\ln x}$ (2017-w)

16) If x $\sqrt{1+y}$ + y$\sqrt{1+x}$ = 0 Prove that (1+x^2) $\frac{dy}{dx}$ + 1 = 0 (for x $\neq y$)

17) If y = log $(\frac{1+\sqrt{x}}{1-\sqrt{x}})$, then find $\frac{dy}{dx}$.

18) If $\cos^{-1}(\frac{x^2-y^2}{x^2+y^2}) = \tan^{-1} a$, Prove that $\frac{dy}{dx} = \frac{y}{x}$.

19) Find $\frac{dy}{dx}$ = ? If $(\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{a})^{\frac{2}{3}}$ = 1.

20) If $e^{x+y} - x$ = 0 prove that $\frac{dy}{dx} = \frac{1-x}{x}$

21) If y = $(\sqrt{x})^{\sqrt{x}^{\sqrt{x}^{\cdots\cdots}}}$ then prove that $\frac{dy}{dx} = \frac{y^2}{2-y\log x}$.

22) Find $\frac{dy}{dx}$ if x = $\frac{2at}{1+t^2}$, y = $\frac{2bt}{1-t^2}$

23) Differentiate $sin^2 x$ w.r.t $(\ln x)^2$

24) Differentiate $\sin^{-1}(\frac{2x}{1+x^2})$ w.r.t $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

25) Differentiate $\tan^{-1} x$ w.r.t $\tan^{-1}\sqrt{1+x^2}$

26) If x = $\frac{a(1-t^2)}{1+t^2}$ and y = at $(\frac{1-t^2}{1+t^2})$ then find $\frac{dy}{dx}$

27) If sin (xy) + $\frac{x}{y}$ = x^2 – y, then find $\frac{dy}{dx}$

28) If $\sqrt{1-x^2} + \sqrt{1-y^2}$ = a(x-y) , prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1-x^2}$

29) Differentiate $\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$ w.r.t. x.

30) If y = log (x+$\sqrt{x^2-1}$) then find $\frac{dy}{dx}$.

31) Find $\frac{dy}{dx}$ of each of the following

(i) $\sin^{-1}(2ax\sqrt{1-a^2x^2})$

(ii) $[\left(\frac{1+t^2}{1-t^2}\right)^2 - 1]^{\frac{1}{2}}$

(iii) $\tan^{-1}(\frac{x}{1+\sqrt{1-x^2}})$

(iv) $x^2\cos^{-1}(\frac{\sqrt{x}-1}{\sqrt{x}+1}) + x^2\operatorname{cosec}^{-1}(\frac{\sqrt{x}+1}{\sqrt{x}-1})$

(v) $\sin^{-1}(3x-4x^3)$

(vi) $\tan^{-1}(\frac{4x}{1-4x^2})$

(vii) $\cos^{-1}(\frac{1-x^2}{1+x^2})$

(viii) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$

ANSWER

(1) -1 **(2)** -cot x **(3)**$-xsinx$ **(4)**$\frac{1}{t}$

(5) $\frac{-\sqrt{1-x^2}}{1+x^2}$ **(6)** $x^{\sin^{-1}x}\,[\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}]$ (7)1 (**8**)-1

(9) - $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ **(10)**$\frac{x\sqrt{x^2-1}}{x^2+1}$

11) (i) $\frac{10\tan^{-1}5x}{1+25x^2}$ (ii)$\frac{16x^3(\sin^{-1}x^4)^3}{\sqrt{1-x^8}}$ (iii) $\frac{\sin\sqrt{x}}{2\sqrt{x}(1+cos^2\sqrt{x})}$

(iv) $\frac{1}{x\,\log_7 x(\log_e 7)^2}$ (v) $2x\,e^{x^2}\cos(e^{x^2})$

12) -1 **(13)** $\frac{1}{2}$ **(14)** $\frac{x-y}{x(1+\ln(xy))}$ **(15)** $(\tan x)^{\ln x}\,[\frac{1}{x}\ln\tan x+\frac{sec^2x\;lnx}{tanx}]$

(17) $\frac{1}{\sqrt{x}(1-\sqrt{x})}$ **(19)** $-(\frac{ay}{bx})^{\frac{1}{3}}$ **(22)** $\frac{b(1+t^2)^3}{a(1-t^2)^3}$ **(23)** $\frac{x\sin 2x}{2\ln x}$

(24) $\frac{-4\sqrt{1-x^2}}{1+x^2}$ **(25)** $\frac{2+x^2}{x\sqrt{1+x^2}}$ **(26)** $\frac{t^4+4t^2-1}{4t}$ **(27)** $\frac{2xy^2-y-y^3\cos(xy)}{xy^2.\cos(xy)-x+y^2}$

(29) $\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$ [2x + $\frac{1}{(1+x^2)\tan^{-1}x}$ - $\frac{x}{(1+x^2)}$] **(30)**$\frac{1}{\sqrt{x^2-1}}$

31) (i) $\frac{2a}{\sqrt{1-a^2x^2}}$ (ii) $\frac{2+2t^2}{(1-t^2)^2}$ (iii) $\frac{1}{2}\frac{1}{\sqrt{1-x^2}}$ (iv) πx

(v) $\frac{3}{\sqrt{1-x^2}}$ (vi) $\frac{4}{1+4x^2}$ (vii) $\frac{2}{1+x^2}$ (viii) - $\frac{4}{(e^x-e^{-x})^2}$

Successive Differentiation

If f is a differential function of x , then the derivative of f(x) may be again differentiable w.r.t x. If f'(x) = $\frac{df}{dx}$ then $f'(x)$is called first derivative of f.

If f'(x) is differentiable , and $\frac{df'(x)}{dx}$ =f''(x), then f ''(x) is called the 2nd order derivative of f (x) w.r.t x.

The above process can be successively continued to obtained derive functions of higher orders.

Notations

1st Order derivatives → $\frac{dy}{dx}$, y' , y_1 , Dy , *f '(x)*

2nd Order derivatives → $\frac{d^2y}{dx^2}$, y'' , y_2 , D^2y, *f ''(x)*

3rd Order derivatives → $\frac{d^3y}{dx^3}$, y''' , y_3 , D^3y , *f '''(x)*

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nth Order derivatives → $\frac{d^ny}{dx^n}$, $y^{(n)}$, y_n , D^ny , f^n(x)

Example – 1

Find 2nd order derivatives of following function.

(i) y = x^5 + 4x^3 - 2x^2 +1 (ii) y = $\log_e x$

(iii) y = $\sqrt{x^2+1}$ (iv) y = $\frac{1}{\sqrt{x}}$

(i) y_1 = $\frac{dy}{dx}$ = $\frac{d}{dx}$(x^5 + 4x^3 - 2x^2 +1) = 5x^4+12x^2-4x+0

= 5x^4+12x^2-4x

y_2 = $\frac{d^2y}{dx^2}$ = $\frac{d}{dx}$ ($\frac{dy}{dx}$) = $\frac{d}{dx}$(5x^4+12x^2-4x)

= 20x^3 + 24x – 4 (Ans)

(ii) y_1 = $\frac{d}{dx}$($\log_e x$) = $\frac{1}{x}$

y_2 = $\frac{dy_1}{dx}$ = $\frac{d(\frac{1}{x})}{dx}$ = - $\frac{1}{x^2}$ (ans)

(iii) y = $\sqrt{x^2+1}$

$y_1 = \frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)$ (Chain Rule)

$= \frac{2x+0}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

$y_2 = \frac{dy_1}{dx} = \frac{d(\frac{x}{\sqrt{x^2+1}})}{dx}$

$= \frac{1.\sqrt{x^2+1} - x \cdot \frac{1}{2\sqrt{x^2+1}}\frac{d(x^2+1)}{dx}}{(\sqrt{x^2+1})^2}$ (applying division formula of derivative)

$= \frac{\sqrt{(x^2+1)} - \frac{x}{2\sqrt{x^2+1}}2x}{x^2+1}$

$= \frac{(x^2+1)-x^2}{\sqrt{x^2+1}\,(x^2+1)} = \frac{1}{(x^2+1)^{3/2}}$ (Ans)

(iv) $y_1 = \frac{d}{dx}(\frac{1}{\sqrt{x}}) = \frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

$y_2 = \frac{dy_1}{dx} = -\frac{1}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1} = \frac{3}{4}x^{-5/2} = \frac{3}{4x^{5/2}}$

Example – 2

Find y_1 and y_2 if y = log(sin x) (2018-S)

Ans.

$y_1 = \frac{d}{dx}\log(\sin x) = \frac{1}{\sin x}cosx = cotx$

$y_2 = \frac{dy_1}{dx} = \frac{d}{dx}(cotx) = -cosec^2x$ (Ans)

Example – 3

If x = at^2 , y = 2at then find $\frac{d^2y}{dx^2}$

Ans.

$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{\frac{d}{dt}(2at)}{\frac{d}{dt}(at^2)} = \frac{2a}{2at} = \frac{1}{t}$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dt}(\frac{dy}{dx}).\frac{dt}{dx}$

$= \frac{\frac{d}{dt}(\frac{1}{t})}{\frac{dx}{dt}} = \frac{-\frac{1}{t^2}}{2at} = -\frac{1}{2at^3}$

Example – 4

If x = a $(\theta - sin\theta)$, y = a (1+cosθ) then find $\frac{d^2y}{dx^2}$

Ans.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-cos\theta)} = - \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2}} = - \cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{d\theta}(\frac{dy}{dx})}{\frac{dx}{d\theta}}$$ {as $\frac{dy}{dx}$ is function of θ}

$$= \frac{\frac{d}{d\theta}(-\cot\frac{\theta}{2})}{\frac{dx}{d\theta}} = \frac{cosec^2\frac{\theta}{2}.\frac{1}{2}}{a(1-cos\theta)} = \frac{1}{2a}\frac{cosec^2\frac{\theta}{2}}{2sin^2\frac{\theta}{2}} = \frac{1}{4a}cosec^2\frac{\theta}{2}.\ cosec^2\frac{\theta}{2}$$

$$= \frac{1}{4a}cosec^4\frac{\theta}{2}$$

Example – 4

Find $\frac{d^2y}{dx^2}$ from the equation $x^2 + y^2 = a^2$

Ans.

Given $x^2 + y^2 = a^2$ --------------------- (1)

Differentiate both sides,

$2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{x}{y}$ ---------------------------------- (2)

Again differentiating w.r.t x

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\{\frac{y.1 - x\frac{dy}{dx}}{y^2}\}$ { applying division formula }

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\{\frac{y - x\,(\frac{-x}{y})}{y.y^2}\}$ {Form (2)}

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\{\frac{y^2+x^2}{y.y^2}\} = -\frac{a^2}{y^3}$ {Form (1)}

Example – 5

If x = 3t - t^3 , y = t + 1 , find $\frac{d^2y}{dx^2}$ at t = 2.

Ans.

Given *y = t + 1* , *x = 3t* - t^3.

$\Rightarrow \frac{dy}{dt}$ = 1 and $\frac{dx}{dt}$ = 3 - 3t^2

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{3-3t^2} = \frac{1}{3(1-t^2)}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{1}{3}\left(\frac{-1}{(1-t^2)^2}\right)(-2t)}{3-3t^2} = \frac{\frac{2}{3}\frac{t}{(1-t^2)^2}}{3(1-t^2)}$

$= \frac{2}{9}\frac{t}{(1-t^2)^3}$

Now $\frac{d^2y}{dx^2}]_{t=2} = \frac{2}{9}\frac{2}{(1-2^2)^3} = \frac{4}{9(-3)^3} = \frac{-4}{243}$

Example – 6

If y = $e^{ax}\sin bx$, then prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0$ **[2017-w]**

Ans.

Given y = $e^{ax}\sin bx$ ------------------ (1)

Differentiate both sides,

$y_1 = ae^{ax}\sin bx + e^{ax}\,b\cos bx$

$\Rightarrow y_1$ = ay + b$e^{ax}\cos bx$ --------------(2)

Differentiate w.r.t x,

$\Rightarrow y_2 = ay_1 + ba\,e^{ax}\cos bx + b\,e^{ax}b(-\sin bx)$

$\Rightarrow y_2 = ay_1 + ab\,e^{ax}\cos bx - b^2y$

$\Rightarrow y_2 = ay_1 + a(y_1 - ay) - b^2y$ {from (2)}

$\Rightarrow y_2 = ay_1 + ay_1 - a^2y - b^2y$

$\Rightarrow$ $\boxed{y_2 - 2ay_1 + (a^2+b^2)y = 0}$ (proved)

Example – 7

If y = $e^{m\cos^{-1}x}$ then show that (1-x^2)$\frac{d^2y}{dx^2}$ - x $\frac{dy}{dx}$ - m^2y = 0 (2018-S)

Ans. y = $e^{m\cos^{-1}x}$

$\Rightarrow \frac{dy}{dx} = e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) = -\frac{my}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}$ = -my ----------------(1)

Differentiate w.r.t x

$\Rightarrow \frac{1(-2x)}{2\sqrt{1-x^2}}\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} = -m\frac{dy}{dx}$

$\Rightarrow \frac{-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}}{\sqrt{1-x^2}} = -m\frac{dy}{dx}$

$\Rightarrow -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = -m\frac{dy}{dx}\sqrt{1-x^2}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ = m (-my) = m^2y{from (1)}

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

Example – 8

If y = ax sin x, then $x^2y_2 - 2xy_1 + (x^2+2)y = 0$ **(2016-S)**

Ans. y = ax sin x --------------------- (1)

Differentiate w.r.t x,

$\Rightarrow y_1$ = a [1. sin x +x. cos x]

$\Rightarrow y_1$ = a (sin x +x cos x) -------------------- (2)

Differentiate w.r.t x,

$\Rightarrow y_2$ = a (cos x + 1. Cos x – x . sin x)

$\Rightarrow y_2$ = 2a cos x – ax sin x --------------------- (3)

Now L.H.S = x^2y_2 - 2x y_1 + (x^2+2)y { applying equation (1),(2) and(3) }

= 2 a x^2 cos x - ax^3sin x – 2ax sin x – 2 a x^2 cos x + ax^3sin x + 2ax sin x

= 0 = R.H.S (Proved)

Exercise

Question with short answers (2marks)

1) Find y_2 for following

(i) y = $x^2 + \sqrt{x}$ (ii) y = $e^x \sin x$

Question with long answers (5 marks)

2) If x = 2 cos t – cos 2t , y = 2 sin t – sin 2t then find y''.

3) Find y_2 y = tan x + sec x

4) If y = $\sin^{-1} x$, then show that (1-x^2) y_2 - x y_1 = 0 **[2017-w]**

5) If y = A cos nx + B sin nx then show that $\frac{d^2y}{dx^2}$ + $n^2 y$ = 0

6) If y = log (x+ $\sqrt{1+x^2}$) , Prove that (1 +x^2) y_2 + x y_1 = 0

Question with long answers (10 marks)

7) If y = sin (m $\sin^{-1} x$) prove that (1-x^2) y_2 - x y_1 + $m^2 y$ = 0

8)If y = $e^{m \sin^{-1} x}$ prove that (1-x^2) y_2 - x y_1 = m^2y (2017-W, 2017-S)

Ans.

1) (i) 2- $\frac{1}{4x^{3/2}}$ (ii) 2 $e^x \cos x$

2) $\frac{3}{8 \sin\frac{t}{2} cos^2\frac{3t}{2}}$ 3) $\frac{\cos x}{(1-\sin x)^2}$

Partial Differentiation

The functions studied so far are of a single independent variable. There are functions which depends on two or more variables. Example, the pressure(P) of a given mass of gas is dependent on its volume(v) and temperature (T).

Functions of two variable

A function f : X x Y to Z is a function of two variables if there exist a unique element z = f(x,y) in Z corresponding to every pair (x,y) in X x Y.

Domain of f is $X \times Y$.

$f(X \times Y)$ is the range of f. { $f(X \times Y)$ C Z }

Notation : - z = f(x,y) means z is a function of two variables x and y.

Limit of a function of two variable

A function f(x,y) tends to limit l as (x,y) → (a,b), .If given $\epsilon > 0$, there exist δ> 0 such that |*f(x,y)-l* | <ϵ whenever o < |(x,y) – (a,b)| <δ .

Continuity

A function f(x,y) is said to be continuous at a point (a,b) if

(i) f(a,b) is defined

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Finding limits and testing continuity of functions of two variable is beyond our syllabus so we have to skip these topics here.

Partial derivatives

Let z = f(x,y) be function of two variables.

If variable x undergoes a chance δx, while y remains constant, then z undergoes a changes written as δz

Now, δz = f(x+ δx,y) – f(x,y)

If $\frac{\delta z}{\delta x}$ exist as δx→ 0, then we write the partial derivative of z w.r.t x as

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x = \lim_{\delta x\to 0} \frac{f(x+\delta x,y)-f(x,y)}{\delta x}$$

Similarly partial derivative of z w.r.t y ,

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = z_y = \lim_{\delta y \to 0} \frac{f(x,y+\partial y) - f(x,y)}{\delta y}$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ symbols are used to notify the partial differentiation.

Note

As from above theory it is clear when partial differentiation w.r.t x is taken , then y is treated as constant and vice – versa. (All the formulae and techniques used in derivative chapter remain same here)

2nd Order Partial Differentiation

If we differentiate the $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ w.r.t x or y , then we set higher order partial derivatives as follows.

1st Order Partial Derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

2nd Order Partial Derivatives

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = z_{xx} = f_{xx}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) = z_{yx} = f_{yx}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) = z_{xy} = f_{xy}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = z_{yy} = f_{yy}$

Note: $f_{yx} = f_{xy}$ *when partial derivatives are continuous .*

Example -1

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

(i) z = $2x^2y + xy^2 + 5xy$.

(ii) z = $\tan^{-1}(\frac{x}{y})$ **[2018-S]**

(iii) z = $e^y \tan x$ **[2019-W]**

(iv) z = $\log(x^2 + y^2)$ **[2015-S]**

(v) z = $\sin^{-1}(\frac{x}{y})$ **[2014-S]**

(vi) z = f($\frac{y}{x}$) **[2017-S]**

(vii) $x^y + y^x$

Ans.

(i) z = $2x^2y$ + xy^2 + 5xy

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(5xy)$ (Here y is treated as constant)

$= 2y\frac{\partial}{\partial x}(x^2) + y^2\frac{\partial}{\partial x}(x) + 5y\frac{\partial}{\partial x}(x)$

$= 2y.2x + y^2.1 + 5y.1$

$= 4xy + y^2 + 5y.$

$\frac{\partial z}{\partial y} = 2x^2\frac{\partial y}{\partial y} + x\frac{\partial y^2}{\partial y} + 5x\frac{\partial y}{\partial y}$

$= 2x^2 + x.2y + 5x = 2x^2 + 2xy + 5x$

(ii) z = $\tan^{-1}(\frac{x}{y})$

$\frac{\delta z}{\delta x} = \frac{1}{1+(\frac{x}{y})^2}\frac{\partial}{\partial x}(\frac{x}{y}) = \frac{1}{\frac{y^2+x^2}{y^2}} \cdot \frac{1}{y}$

$= \frac{y^2}{y(x^2+y^2)} = \frac{y}{x^2+y^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{x}{y})^2}\frac{\partial}{\partial y}(\frac{x}{y}) = \frac{1}{\frac{y^2+x^2}{y^2}}(\frac{-x}{y^2})$

$= \frac{-x}{x^2+y^2}$

(iii) z = $e^y \tan x$

$\frac{\partial z}{\partial x} = e^y\frac{\partial}{\partial x}(\tan x) = \frac{e^y}{1+x^2}$

$\frac{\partial z}{\partial y} = \frac{\partial(e^y)}{\partial x}\tan x = e^y \tan x$

(iv) z = log $(x^2 + y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial x}(x^2 + y^2) = \frac{2x}{x^2+y^2}$ {$\frac{\partial}{\partial x}y^2$= 0 As y is constant}

$\frac{\partial z}{\partial y} = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial y}(x^2 + y^2) = \frac{2y}{x^2+y^2}$

(v) z = $\sin^{-1}(\frac{x}{y})$

$\frac{\delta z}{\delta x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{\partial}{\partial x}(\frac{x}{y}) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}.\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$

$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{\partial}{\partial y}(\frac{x}{y}) = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}.(\frac{-x}{y^2})$

$= -\frac{x}{y\sqrt{y^2-x^2}}$

(vi) z = f($\frac{y}{x}$)

$\frac{\partial z}{\partial x}$ = f ‘ ($\frac{y}{x}$) $\frac{\partial}{\partial x}(\frac{y}{x})$ = f ‘ ($\frac{y}{x}$) . ($\frac{-y}{x^2}$)

= $\frac{-y}{x^2}$ f ‘ ($\frac{y}{x}$)

$\frac{\partial z}{\partial y}$ = f ‘ ($\frac{y}{x}$) $\frac{\partial}{\partial y}(\frac{y}{x})$ = f ‘ ($\frac{y}{x}$). $\frac{1}{x}$

= $\frac{1}{x}$ f ‘ ($\frac{y}{x}$)

vii) z = $x^y + y^x$

$\frac{\partial z}{\partial x}$ = y x^{y-1} + $y^x \ln y$ (y is a constant here)

$\frac{\partial z}{\partial y}$ = x^yln x + x y^{x-1} (As x is treated as constant)

Example-2 . Find f_{xx} and f_{yx} where f(x,y) = $x^3 + y^3 + 3xy$

Ans: - $f_x = 3x^2 + 3y$, $f_y = 3y^2 + 3x$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = 6x + 0 = 6x$

$f_{yx} = \frac{\partial}{\partial y}(f_x) = 0 + 3 = 3$

Example – 3

If z = log $(x^2 + y^2)$ + $\tan^{-1}(\frac{y}{x})$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Ans. $\frac{\partial z}{\partial x} = \frac{1}{(x^2+y^2)}2x + \frac{1}{1+\frac{y^2}{x^2}}(-\frac{y}{x^2})$

$= \frac{2x}{x^2+y^2} + \frac{x^2}{x^2+y^2}(-\frac{y}{x^2}) = \frac{2x-y}{x^2+y^2}$

$\frac{\partial z}{\partial y} = \frac{1}{(x^2+y^2)} 2y + \frac{1}{1+\frac{y^2}{x^2}} (\frac{1}{x})$

$= \frac{2y}{(x^2+y^2)} + \frac{x^2}{(x^2+y^2).x}$

$= \frac{2y+x}{(x^2+y^2)}$

Now $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{(x^2+y^2)(2-0)-(2x-y)(2x+0)}{(x^2+y^2)^2}$

$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = \frac{(x^2+y^2)(2+0)-(2y+x)(0+2y)}{(x^2+y^2)^2}$

$= \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$

Now $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2-2x^2+2xy+2x^2-2y^2-2xy}{(x^2+y^2)^2}$

$= \frac{0}{(x^2+y^2)^2} = 0$ (Proved)

Homogenous function and Euler's theorem

Homogenous function

A function f (x,y) is said to be homogenous in x and y of degree n iff (tx, ty) = $t^n f(x,y)$ where t is any constant.

Example – 4

Test whether the following functions are homogenous or not. If homogenous then find their degree.

(i) $2xy^2 + 3x^2y$ (ii) $\sin^{-1}(\frac{x}{y})$

(iii) $\frac{3x^2+2y^2}{x+y}$ (iv) $x^2 + 2xy + 4x$

Ans.

(i) Let f(x, y) = $2xy^2 + 3x^2y$

f(tx, ty) = $2(tx)(ty)^2 + 3(tx)^2(ty)$

$= 2txt^2y^2 + 3t^2x^2ty$

$= t^3(2xy^2 + 3x^2y) = t^3 f(x,y)$

Hence f(x,y) is a homogenous function of degree 3.

(ii) Let f(x, y) = $\sin^{-1}(\frac{x}{y})$

f(tx, ty) = $\sin^{-1}(\frac{tx}{ty})$ = $\sin^{-1}\frac{x}{y}$ = $t^0 \sin^{-1}(\frac{x}{y})$ = t^0f(x, y)

Hence f(x, y) is a homogenous function of degree '0'.

(iii) f(x, y) = $\frac{3x^2+2y^2}{x+y}$

f(tx, ty) = $\frac{3(tx)^2+2(ty)^2}{tx+ty}$ = $\frac{t^2}{t}\frac{(3x^2+2y^2)}{x+y}$ = t f(x, y)

Hence f (x, y) is a homogenous function of degree 1.

(iv) f(x , y) = $x^2 + 2xy + 4x$

f(tx, ty) = $(tx)^2$ + 2 (tx) (ty) + 4(tx)

= t (tx^2 + 2t xy + 4x)

So here f (tx, ty) cannot be expressed as $t^n f(x, y)$

Hence f(x,y) is not a homogenous function.

Note

(i) If each term in the expression of a function is of the same degree then the function is homogenous.

(ii) If z is a homogenous function of x and y of degree n , then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also homogenous of degree n-1.

(iii) If z = f(x, y) is a homogenous function of degree n , then we can write it as

z = $x^n \Phi(\frac{y}{x})$

e.g. In example - 4(i) $2xy^2 + 3x^2y$ is homogenous function of degree 3.

Now $f(x,y) = 2xy^2 + 3x^2y = x^3(2(\frac{y}{x})^2 + 3(\frac{y}{x})) = x^3\Phi(\frac{y}{x})$

Similarly in Example – 4 (iii), f(x,y) is of degree 1.

Now $f(x, y) = \frac{3x^2+2y^2}{x+y} = \frac{x^2}{x}(\frac{3+2(\frac{y}{x})^2}{1+\frac{y}{x}}) = x(\frac{3+2(\frac{y}{x})^2}{1+\frac{y}{x}}) = x\,\Phi(\frac{y}{x})$

Euler's theorem

If z is a homogenous function of degree n , then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$. **[2014-S]**

Proof: -

Since z is a homogenous function of degree n , so z can be written as

$$z = x^n \Phi(\frac{y}{x})$$

Now $\frac{\partial z}{\partial x}$ = n $x^{n-1}\,\Phi(\frac{y}{x})$ + $x^n\Phi'(\frac{y}{x}).\frac{\partial}{\partial x}(\frac{y}{x})$

= n $x^{n-1}\,\Phi(\frac{y}{x})$ + $x^n\Phi'(\frac{y}{x})$ (- $\frac{y}{x^2}$)

= n $x^{n-1}\,\Phi(\frac{y}{x})$ - x^{n-2} y $\Phi'(\frac{y}{x})$ ------------ (1)

Similarly, $\frac{\partial z}{\partial y}$ = $x^n\Phi'(\frac{y}{x})\frac{\partial}{\partial y}(\frac{y}{x})$

= $x^n\Phi'(\frac{y}{x})$ $(\frac{1}{x})$ = $x^{n-1}\,\Phi(\frac{y}{x})$ ----------- (2)

Now x × Equation (1) + y × Equation (2)

⇒$x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = x {n $x^{n-1}\,\Phi(\frac{y}{x})$ - x^{n-2} y $\Phi'(\frac{y}{x})$ } + y $x^{n-1}\,\Phi'(\frac{y}{x})$

= n $x^n\Phi(\frac{y}{x})$ - $x^{n-1}y\,\Phi'(\frac{y}{x})$ + $x^{n-1}\,y\,\Phi'(\frac{y}{x})$

= n $x^n\Phi(\frac{y}{x})$ = nz (proved)

Example – 5

Verify Euler's theorem for z = $\frac{y}{x}$ **[2014-S]**

Ans. $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\frac{y}{x}) = -\frac{y}{x^2}$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\frac{y}{x}) = \frac{1}{x}$

Here z = f(x , y) = $\frac{y}{x}$

F(tx, ty) = $\frac{ty}{tx} = \frac{y}{x}$ = f(x, y)

Hence f(x, y) is a homogenous function of degree 0.

Statement of Euler's theorem is $x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = n z (here n=0)

=>$x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 0 . Z = 0

Now we have to verify it.

From above

L.H.S = $x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = x (- $\frac{y}{x^2}$) + y . $\frac{1}{x}$ = - $\frac{y}{x}$ + $\frac{y}{x}$ = 0 = R.H.S

Hence Euler's theorem is verified.

Example – 6

Verify Euler's theorem for z = x^2y^2 + 4xy^3 - 3x^3y

Ans. Here z = f(x, y) = x^2y^2 + 4xy^3 - 3x^3y

F (tx, ty) = $t^2x^2t^2y^2$ + 4txt^3y^3 - 3t^3x^3y

= $t^4(x^2y^2 + 4xy^3 - 3x^3y) = t^4$f(x, y)

Hence z is homogenous function of degree 4.

Here n = 4.So, the statement of Euler's theorem is

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4z$

Now we have to verify it

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y^2 + 4xy^3 - 3x^3y)$

= $2xy^2 + 4y^3 - 3(3x^2)y$

= $2xy^2 + 4y^3 - 9x^2y$ ----------------- (1)

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y^2 + 4xy^3 - 3x^3y)$

= $2x^2y + 12xy^2 - 3x^3$ ------------------ (2)

L.H.S = $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

= $x(2xy^2 + 4y^3 - 9x^2y) + y(2x^2y + 12xy^2 - 3x^3)$ {from (1) and (2)}

= $2x^2y^2 + 4xy^3 - 9x^3y + 2x^2y^2 + 12xy^3 - 3x^3y$

= $4x^2y^2 + 16xy^3 - 12x^3y$

= $4(x^2y^2 + 4xy^3 - 3x^3y)$ = 4z (verified)

Example – 7

If z = $\sin^{-1}(\frac{xy}{x+y})$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$ **[2017-S, 2018-S, 2019-W]**

Ans.

Let z = $\sin^{-1}(\frac{xy}{x+y}) = \sin^{-1}u$

Now u = $\frac{xy}{x+y}$

u = (tx , ty) = $\frac{txty}{tx+ty}$ = $\frac{t^2}{t}(\frac{xy}{x+y})$ = tu

Hence u is homogenous function of degree 1.

So by Euler's theorem

x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 1 . u = u ----------- (1)

As z = $\sin^{-1} u$

⇒ u = sin z

$\frac{\partial u}{\partial x}$ = $\frac{\partial}{\partial x}$(sin z) = cos z $\frac{\partial z}{\partial x}$ ------------ (2)

And $\frac{\partial u}{\partial y}$ = $\frac{\partial}{\partial y}$(sin z) = cos z $\frac{\partial z}{\partial y}$ ---------- (3)

From (1), (2) and (3)

⇒ x cos z $\frac{\partial z}{\partial x}$ + y cos z $\frac{\partial z}{\partial y}$ = sin z

⇒$x \frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = $\frac{\sin z}{\cos z}$ = tan z (Proved)

Example – 8

If u = $\sin^{-1}(\frac{x}{y})$ + $\tan^{-1}(\frac{y}{x})$ show that x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 0

Ans.

If u = $\sin^{-1}(\frac{x}{y})$ + $\tan^{-1}(\frac{y}{x})$

u (tx, ty) = $\sin^{-1}(\frac{tx}{ty})$ + $\tan^{-1}(\frac{ty}{tx})$

= $\sin^{-1}(\frac{x}{y})$ + $\tan^{-1}(\frac{y}{x})$

= u (x , y)

Hence u is a homogenous function of degree '0'

So by Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

Example – 9

If z = $\tan^{-1}(\frac{x^3+y^3}{x+y})$, show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = sin 2z **[2017-w]**

Ans. Let z = $\tan^{-1}$, where u = $(\frac{x^3+y^3}{x+y})$

Now u (tx, ty) = $\frac{t^3x^3+t^3y^3}{tx+ty}$ = t^2 $(\frac{x^3+y^3}{x+y})$ = t^2u

Hence u is a homogenous function of degree 2.

So by Euler's theorem

$x\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 2u ----------- (1)

Now z = $\tan^{-1}u$

⇒ u = tan z -------------------------------- (2)

$\frac{\partial u}{\partial x}$ = $\frac{\partial}{\partial x}(\tan z)$ = $sec^2z\frac{\partial z}{\partial x}$ -------------- (3)

And $\frac{\partial u}{\partial y}$ = $\frac{\partial}{\partial y}(\tan z)$ = $sec^2z\frac{\partial z}{\partial y}$ ---------(4)

From (1), (2), (3) and (4)

⇒$x\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 2u

⇒$xsec^2z\frac{\partial z}{\partial x}$ + y $sec^2z\frac{\partial z}{\partial y}$ = 2 tan z

⇒$x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 2 $\frac{\tan z}{sec^2z}$ = 2 tan z cos^2z

= 2 $\frac{\sin z}{\cos z}cos^2z$ = 2 sin z cos z

= sin 2 z (proved)

Example – 10

If z is a homogenous function of x and y of degree n and $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are continuous , then show that $x^2\frac{\partial^2 z}{\partial x^2}$ + 2xy $\frac{\partial^2 z}{\partial x\partial y}$ + $y^2\frac{\partial^2 z}{\partial y^2}$ = n(n-1)z

Proof

Given z is a homogenous function of degree n

So by Euler's theorem

$x\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = nz ------------ (1)

Differentiating (1) w.r.t x ,

1. $\frac{\partial z}{\partial x}$ + $x\frac{\partial^2 z}{\partial x^2}$ + y $\frac{\partial^2 z}{\partial x \partial y}$ = n $\frac{\partial z}{\partial x}$

⇒$x\frac{\partial^2 z}{\partial x^2}$ + y $\frac{\partial^2 z}{\partial x \partial y}$ = (n-1) $\frac{\partial z}{\partial x}$ ----------- (2)

Differentiating (1) w.r.t y ,

$x\frac{\partial^2 z}{\partial y \partial x}$ + 1. $\frac{\partial z}{\partial y}$ + y $\frac{\partial^2 z}{\partial y^2}$ = n $\frac{\partial z}{\partial y}$

⇒$x\frac{\partial^2 z}{\partial x \partial y}$ + y $\frac{\partial^2 z}{\partial y^2}$ = (n-1)$\frac{\partial z}{\partial y}$ ------------ (3)

{As $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are continuous}

{⇒$\frac{\partial^2 z}{\partial x \partial y}$ = $\frac{\partial^2 z}{\partial y \partial x}$ }

Equ^n(2) X x + Equ^n(3) X y

⇒$x^2\frac{\partial^2 z}{\partial x^2}$ + xy $\frac{\partial^2 z}{\partial x \partial y}$ + xy $\frac{\partial^2 z}{\partial x \partial y}$ + $y^2\frac{\partial^2 z}{\partial y^2}$ = x(n-1) $\frac{\partial z}{\partial x}$ + y (n-1)$\frac{\partial z}{\partial y}$

⇒$x^2\frac{\partial^2 z}{\partial x^2}$ + 2xy $\frac{\partial^2 z}{\partial x \partial y}$ + $y^2\frac{\partial^2 z}{\partial y^2}$ = (n-1) { x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ }

= (n-1) nz

= n (n-1)z (proved)

Exercise

Question with short answers (2 marks)

1) if z = sin $\frac{x}{y}$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

2) If f(x,y) = $\sqrt{x^2+y^2}$, find f_x , f_y .

3) If f(x,y) = log (x^2+y^2- 2xy) find f_{xx} , f_{yx} , f_{xy}

4) If z = f(x,y) , then find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

5) Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if z = $xe^y + ye^x$

Questions with long answers (5 marks)

6) Given f(u,v) = $\frac{2u-3v}{u^2+v^2}$, find f_u (2,1) and f_v(2,1)

7) If z = $\frac{x-y}{x+y}$, then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 0

8) If z = x^2y + 3xy^2 - $\frac{x}{y}$. Find partial derivatives of 2nd order.

9) Verify Euler's theorem for u = $x^2 \log(\frac{y}{x})$

10) If z = xy f ($\frac{y}{x}$) , then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 2z

11) If u = $\sin^{-1}(\frac{x+y}{\sqrt{x}+\sqrt{y}})$ show that x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = $\frac{1}{2}$ tan u

12) If z = $\ln(\frac{x^2+y^2}{x+y})$ then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 1

13) If z = $\cos^{-1}(\frac{x^2+y^2}{x+y})$, then show that x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = - cot z

Answers

1) $\frac{1}{y}$ cos ($\frac{x}{y}$) , $\frac{-x}{y^2}$ cos ($\frac{x}{y}$) 2) $\frac{x}{\sqrt{x^2+y^2}}$, $\frac{y}{\sqrt{x^2+y^2}}$ 3) $\frac{-2}{(x-y)^2}$, $\frac{2}{(x-y)^2}$, $\frac{2}{(x-y)^2}$

4) y f'(xy) , xf'(xy) 5) e^y + y e^x , xe^y + e^x 6) $\frac{6}{25}$, $\frac{-17}{25}$

8) Z_{xx} =2y , Z_{yx} = 2x + 6y+ $\frac{1}{y^2}$, Z_{xy} = 2x + 6y+ $\frac{1}{y^2}$, Z_{yy} = 6x - $\frac{2x}{y^3}$.

INTEGRATION

Introduction

Calculus deals with some important geometrical problem related to draw a tangent of a curve and determine area of a region under a curve. In order to solve these problems we use differentiation and integration respectively.

In the previous lesson, we have studied derivative of a function. After studying differentiation it is natural to study the inverse process called integration.

Objectives

After completion of this topic you will able to

1. Explain integration as inverse process of differentiation.

2. State types of integration.

3. State integral of some standard functions like x^n , sinx, cosx,.... $sin^{-1}x$,...... a^x etc.

4. State properties of integration.

5. Find integration of algebraic, trigonometric, inverse trigonometric functions using standard integration formulae.

6. Evaluate different integrals by applying substitution method and integration by parts method.

Expected Background Knowledge

1. Trigonometry

2. Derivative

Integration (Primitive or Anti derivative)

Integration is the reverse process of differentiation.

If $\frac{df(x)}{dx}$= g(x), then the integration of g(x) w.r.t x is $\int g(x)dx = f(x) + c$

- ➔ The Symbol $\int$. is used to denote the operation of integration called as Integral sign.
- ➔ The function (here g(x)) is called the integrand.
- ➔ 'dx' denote that the Integration is to be performed w.r.t x (x is the variable of Integration).
- ➔ 'c' is the constant of Integration (which gives family of curves)
- ➔ Integrate means to find the Integral of the function and the process is known as Integration

Types of Integration

Integration are of two types:- i) Indefinite ii) definite

The integration written in the form $\int g(x)dx$ is called indefinite integral.

The integration written in the form $\int_a^b g(x)dx$ is called definite integral.

In this chapter we only discuss the indefinite integrals. The definite integrals will be discussed in the next chapter.

Algebra of Integrals

i. $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int f(x)dx \pm \int g(x)dx$

ii. $\int \lambda f(x)dx = \lambda \int f(x)dx$ for any constant λ.

iii. $\frac{d}{dx}(\lambda \int f(x)dx) = \lambda\frac{d}{dx}(\int f(x)dx) = \lambda f(x)$

Simple Integration Formula of some standard functions

i) $\int k$ dx = kx + c

ii) $\int x^n$ dx = $\frac{x^{n+1}}{n+1}$ + c, n≠-1

iii) $\int \frac{1}{x}$dx =ln|x| + c

iv) $\int a^x dx = \frac{a^x}{lna} + c$

v) $\int e^x dx = e^x + c$

vi) $\int \sin x dx = -\cos x + c$

vii) $\int \cos x dx = \sin x + c$

viii) $\int sec^2 xdx = tanx + c$

ix) $\int cosec^2xdx = -cotx + c$

x) $\int secxtanxdx = secx + c$

xi) $\int \operatorname{cosec} x \cot x dx = -cosecx + c$

xii) $\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + c$

xiii) $\int \frac{-1}{\sqrt{1-x^2}}dx = cos^{-1}x + c$

xiv) $\int \frac{1}{1+x^2} dx = tan^{-1}x + c$ xv) $\int \frac{-1}{1+x^2} dx = cot^{-1}x + c$

xvi) $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1} x + c$ xvii) $\int \frac{-1}{x\sqrt{x^2-1}} dx = cosec^{-1}x + c$

Methods of integration

1. Integration by using standard formula.
2. Integration by substitution.
3. Integration by parts.

1. INTEGRATION BY USING FORMULAS:-

Example -1 Evaluate the following

(i) $\int (5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x})\, dx$

Ans :- $\int (5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x})\, dx$

$= 5\int x^3 dx + 2\int x^5 dx - 7\int x dx + \int x^{-1/2} dx + 5\int \frac{dx}{x}$ { by algebra of integration}

$= 5\, x\frac{x^{3+1}}{3+1} + 2 \times \frac{x^{5+1}}{5+1} - 7 \times \frac{x^{1+1}}{1+1} + \frac{x^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + 5 \ln|x| + c$

$= 5\, x\frac{x^4}{4} + 2 \times \frac{x^6}{6} - 7 \times \frac{x^2}{2} + \frac{x^{1/2}}{\frac{1}{2}} + 5 \ln|x| + c$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2x^{1/2} + 5lnx + c$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7x^2}{2} + 2\sqrt{x} + 5lnx + c$

(ii) $\int (\frac{3x^4 - 5x^3 + 4x^2 - x + 2}{x^3})\, dx$

Ans :- $\int (\frac{3x^4 - 5x^3 + 4x^2 - x + 2}{x^3})\, dx$

$= \int \frac{3x^4}{x^3} dx - \int \frac{5x^3}{x^3} dx + \int \frac{4x^2}{x^3} dx - \int \frac{x}{x^3} dx + \int \frac{2}{x^3} dx$

$= 3\int x dx - 5\int dx + 4\int \frac{dx}{x} - \int x^{-2} dx + 2\int x^{-3} dxdx$

$= 3\, x\frac{x^{1+1}}{1+1} - 5x + 4lnx - \frac{x^{-2+1}}{-2+1} + \frac{2x^{-3+1}}{-3+1} + c$

$= 3\, x\frac{x^2}{2} - 5x + 4lnx + \frac{1}{x} - \frac{1}{x^2} + c$

$= \frac{3x^2}{2} - 5x + 4lnx + \frac{1}{x} - \frac{1}{x^2} + c$

(iii) $\int(4\cos x - 3e^{x} + \frac{2}{\sqrt{1-x^2}})$dx

Ans :- $\int(4\cos x - 3e^{x} + \frac{2}{\sqrt{1-x^2}})$dx

= $4\int \cos x dx - 3\int e^{x}\, dx + 2\int \frac{dx}{\sqrt{1-x^2}}$

= $4\sin x - 3e^{x}$ $+2sin^{-1}x + c$

(iv) $\int 6x^3\,(x+5)^2 dx$

Ans :- $\int 6x^3\,(x+5)^2 dx$

= $\int 6x^3(x^2 + 10x + 25)dx$

= $\int(6x^5 + 60x^4 + 150x^3)dx$

= $6\int x^5\, dx + 60\int x^4\, dx + 150\int x^3\, dx$

= $6 \times \frac{x^6}{6} + 60 \times \frac{x^5}{5} + 150\ \times \frac{x^4}{4} + c$

= $x^6 + 12x^5 + \frac{75}{2}x^4 + c$

(v) $\int 5tan^2xdx$

= $\int 5tan^2xdx$ = $\int 5(sec^2x - 1)dx$

= $5\int sec^2xdx - 5\ \int 1.dx$

= 5tanx – 5x + c

(vi) $\int sin^2\frac{x}{2}dx$

Ans :- $\int sin^2\frac{x}{2}dx$

= $\int(\frac{1-cosx}{2})dx$= $\frac{1}{2}[\int dx - \ \int cosxdx]$ { 1- cosx = 2 $sin^2\frac{x}{2}$ }

= $\frac{1}{2}[x - sinx]$+ c

(vii) $\int\frac{sinx}{1+sinx}$ dx

Ans :- $\int\frac{sinx}{1+sinx}$ dx

=$\int\frac{(1-sinx)sinx}{(1+sinx)(1-sinx)}dx$ = $\int\frac{(1-sinx)sinx}{1-sin^2x}\ dx$

= $\int(\frac{\sin x - sin^2x}{cos^2x})\ dx$ = $\int\frac{sinx}{cos^2\ x}dx-\int tan^2$xdx

$= \int \frac{sinx}{cosx} \cdot \frac{1}{cosx} dx - \int (sec^2x - 1)dx$

$= \int tanx.secxdx - \int sec^2xdx + \int dx$

$= secx - tanx + x + c$

(viii) $\int \frac{1}{sin^2x.cos^2x} dx$

Ans :- $\int \frac{1}{sin^2x.cos^2x} dx$

$= \int \frac{sin^2x+cos^2x}{sin^2x\,.cos^2x} dx$

$= \int \frac{sin^2x}{sin^2xcos^2x} dx + \int \frac{cos^2x}{sin^2xcos^2x} dx$

$= \int sec^2\,xdx + \int cosec^2xdx$

$= \tan x - cotx + c$

(ix) $\int tan^{-1}\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\}dx$

Ans :- $\int tan^{-1}\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\}dx$ $(\because 1 - cos2x = 2sin^2x \text{ and } 1 + cox2x = 2cos^2x)$

$= \int tan^{-1}\{\sqrt{\frac{2sin^2x}{2cos^2x}}\}\, dx = \int tan^{-1}(\sqrt{tan^2x})dx$

$= \int tan^{-1}(tanx)dx \quad (\because tan^{-1}(tanx) = x)$

$= \int xdx = \frac{x^2}{2} + c$

(x) $\int \frac{secx}{secx+tanx} dx$

Ans:- $\int \frac{secx}{secx+tanx} dx$

$= \int \frac{secx(secx-tanx)}{(secx+tanx)(secx-tanx)} dx = \int \frac{sec^2x-secxtanx}{sec^2x-\,tan^2x} dx \quad \{sec^2x - \; tan^2x=1\}$

$= \int sec^2x\,dx - \int secx\,tanx\,dx$ = tanx – secx + c

(xi) $\int a^x e^x dx$

Ans:- $\int a^x e^x dx = \int (ae)^x dx$ { we know $\int a^x dx = \frac{a^x}{lna}$ here ae is in place of a}

$= \frac{(ae)^x}{\ln(ae)} + c = \frac{a^x e^x}{\ln(ae)} + c$ (Ans)

(xii) $\int \sqrt{1+cos2x}\,dx$ **(2017-S , 2018-S)**

Ans: - $\int \sqrt{1+cos2x}\,dx$ **=** $\int \sqrt{2cos^2x}\,dx$

$= \int \sqrt{2}cosx$dx

= $\sqrt{2}$ sinx + c

2. INTEGRATION BY SUBSTITUTION:-

When the integral $\int f(x)dx$ cannot be determined by the standard formulae then we may reduce it to another form by changing the independent variable 'x' by another variable t (as x=ɸ(t)) which can be integrated easily. This is called substitution method.

$\int f(x)dx = \int f(x)\frac{dx}{dt}dt \quad = \int f[\phi(t)]\phi'(t)dt,$ where x=ɸ(t).

The substitution x=ɸ(t) depends upon the nature of the given integral and has to be properly chosen so that integration is easier after substitution. The following types of substitution are very often used in Integrations.

TYPE – I

$\int f(ax+b)dx$

$Put\ ax+b\ =\ t$

$adx = dt$

$=> dx=\frac{1}{a}dt$

$\therefore \int f(ax+b)dx\ = \int f(t)\frac{1}{a}dt\ =\ \frac{1}{a}\int f(t)dt$

TYPE – II

$\int x^{n-1}f(x^n)dx$

Put $x^n\ = t$

$nx^{n-1}dx = dt$

$=>x^{n-1}dx = \frac{dt}{n}$

$\therefore \quad \int x^{n-1}f(x^n)dx\ = \int f(t)\frac{dt}{n} = \frac{1}{n}\int f(t)dt$

TYPE – III

$\int \{f(x)\}^n . f'(x) dx$

Put f(x)=t

Differentiate both sites w.r.t *x,*

$f'(x) = \frac{dt}{dx}$

=>$\int \{f(x)\}^n . f'(x) dx = \int t^n \, dt = \frac{t^{n+1}}{n+1} + c$

$= \frac{[f(x)]^{n+1}}{n+1} + c \quad (\because t = f(x))$

TYPE-IV

$\int \frac{f^1(x)}{f(x)} dx$

Put f(x)=t

=>$f^1(x) dx = dt$

$\therefore \int \frac{f^1(x)}{f(x)} dx = \int \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c \; (\because f(x) = t)$

SOME USE FULL RESULTS

1. $\int \frac{dx}{ax+b}$

Ans :- Put ax+b = t

Differentiate both sites w.r.t x,

a = $\frac{dt}{dx}$

=> dx = $\frac{dt}{a}$

$\therefore \int \frac{dx}{ax+b} = \int \frac{dt/a}{t} = \frac{1}{a} \int \frac{dt}{t} = \frac{1}{a} \ln|t| = \frac{1}{a} \ln|ax+b| + c.$

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + c$$

2. $\int cotxdx$

Ans:- $\int cotxdx$

= $\int \frac{cosx}{sinx}dx$

Put $sinx = t$

Differentiate both sites w.r.t x,

$$cosx = \frac{dt}{dx}$$

=> $dt = cosxdx$

$\therefore \int \frac{cosxdx}{sinx} = \int \frac{dt}{t} = \ln|t| = \ln|sinx| + c$

$$\boxed{\int cotxdx = \ln|sinx| + c}$$

3. $\int tanxdx$

Ans :- $\int tanxdx$

= $\int \frac{secxtanx}{secx}$dx (multiply & divide by sec x)

Put secx = t

Differentiate both sites w.r.t x.

sec x tan x = $\frac{dt}{dx}$

=> $secx\ tanx\ dx = dt$

$\int \frac{\sec x \tan x}{\sec x} dx = \int \frac{dt}{t} = \ln|t| = \ln|secx| + c$

$$\boxed{\int tanxdx = \ln|secx| + c}$$

4. $\int cosecxdx$

Ans :- $\int cosecxdx$

= $\int \frac{1}{sinx}dx$

= $\int \frac{1}{2sin\frac{x}{2}\cos\frac{x}{2}}$ dx

Divide numerator & denominator by $cos^2 x/2$

$$\int \frac{\frac{1}{cos^2 x/2}}{\frac{2sin\frac{x}{2}cos\frac{x}{2}}{cos^2 x/2}} dx = \int \frac{sec^2 x/2}{2tan x/2} dx$$

Let $\tan\frac{x}{2} = t$

$\Rightarrow sec^2 x/2 \times \frac{1}{2} dx = dt$

$\Rightarrow sec^2 x/2\, dx = 2dt$

$= \int \frac{sec^2 x/2}{2tan\frac{x}{2}} dx$

$= \int \frac{2dt}{2t} = \int \frac{dt}{t} = \ln|t| + c$

$= ln\left|tan\frac{x}{2}\right| + c$

$$\boxed{\int cosecx dx = \ln|tan\frac{x}{2}| + c}$$

Now $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2sin^2\frac{x}{2}}{sinx} = \frac{1-cosx}{sinx}$ = cosecx – cotx

Hence $\int cosecxdx$= ln I cosecx-cotx I + c

5. $\int secxdx$

Ans:-$\int secxdx$

$= \int cosec(\frac{\pi}{2} + x)dx \quad (\because\ cosec\left(\pi/2 + x\right) = secx)$

$= \ln|\tan(\frac{\pi}{2} + \frac{x}{2})| + c \quad (\because \int cosecxdx = \ln\left|tan\frac{x}{2}\right| + c)$

$$\boxed{\int secxdx = \ln|[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)]| + c}$$

As $\tan(\frac{\pi}{4} + \frac{x}{2})$ = secx+ tanx (we can easily verify it by applying trigonometric formulae.

Hence **$\int secxdx$ = ln I secx+tanx I + c**

BY APPLYING ABOVE FORMULA WE OBTAIN FOLLOWING

1. $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$$

Proof : -$\int \cos(ax+b)dx$

Put ax+b=θ

Differentiate both sites w.r.t x.

a = $\frac{d\theta}{dx}$

$=> adx = d\theta => dx = \frac{d\theta}{a}$

$\therefore \int \cos(ax+b)\,dx = \int \cos\theta \times \frac{d\theta}{a}$

$= \frac{1}{a}\int cos\theta d\theta = \frac{1}{a}\, sin\theta + c$

$= \frac{1}{a}\sin(ax+b)+c$

Similarly we can get the following results.

2. $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$$

3. $$\int sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$$

4. $$\int cosec^2(ax+b)dx = -\frac{1}{a}cot(ax+b)$$

5. $$\int \sec(\mathrm{ax+b})\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c$$

6. $$\int cosec(ax+b)\cot(ax+b)dx = \frac{-1}{a}cosec\,(ax+b)+c$$

7. $$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$$

8. $\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a} sin^{-1}(ax+b) + c = -\frac{1}{a} cos^{-1}(ax+b) + c$

9. $\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a} tan^{-1}(ax+b) + c = -\frac{1}{a} cot^{-1}(ax+b) + c$

10. $\int (ax+b)^n dx, n \neq -1 = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

11. $\int \frac{dx}{(ax+b)\sqrt{(ax+b)^2-1}} = \frac{1}{a} sec^{-1}(ax+b) + c$ = - $\frac{1}{a} cosec^{-1}(ax+b)$ + c

12. $\int a^{mx+b} dx = \frac{1}{m}\frac{a^{mx+b}}{lna}$+ c

The above results of substitution may be used directly to solve different integration problem.

Example – 2 integrate the following

(i) $\int \boldsymbol{xsinx^2dx}$

Ans :- $\int xsinx^2dx$ { Let $x^2 = t$ then $2x = \frac{dt}{dx}$ =>$2xdx = dt$}

=$\int sint\frac{dt}{2}$=$\frac{1}{2}\int sintdt$ =$\frac{-1}{2}$ $cost + c$ = $\frac{-1}{2}cosx^2 + c$ (ans)

(ii) $\int \boldsymbol{(x-2)\sqrt{(x^2-4x+7)dx}}$

Ans :- $\int (x-2)\sqrt{(x^2-4x+7)dx}$

Let $x^2 - 4x + 7 = t^2$

Differentiate both sides w.r.t x

$2x - 4 = 2t\frac{dt}{dx}$

=> $(2x-4)dx = 2tdt$

=> $2(x-2)dx = 2tdt$

=> $(x-2)dx = tdt$

Now $\int (x-2)\sqrt{x^2-4x+7}\,dx =$ $\int \sqrt{t^2}\,tdt$

$= \int t \times t\,dt = \int t^2 dt = \frac{t^3}{3} + c$

$= \frac{(x^2-4x+7)^{3/2}}{3} + c$ $\left(\because\ t = \sqrt{x^2-4x+7} = (x^2-4x+7)^{1/2}\right).$

(iii) $\int (\mathbf{3x+5})^{7}\,\boldsymbol{dx}$

Ans: $\int (3x+5)^7 dx$

Put 3x+5 =t

Differentiate w.r.t x

$3 = \frac{dt}{dx} \Rightarrow 3dx = dt \Rightarrow dx = \frac{1}{3}dt$

$\therefore \int (3x+5)^7 dx =$ $\frac{1}{3}\int t^7\,dt = \frac{1}{3} \times \frac{t^8}{8} + c$

$=$ $\frac{1}{3} \times \frac{(3x+5)^8}{8} + c = \frac{(3x+5)^8}{24} + c$

(iv) $\int \frac{\mathbf{x^4+4x^3}}{\mathbf{x^5+5x^4+7}}\,\boldsymbol{dx}$

Ans :- $\int \frac{x^4+4x^3}{x^5+5x^4+7}dx$

Put $x^5+5x^4+7 = t$

Differentiate w.r.t x

$5x^4 + 20x^3 = \frac{dt}{dx}$

=> $(5x^4+20x^3)dx = dt$

=> $5(x^4+4x^3)\,dx = dt$

=> $(x^4+4x^3)dx = \frac{dt}{5}$

$\therefore \int \frac{x^4+4x^3}{x^5+5x^4+7}dx = \frac{1}{5}\int \frac{dt}{t} = \frac{1}{5}\ln|t| + c$

$= \frac{1}{5}\ln\left|x^5 + 5x^4 + 7\right| + c$

(v) $\int \boldsymbol{sin^7x\, cosx\, dx}$

Ans :- $\int sin^7x\, cosx\, dx$

Put $sinx = \theta$

Differentiating both sides w.r.t x

$cosx = \frac{d\theta}{dx}$

=> $cosxdx = d\theta$

$\therefore$ $\int sin^7 xcosx$ dx = $\int \theta^7\, d\theta = \frac{\theta^8}{8} + c$

= $\frac{sin^8x}{8} + c$

(vi) $\int \boldsymbol{2e^{tan^2x}\, tanx\, sec^2x\, dx}$

Ans :- $\int 2e^{tan^2x}\, tanx\, sec^2x\, dx$

Put $tan^2x = \theta$

Differentiating both sides w.r.t x,

$2\, tanx.sec^2x = \frac{d\theta}{dx}$

=> $2\, tanx\, sec^2xdx = d\theta$

$\therefore$ $\int 2e^{tan^2x}\, tanx\, sec^2x\, dx$ = $\int e^{\theta} d\theta$ = $e^{\theta} + c$

= $e^{tan^2x} + c$

(vii) $\int \frac{3(lnx)^2}{x} dx$

Ans :- $\int \frac{3(lnx)^2}{x} dx$

Put $lnx = t$ =>$\frac{1}{x} = \frac{dt}{dx}$ => $\frac{dx}{x} = dt$

$\therefore$ $\int \frac{3(lnx)^2}{x} dx = 3\int t^2 dt = 3 \times \frac{t^3}{3} + c$ = $(lnx)^3 + c$

viii) Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ (2017-S)

Ans:-$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ (Let t = $e^x + e^{-x}$ => dt = $(e^x - e^{-x})dx$

= $\int \frac{dt}{t}$ = $\ln |t| + c$ = $\ln |e^x + e^{-x}| + c$

ix) Integrate $\int \frac{1}{2-5x} dx$ (2015-S)

Ans:-$\int \frac{1}{2-5x} dx$ (Let 2 - 5x = t => - 5 dx = dt => dx = - $\frac{dt}{5}$)

$= -\frac{1}{5}\int \frac{1}{t} dt = -\frac{1}{5}\, lnt + c = -\frac{1}{5}\ln(2-5x) + c$.

x) Evaluate $\int e^x sin\, e^x dx$ (2019-W)

Ans:-$\int e^x sin\, e^x dx$ (Put $e^x = t => e^x dx = dt$)

$= \int sintdt = -cost + c = -cose^x + c$

INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS

If the integrand is of the form sinmx cosnx ,sinmx sinnx or cosmx cosnx,a trigonometric transformation will help to reduce.it to the sum of sines or cosines of multiple angles which can be easily integrated.

sinmxcosnx $= \frac{1}{2} \times 2sinmx\, cosnx$

$= \frac{1}{2}[\sin(m+n)\,x + \sin(m-n)x]$

sinmx sinnx= $\frac{1}{2}[\cos(m-n)\,x - \cos(m+n)x]$

cosmx cosnx= $\frac{1}{2}[\cos(m-n)\,x + \cos(m+n)x]$

Example – 3

i)Evaluate $\int sin3x\, cos2x\, dx$

Ans :- $\int sin3x\, cos2x\, dx$

$=\frac{1}{2}\int \sin(3x+2x) + \sin(3x-2x)\, dx$

$= \frac{1}{2}\int (sin5x + sinx)dx$

$= \frac{1}{2}\int sin5xdx + \frac{1}{2}\int sinxdx$

$= \frac{1}{2} \times \frac{-cos5x}{5} + \frac{1}{2}(-cosx)$ +c

$= \frac{-1}{10}cos5x - \frac{1}{2}cosx + c$

$= \frac{-1}{10}(cos5x - 5cosx) + c$

ii) Evaluate $\int sin2x\ sinx\ dx$

Ans :- $\int sin2x\ sinx\ dx$

= $\frac{1}{2}\int \cos(2x-x)-\cos(2x+x)dx$

= $\frac{1}{2}\int (cosx-cos3x)dx$

= $\frac{1}{2}\int cosxdx-\frac{1}{2}\int cos3xdx$

= $\frac{1}{2}\times sinx-\frac{1}{2}\times\frac{sin3x}{3}+c$

= $\frac{1}{2}sinx-\frac{1}{6}sin3x+c=\ \frac{1}{6}(3sinx-sin3x)+c$

iii) Evaluate $\int cos4x\ cos3x\ dx$

Ans :- $\int cos4x\ cos3x\ dx$

=$\frac{1}{2}\int (\cos(4x-3x)+\cos(4x+3x)dx\ )$

= $\frac{1}{2}\int (cosx+cos7x)dx$

= $\frac{1}{2}\int cosxdx+\frac{1}{2}\int cos7x$

= $\frac{1}{2}sinx+\frac{1}{2}\times\frac{sin7x}{7}+c$

= $\frac{1}{2}sinx+\frac{1}{14}sin7x+c$

= $\frac{1}{14}(sin7x+7sinx)+c$

iv) Evaluate $\int sin^2xdx$

Ans :- $\int sin^2xdx$

= $\int (\frac{1-cos2x}{2})\,dx$ $\quad (\because sin^2x=\frac{1-cos2x}{2})$

= $\frac{1}{2}\int (1-cos2x)dx$

= $\frac{1}{2}\int dx-\frac{1}{2}\int cos2xdx$

= $\frac{1}{2}\times x-\frac{1}{2}\times\frac{sin2x}{2}+c$

= $\frac{x}{2}-\frac{sin2x}{4}+c\ =\frac{1}{4}(2x-sin2x)+c$

v)Evaluate $\int cos^3x\,dx$

= $\int cos^3x\,dx$

= $\int(\frac{cos3x+3cosx}{4})\,dx$

($\because$ cos3x = $4cos^3x - 3cosx$ => $4cos^3x = cos3x + 3cosx$=>$cos^3x = \frac{cos3x+3cosx}{4}$)

= $\frac{1}{4}\int(cos3x + 3cosx)dx$

= $\frac{1}{4}\int cos3xdx + \frac{3}{4}\int cosxdx$

= $\frac{1}{4} \times \frac{sin3x}{3} + \frac{3}{4} \times sinx + c$

= $\frac{sin3x}{12} + \frac{3sinx}{4} + c$

= $\frac{1}{12}(sin3x + 9sinx) + c$

vi) Evaluate $\int cos^5xdx$

Ans :- $\int cos^5xdx$

= $\int cos^4x.cosxdx$=$\int(cos^2x)^2cosxdx$

= $\int(1 - sin^2x)^2cosxdx$

{ Put $sinx = \theta$ => cosx=$\frac{d\theta}{dx}$ =>$d\theta = cosxdx$}

= $\int(1-\theta^2)^2\,d\theta = \int(1 - 2\theta^2 + \theta^4)d\theta$

= $\int d\,\theta - 2\int\theta^2d\theta + \int\theta^4d\theta$

= $\theta - 2 \times \frac{\theta^3}{3} + \frac{\theta^5}{5} + c$

= $sinx - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c$

vii) Evaluate $\int sin^4x\,cos^3x\,dx$

Ans :- $\int sin^4x\,cos^3x\,dx$

= $\int sin^4x\,cos^2x\,cosx\,dx$

= $\int sin^4x\,(1 - sin^2x)cosxdx$

{ Put sinx =θ => cosx = $\frac{d\theta}{dx}$ =>dθ = cosxdx}

$=\int \theta^4(1-\theta^2)\,d\theta$

$=\int(\theta^4-\theta^6)d\theta = \int \theta^4 d\theta - \int \theta^6 d\theta$

$= \frac{\theta^5}{5}-\frac{\theta^7}{7}+c$

$= \frac{1}{5}sin^5x-\frac{1}{7}sin^7x+c$

viii) Evaluate $\int \frac{cos^3x}{sin^4x}dx$

Ans :- $\int \frac{cos^3x}{sin^4x}dx$

$= \int \frac{cos^2x}{sin^4x}.cosx\,dx$

$= \int \frac{(1-sin^2x)}{sin^4x}cosxdx$

$Put\ sinx = \theta$ => $cosx\,dx = d\theta$

$= \int \frac{1-\theta^2}{\theta^4}\,d\theta = \int(\theta^{-4}-\theta^{-2})\,d\theta$

$= \frac{\theta^{-3}}{-3}-\frac{\theta^{-1}}{-1}+c$

$= -\frac{1}{3\theta^3}+\frac{1}{\theta}+c = \frac{1}{sinx}-\frac{1}{3sin^3x}+c$

$= cosec\,x-\frac{1}{3}cosec^3x+c$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

TRIGONOMETRIC INDENTITIES

$1-sin^2\theta = cos^2\theta(or\ 1-cos^2\theta = sin^2\theta)$

$tan^2\theta+1 = sec^2\theta\ \ (also\ cot^2\theta+1 = cosec^2\theta)$

$sec^2\theta-1 = tan^2\theta\ (also\ cosec^2\theta-1 = cot^2\theta)$

➔ $the\ integrand\ of\ the\ form\ \sqrt{a^2-x^2},\sqrt{x^2+a^2},\sqrt{x^2-a^2}$ can be simplified by putting

X=a sin θ
X=a tanθ
X=a secθ
X=a cosθ
X=a cotθ
X=a cosec θ

Note

1.The integrand of the form a^2 - x^2 can be simplify by putting x= a sin θ (or x = a cos θ)

2. The integrand of the form x^2+a^2 can be simplify by putting x= a tan θ (or x = a cot θ)

3.The integrand of the form x^2 - a^2 can be simplify by putting x= a se θ (or x = a cosec θ)

Example -4

i) Integrate $\int \frac{dx}{\sqrt{a^2-x^2}}$

Ans :- $\int \frac{dx}{\sqrt{a^2-x^2}}$

Let x=a sin θ

Differentiate both sites w.r.t x

dx = a cos θd θ

And x = a sin θ=>$\theta = sin^{-1}\frac{x}{a}$

Hence $\int \frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\frac{a\cos\theta\, d\,\theta}{\sqrt{a^2-a^2 sin^2\theta}}} = \sqrt{\frac{a\, cos\theta}{a\, cos\theta}}\, d\theta = \int d\theta = \theta + c = sin^{-1}\frac{x}{a} + c$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$$

ii) Integrate $\int \frac{dx}{x^2+a^2}$

Ans :- $\int \frac{dx}{x^2+a^2}$

Let$x = a \tan\theta$

diffentiating both sides w.r.t x,

$dx = a\, sec^2\theta\, d\,\theta$

And $x = tan\theta$ => $\theta = tan^{-1}\frac{x}{a}$

Hence $\int \frac{dx}{x^2+a^2} = \int \frac{asec^2\theta d\theta}{a^2tan^2+a^2}$

$=\int \frac{asec^2\theta d\theta}{a^2(tan^2\theta+1)}$

$= \int \frac{sec^2\theta\ d\theta}{a\ sec^2\theta}$

$= \frac{1}{a}\int d\theta\ = \frac{1}{a}\ \theta\ + c = \frac{1}{a}tan^{-1}\frac{x}{a}$ + c

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + c$$

iii) Integrate $\int \frac{dx}{\sqrt{x^2+a^2}}$

Ans :- $\int \frac{dx}{\sqrt{x^2+a^2}}$

$Let\ x\ =\ a\ tan\theta$

Diffentiating w.r.t x we have,

$dx\ =\ asec^2\theta d\theta$

Hence $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{asec^2\theta d\theta}{\sqrt{a^2tan^{2\theta}+a^2}} = \int \frac{asec^2\theta d\theta}{\sqrt{a^2(tan^2\theta+1)}} = \int \frac{asec^2\theta d\theta}{asec\theta}$

= $\int sec\theta d\theta$

= $\ln|sec\theta + tan\theta| + c \quad (\because \int secxdx = \ln|secx + tanx| + c)$

(x = a tanθ => $tan\theta = \frac{x}{a}$ => $sec^2\theta = tan^2\theta + 1 = \frac{x^2}{a^2} + 1$ =>$sec\theta = \sqrt{\frac{x^2+a^2}{a^2}}$)

= $\ln|\sqrt{\frac{x^2+a^2}{a^2}} + \frac{x}{a}|$+c

= $\ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + c$

= $\ln|\frac{x+\sqrt{x^2+a^2}}{a}|$+c

= $\ln\left|x + \sqrt{x^2+a^2}\right| - ln\left|a\right| + c$

= $ln\left|x + \sqrt{x^2+a^2}\right| + k$ (∵ where k = c- ln|a| is a constant)

∴

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + k$$

iv) Integrate $\int \frac{dx}{\sqrt{x^2-a^2}}$

Ans :- $\int \frac{dx}{\sqrt{x^2-a^2}}$

Let $x = a\,sec\theta$ => $dx = a\,sec\theta tan\theta d\theta$

Now $\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{asec\theta tan\theta d\theta}{\sqrt{a^2sec^2\theta-a^2}}$

$= \int \frac{asec\theta tan\theta d\theta}{\sqrt{a^2(sec^2\theta-1)}}$

$= \int \frac{asec\theta tan\theta}{atan\theta}\, d\theta = \int sec\theta\, d\theta$

$= \ln|sec\theta + tan\theta| + c$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c$

{ As $x = a\,sec\theta => sec\theta = {}^{x}/_{a}$ => $\tan\theta = \sqrt{sec^2\theta - 1} = \sqrt{(\frac{x}{a})^2 - 1}$ }

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right| + c = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c$

$= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln|a| + c$

$= \ln\left|x + \sqrt{x^2 - a^2}\right| + k \quad (\because k = c - \ln|a| = constant)$

Hence

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + k$$

v)Integrate $\int \frac{dx}{x\sqrt{x^2-a^2}}$ **(2016-S)**

Let x = $asec\theta$ =>$dx = a \sec\theta \tan\theta\ d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

Now $\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\,sec\theta tan\theta d\theta}{asec\theta\sqrt{a^2sec^2\theta-a^2}}$

$= \int \frac{asec\theta tan\theta\, d\theta}{asec\theta\sqrt{a^2(sec^2\theta-1)}} = \int \frac{asec\theta tan\theta\, d\theta}{asec\theta\sqrt{a^2tan^2\theta}}$

$= \int \frac{asec\theta tan\theta}{asec\theta\ atan\theta} d\theta = \int \frac{asec\theta tan\theta}{a^2sec\theta tan\theta} d\theta$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}sec^{-1}\frac{x}{a} + c$$

$$\therefore \quad \boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a} + c}$$

vi) Integrate $\int \frac{dx}{x^2-a^2}$

Ans :- $\int \frac{dx}{x^2-a^2}$

$Let\ x = a\ sec\theta$ $\Rightarrow dx = a\ sec\theta\ tan\theta\ d\theta$

Now $\int \frac{dx}{x^2-a^2} = \int \frac{asec\theta tan\theta d\theta}{a^2sec^2\theta - a^2}$

$$= \int \frac{a\ sec\theta\ tan\theta\ d\theta}{a^2(sec^2\theta-1)}$$

$$= \int \frac{sec\theta tan\theta d\theta}{atan^2\theta}$$

$$= \frac{1}{a}\int \frac{sec\theta}{tan\theta}d\theta = \frac{1}{a}\int \frac{\frac{1}{cos\theta}}{\frac{sin\theta}{cos\theta}}d\theta$$

$$= \frac{1}{a}\int \frac{1}{sin\theta}d\theta$$

$$= \frac{1}{a}\int cosec\theta\ d\theta$$

$$= \frac{1}{a}\ln|cosec\theta - cot\theta| + c$$

{ As $x = a\ sec\theta \Rightarrow sec\theta = \frac{x}{a} \Rightarrow cos\theta = \frac{a}{x} \Rightarrow sin\theta = \sqrt{1-cos^2\theta} = \sqrt{1-(\frac{a}{x})^2} = \sqrt{\frac{x^2-a^2}{x^2}}$

$\Rightarrow sin\theta = \frac{\sqrt{x^2-a^2}}{x}$ $\Rightarrow cosec\theta = \frac{x}{\sqrt{x^2-a^2}}$ $and\ cot\theta = \frac{cos\theta}{sin\theta} = \frac{a}{\sqrt{x^2-a^2}}$ }

$$= \frac{1}{a}\ln\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c$$

$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x+a}\sqrt{x-a}}\right| + c$

$= \frac{1}{a}\ln\left|\frac{\sqrt{x-a}}{\sqrt{x+a}}\right| + c$

$= \frac{1}{a}\ln|\frac{x-a}{x+a}|^{\frac{1}{2}} + c$ $(\because \log_a m^n = n\log_a m)$

$= \frac{1}{2a}\ln|\frac{x-a}{x+a}| + c$

$\therefore$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

vii) Integrate $\int \frac{dx}{a^2-x^2}$

Ans :- $\int \frac{dx}{a^2-x^2}$

$Let\ x = a\sin\theta$ => $dx = a\ cos\theta d\theta$

Now $\int \frac{dx}{a^2-x^2} = \int \frac{a\ cos\theta\ d\theta}{a^2-a^2 sin^2\theta}$

$= \int \frac{a\ cos\theta\ d\theta}{a^2(1-sin^2\theta)} = \int \frac{cos\theta d\theta}{acos^2\theta}$

$= \frac{1}{a}\int \frac{1}{cos\theta}d\theta = \frac{1}{a}\int sec\theta d\theta$

{ As $x = a\ sin\theta \Rightarrow sin\theta = \frac{x}{a} \Rightarrow cos\theta = \sqrt{1-sin^2\theta} = \sqrt{1-\frac{x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$

$\Rightarrow sec\theta = \frac{1}{cos\theta} = \frac{a}{\sqrt{a^2-x^2}} \Rightarrow tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{a^2-x^2}}$ }

$= \frac{1}{a}\ln|sec\theta + tan\theta| + c$

$= \frac{1}{a}\ln|\frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}| + c$

$= \frac{1}{a}\ln|\frac{a+x}{\sqrt{a^2-x^2}}| + c$

$= \frac{1}{a}\ln|\frac{a+x}{\sqrt{a+x}\sqrt{a-x}}| + c$

$= \frac{1}{a}\ln\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + c$

$= \frac{1}{a}\ln|\frac{a+x}{a-x}|^{\frac{1}{2}} + c$ $(\because \log m^n = n\log m)$

$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

∴
$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

These 7 results deduced in Example-4 are sometimes used to find the integration of some other functions. Some examples are given below.

Example-5 :-Integrate $\int \frac{dx}{\sqrt{25-16x^2}}$

Ans :- $\int \frac{dx}{\sqrt{25-16x^2}}$ (As $\int \frac{dx}{\sqrt{25-16x^2}} = \int \frac{dx}{\sqrt{16(\frac{25}{16}-x^2)}} = \frac{1}{4}\int \frac{dx}{\sqrt{(\frac{5}{4})^2-x^2}}$)

$= \frac{1}{4}\int \frac{dx}{\sqrt{(\frac{5}{4})^2-x^2}}$

$= \frac{1}{4}sin^{-1}\frac{x}{\frac{5}{4}} + c$ (using formula$\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + c$, $here\ a = 5/4$)

$= \frac{1}{4}sin^{-1}\frac{4x}{5} + c$

Example – 6: - Integrate $\int \frac{e^x}{e^{2x}+9}dx$

Ans :- $\int \frac{e^x}{e^{2x}+9}dx$

$= \int \frac{e^x}{(e^x)^2+3^2}dx$ { Let $e^x = t$ =>$e^x dx = dt$ }

Now $\int \frac{e^x}{(e^x)^2+3^2}dx = \int \frac{dt}{t^2+3^2} = \frac{1}{3}\tan^{-1}\frac{t}{3} + c$

{ as $\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + c$, here a= 3}

$= \frac{1}{3}\tan^{-1}\frac{e^x}{3} + c$

Example – 7:- Integrate $\int \frac{dx}{x\sqrt{x^8-4}}$

Ans :- $\int \frac{dx}{x\sqrt{x^8-4}}$ (multiplying numerator and denominator by $4x^3$)

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$

$= \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}dx$ (Let $x^4 = t$ =>$4x^3 = \frac{dt}{dx}$)

$= \quad \frac{1}{4}\int \frac{4x^3dx}{x^4\sqrt{(x^4)^2-4}}$

$= \frac{1}{4}\int \frac{dt}{t\sqrt{t^2-2^2}} = \frac{1}{4} \times \frac{1}{2}\sec^{-1}\frac{t}{2} + c$ (using formula $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a}$, here a=2)

$= \frac{1}{8}\sec^{-1}(\frac{x^4}{2}) + c$

Example –8 ; -

Integrate $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$

Ans :- $\int \frac{x+5}{\sqrt{x^2+6x-7}}dx$

$= \int \frac{x+5}{\sqrt{x^2+2.3.x+3^2-9-7}}$

$= \quad \int \frac{x+3+2}{\sqrt{(x+3)^2-16}}$ { Let x+3 = t =>dx = dt }

$= \quad \int \frac{t+2}{\sqrt{t^2-16}}dt$

$= \quad \int \frac{tdt}{\sqrt{t^2-16}} + 2\int \frac{dt}{\sqrt{t^2-16}}$

$= I_1 + I_2$ ....(1)

$I_1 = \int \frac{tdt}{\sqrt{t^2-16}} = \int \frac{dz}{2\sqrt{z}}$ (putting $t^2 - 16 = z$ => 2t dt = dz => t dt = $\frac{dz}{2}$)

$= \frac{1}{2}.2.z^{\frac{1}{2}} + c_1 = \sqrt{z} + c_1 = \sqrt{t^2-16} + c_1$

$= \sqrt{(x+3)^2-16} + c_1$......................(2)

$I_2 = 2\int \frac{dt}{\sqrt{t^2-16}}$ (applying formula $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + k$, where a=4)

$= 2\ln|t + \sqrt{t^2-16}| + c_2$

$= 2\ln|(x+3) + \sqrt{(x+3)^2-16}| + c_2$ ....(3)

From (1),(2) and (3) we have,

$\int \frac{x+5}{\sqrt{x^2+6x-7}}dx = I_1 + I_2$

$= \quad \sqrt{(x+3)^2-16} + c_1 + 2\ln|x+3+\sqrt{(x+3)^2-16}| + c_2$

$= \quad \sqrt{x^2+6x-7} + 2\ln\left|x+3+\sqrt{x^2+6x-7}\right| + c$ (where $c_1 + c_2 = c$ is a constant)

Example-9 : -Integrate $\int \frac{dx}{\sqrt{x}\sqrt{x-a^2}}$ **(2016-S)**

Ans: - $\int \frac{dx}{\sqrt{x}\sqrt{x-a^2}}$ (put $\sqrt{x} = t$ $=> \frac{1}{2\sqrt{x}}dx = dt$ $=> \frac{dx}{\sqrt{x}} = 2dt$)

$= \int \frac{2dt}{\sqrt{t^2-a^2}} = 2\ln\left(t+\sqrt{t^2-a^2}\right)+c$ (Applying $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+k$)

$= 2\ln\left(\sqrt{x}+\sqrt{x-a^2}\right)+c$

Example-10: - Evaluate $\int \frac{dx}{2x^2+x-1}$

Ans:- $\int \frac{dx}{2x^2+x-1} = \int \frac{dx}{2(x^2+\frac{x}{2}-\frac{1}{2})} = \frac{1}{2}\int \frac{dx}{x^2+2.x.\frac{1}{4}+(\frac{1}{4})^2-\frac{1}{16}-\frac{1}{2}}$

$= \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-\frac{9}{16}} = \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-(\frac{3}{4})^2}$ (applying $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$)

$= \frac{1}{2}\frac{1}{2.\frac{3}{4}} \; ln\left|\frac{(x+\frac{1}{4})-\frac{3}{4}}{(x+\frac{1}{4})+\frac{3}{4}}\right|+c = \frac{1}{3}\ln\left|\frac{\frac{4x+1-3}{4}}{\frac{4x+1+3}{4}}\right|+c$

$= \frac{1}{3}\ln\left|\frac{4x-2}{4x+4}\right|+c = \frac{1}{3}\ln\left|\frac{2x-1}{2x+2}\right|+c$

3.INTEGRATION BY PARTS:-

If v& w are two differentiation function of x,then

$$\frac{d}{dx}(vw) = v\frac{dw}{dx}+w\frac{dv}{dx}$$

Or $v\frac{dw}{dx} = \frac{d}{dx}(vw) - w\frac{dv}{dx}$

Integrating both sites,

$$\int v\frac{dw}{dx}dx = \int \frac{d}{dx}(vw)dx - \int w\frac{dv}{dx}dx$$

$$= wv - \int w\frac{dv}{dx}dx$$

Let u = $\frac{dw}{dx}$ $then\ w = \int udx$

Then the above result canbe written as $\int \boldsymbol{uv}\ \boldsymbol{dx} = (\int \boldsymbol{udx})\boldsymbol{v}$- $\int(\int \boldsymbol{udx}) \times \frac{\boldsymbol{dv}}{\boldsymbol{dx}})\boldsymbol{dx}$.

This rule is called integration by parts and is used to integrate the product of two functions

Integration of the product of two functions

= $(integral\ of\ first\ function) \times second\ function - integral\ of\ (integral\ of\ first\ \times derivative\ of\ second)$

Int.of product =(int.first)×second – $\int (int. first)(der. second)dx.)$

➔ Before applying integration by parts we have follow some important things which are listed below.

1. In above formula there are two functions one is u and other one is v. The function 'u' is called the 1st function where as 'v' is called as the 2nd function.

2. The choice of 1st function is made basing on the order ETALI . The meaning of these letters is given below.

E – Exponential function

T – Trigonometric function

A – Algebraic function

L – Logarithmic function

I – inverse trigonometric function

The following table-1 gives a proper choice of 1st and 2nd function in certain cases. Here mϵN, n may be zero or any positive integer.

Table-1

Function to be integrated	**first function**	**second function**
$x^n e^x$	e^x	x^n
$x^n sinx$	$sinx$	x^n
$x^n cosx$	$cosx$	x^n
$x^n (lnx)^m$	x^n	$(lnx)^m$
$x^n \sin^{-1} x$	x^n	$\sin^{-1} x$
$x^n\, cos^{-1} x$	x^n	$\cos^{-1} x$
$x^n\, tan^{-1} x$	x^n	$tan^{-1} x$

Example – 11

Integrate $\int x\, cosx\, dx$

Ans :- $\int x\, cosx\, dx$ { from table-1, 1st function = cosx and 2nd function = x }

= $\{\int(\cos x\, dx)\}\, x - \int(\int cosx\, dx) \times \frac{dx}{dx}.dx$

= $x \sin x - \int sinx.1.dx$

= $xsinx + cosx + c$

Example – 12

Integrate $\int x^2 e^x dx$

Ans:-$\int x^2 e^x dx$ { 1st function = e^x and 2nd function = x^2 }

=$(\int e^x dx).x^2 - \int(\int e^x dx)\frac{d}{dx}(x^2)dx$

=$x^2 e^x - \int e^x \times 2xdx$

= $x^2 e^x - 2\int xe^x dx$ { again by parts is applied taking e^x as 1st and x as 2nd function.}

= $x^2 e^x - 2[(\int e^x dx).x - \int(\int e^x dx).1.dx]$

= $x^2 e^x - 2xe^x + 2e^x + c$ = $(x^2 - 2x + 2)\, e^x + c$

Example –13

Integrate $\int \tan^{-1} xdx$

Ans :- $\int \tan^{-1} xdx$

{ There is no direct formula for $\tan^{-1} x$ and two functions are not multiplied with each other in this integral. This type of integration can be solved by using integration by parts by writing $\tan^{-1} x$ as 1. $\tan^{-1} x$ where '1' represent an algebraic function. }

=$\int 1.\tan^{-1} x\, dx$ =$(\int 1\, dx).\tan^{-1} x - \int(\int 1.dx).\frac{d}{dx}(\tan^{-1} x)dx$

= $x \tan^{-1} x - \int x.\frac{1}{1+x^2}dx$

{ Let $1 + x^2 = t$ => $2xdx = dt$

Now $\int x.\frac{1}{1+x^2}dx = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\, lnt + c = \frac{1}{2}\ln(1 + x^2) + c$ }

= $x \tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + c$

Example –14

Integrate $\int lnx\, dx$ **(2016-S)**

Ans : - $\int lnx\, dx$

$= \int 1. lnx\, dx$ (Taking 1 as 1st function and lnx as 2nd function)

$= (\int 1. dx)\, lnx - \int (\int 1. dx) \frac{d}{dx}(lnx)\, dx$

$= xlnx - \int x. \frac{1}{x}\, dx \;\; = xlnx - \int dx$

$= xlnx - x + c \;=\; x(lnx - 1) + c$

Example-15:-

Integrate $\int (lnx)^2 dx$

Ans :- $\int (lnx)^2 dx$

$= \int 1 \times (lnx)^2 dx$

$= (\int 1. dx). (lnx)^2 - \int (\int 1. dx) \frac{d}{dx}(lnx)^2 dx$

$= x(lnx)^2 - \int x. \frac{2lnx}{x} dx.$

$= x(lnx)^2 - 2\int 1 \times lnxdx$

$= x(lnx)^2 - 2[x. lnx - \int x. \frac{1}{x}\, dx]$

$= x(lnx)^2 - 2xlnx + 2\int dx$

$= x(lnx)^2 - 2xlnx + 2x + c$

$= x[(lnx)^2 - 2(lnx) + 2] + c$

Example – 16 :-

Evaluate $\int xtan^{-1}x\, dx$ **(2017-W, 2017-S)**

Ans:- $\int xtan^{-1}x\, dx$

$= (\int xdx) tan^{-1}x - \int (\int x\, dx) \frac{d}{dx}(tan^{-1}x)\, dx$

$= \frac{x^2}{2} tan^{-1}x - \int \frac{x^2}{2} \frac{1}{1+x^2}\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2}\int(1 - \frac{1}{1+x^2})\, dx$

$= \frac{x^2}{2} tan^{-1}x - \frac{1}{2}(x - tan^{-1}x) + c$

$= (\frac{x^2+1}{2})\, tan^{-1}x - \frac{1}{2}x + c$

Note: - *When the Integrand is of the form* $e^x\{ f(x) + f'(x)\}$*, the integral is* $e^x f(x)$*,which can be verified by using integration by parts as given below.*

$\int e^x f(x)dx = e^x f(x) - \int(\int e^x dx)\frac{d}{dx}f(x)dx$ (*choosing* e^x $as\ 1st\ function\ and\ f(x)\ as\ 2nd$)

$= e^x f(x) - \int e^x f'(x)dx + c$

$=> \int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x) + c$

Hence $\int \boldsymbol{e^x\{f(x) + f'(x)dx\}dx = e^x f(x) + c}$

Example-17: -

Integrate $\int \frac{e^x}{x}(1 + xlnx)dx$ **(2017-S)**

Ans:- $\int \frac{e^x}{x}(1 + xlnx)dx = \int \frac{e^x}{x}dx + \int e^x(lnx)dx$

$= \int \frac{e^x}{x}dx + \int e^x(lnx)dx$ (Keeping 1st integral fixed we only simplify the 2nd one.)

$= \int \frac{e^x}{x}dx + (\int e^x dx)(lnx) - \int(\int e^x dx)\ \frac{d}{dx}(lnx)dx$

(taking e^x $as\ 1st\ and\ lnx\ as\ 2nd\ function.$)

$= \int \frac{e^x}{x}dx + e^x lnx - \int \frac{e^x}{x}dx + c$

$= e^x lnx + c.$

In some cases, integrating by parts we get a multiple of the original integral on the right hand side, which can be transferred and added to the given integral on the left hand side . After that we can evaluate these integrals. Some examples of such integrals are given below.

Example18: -

Integrate $\int \sqrt{\boldsymbol{x^2 + a^2}}\boldsymbol{dx}$

Ans:-L$et\quad I = \int \sqrt{x^2 + a^2}dx$

$= \int \sqrt{x^2 + a^2} \times 1\ dx$

$= \{\int(1.dx)\}\sqrt{x^2+a^2} - \int(\int 1.dx).\frac{d}{dx}\left(\sqrt{x^2+a^2}\right).dx$

$= x\sqrt{x^2+a^2} - \int x \times \frac{1}{2\sqrt{x^2+a^2}} \times 2x\, dx$

$= x\sqrt{x^2+a^2} - \int x \times \frac{x}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\, dx + \int \frac{a^2}{\sqrt{x^2+a^2}} dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}$

$= x\sqrt{x^2+a^2} - I + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + c$

$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + c$

$\Rightarrow I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

$$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

Example – 19

Integrate $\int sec^2\theta\sqrt{sec^2\theta+3}\, d\theta$

Ans: -$\int sec^2\theta\sqrt{sec^2\theta+3}\, d\theta$

$= \int sec^2\theta\sqrt{tan^2\theta+1+3}\, d\theta$

$= \int sec^2\theta\sqrt{tan^2\theta+4}\, d\theta \;\; \{ tan\theta = t \Rightarrow sec^2\theta\, d\theta = dt\}$

$= \int \sqrt{t^2+2^2}\, dt$ (From example-18 , putting a =2)

$= \frac{t}{2}\sqrt{t^2+2^2} + \frac{2^2}{2}\ln\left|t+\sqrt{t^2+2^2}\right| + c$

$= \frac{t}{2}\sqrt{t^2+4} + \frac{4}{2}\ln\left|t+\sqrt{t^2+4}\right| + c$

$= \frac{tan\theta}{2}\sqrt{tan^2\theta+4} + 2\,\ln\left|tan\theta+\sqrt{tan^2\theta+4}\right| + c$

Example-20 : -

Integrate $\int \sqrt{x^2 - a^2}dx$ (2014-S)

Ans:- Let I = $\int \sqrt{x^2 - a^2}dx$

$= \int \sqrt{x^2 - a^2} \times 1\, dx$

$= (\int 1.dx)\sqrt{x^2 - a^2} - \int (\int 1.dx).\frac{d}{dx}(\sqrt{x^2 - a^2})dx$

$= x\sqrt{x^2 - a^2} - \int x \times \frac{1}{2\sqrt{x^2 - a^2}} \times 2x\, dx$

$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}dx$

$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}dx$

$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}}dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}dx$

$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$

=> I = $x\sqrt{x^2 - a^2} - I - a^2 \ln\left|x + \sqrt{x^2 - a^2}\right| + c$

=> 2I = $x\sqrt{x^2 - a^2} - a^2 \ln\left|x + \sqrt{x^2 - a^2}\right| + c$

=> I = $\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

=> $$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

Example – 21

Integrate $\int \sqrt{a^2 - x^2}\, dx$

Ans - let I = $\int \sqrt{a^2 - x^2}\, dx$

$= \int \sqrt{a^2 - x^2} \times 1dx$

$= (\int 1.dx)\sqrt{a^2 - x^2} - \int (\int 1.dx).\frac{d}{dx}(\sqrt{a^2 - x^2})dx$

$= x\sqrt{a^2 - x^2} - \int x.\frac{1}{2\sqrt{a^2 - x^2}}(-2x)\, dx$

$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}} dx$

$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}} dx$

$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}} dx + a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$

$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\, dx + a^2 \sin^{-1}\frac{x}{a} + c$

=> I = $x\sqrt{a^2-x^2} - I + a^2 \sin^{-1}\frac{x}{a} + c$

=> 2I = $x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a} + c$

=> I = $\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$$\therefore \quad \boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$$

Example – 22

Evaluate $\int a^x\sqrt{a^{2x}-9}\, dx$

Ans:- $\int a^x\sqrt{a^{2x}-9}\, dx$ { Put $a^x = t \Rightarrow a^x lna\, dx = tdt$.}

$= \frac{1}{lna}\int \sqrt{t^2-3^2}\, dt$ { As $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + c$ }

$= \frac{1}{lna}[\frac{t}{2}\sqrt{t^2-3^2} - \frac{3^2}{2}\ln|t+\sqrt{t^2-3^2}|] + c$

$= \frac{1}{lna}[\frac{t}{2}\sqrt{t^2-9} - \frac{9}{2}\ln|t+\sqrt{t^2-9}|] + c$

$= \frac{1}{lna}[\frac{a^x}{2}\sqrt{(a^x)^2-9} - \frac{9}{2}\ln|(a^x+\sqrt{(a^x)^2-9}|] + c$

$= \frac{1}{lna}[\frac{a^x}{2}\sqrt{a^{2x}-9} - \frac{9}{2}\ln|(a^x+\sqrt{a^{2x}-9}| + c$

Example-23 : -

Integrate $\int \sqrt{2x^2+3x+4}\ dx$

Ans: - $\int \sqrt{2x^2+3x+4}\ dx$

$= \sqrt{2}\int \sqrt{x^2+\frac{3}{2}x+2}\ dx$

$$= \sqrt{2}\int\sqrt{x^2+2.\frac{3}{4}x+(\frac{3}{4})^2-(\frac{3}{4})^2+2}\ dx$$

$$= \sqrt{2}\int\sqrt{(x+\frac{3}{4})^2-\frac{9}{16}+2}\ dx$$

$$= \sqrt{2}\int\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}\ dx$$ (put x+ $\frac{3}{4}$= t => dx = dt)

$$= \sqrt{2}\int\sqrt{t^2+(\frac{\sqrt{23}}{4})^2}\ dt$$ (applying result obtained in example-18)

$$= \sqrt{2}\ [\frac{t}{2}\sqrt{t^2+(\frac{\sqrt{23}}{4})^2}+\frac{(\frac{\sqrt{23}}{4})^2}{2}\ln\left|t+\sqrt{t^2+\left(\frac{\sqrt{23}}{4}\right)^2}\right|\ ]+c$$

$$= \sqrt{2}\ [\frac{x+\frac{3}{4}}{2}\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}+\frac{\frac{23}{16}}{2}\ln\left|x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}\right|\ ]+c$$

$$= \sqrt{2}\ [\frac{4x+3}{8}\sqrt{x^2+\frac{3}{2}x+2}+\frac{23}{32}\ln\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right|\ ]+c$$

$$= \frac{4x+3}{8}\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\ln\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right|+c$$ (Ans)

Example-24: -

Integrate $\int e^{ax}\, sinbx\, dx$

Ans :- Let I = $\int e^{ax}\, sinbx\, dx$

$$= (\int e^{ax}\,dx)\,sinbx - \int(\int e^{ax}\,dx)\times\frac{d}{dx}\,(sinbx)\times\ dx$$

$$= \frac{e^{ax}}{a}sinbx - \int\frac{e^{ax}}{a}\times cosbx\times b\,dx$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\int e^{ax}cosbx\,dx$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\,[(\int e^{ax}dx)cosbx - \int(\int e^{ax}\,dx)\times\frac{d}{dx}(cosbx)dx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\,[\frac{e^{ax}}{a}cosbx - \int\frac{e^{ax}}{a}\times(-sinbx)\times b\,dx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a}\,[\frac{e^{ax}cosbx}{a} + \frac{b}{a}\int e^{ax}sinbxdx]$$

$$= \frac{e^{ax}sinbx}{a} - \frac{b}{a^2}e^{ax}cosbx - \frac{b^2}{a^2}\int e^{ax}\,sinbx\,dx$$

$$I = \frac{e^{ax}sinbx}{a} - \frac{b}{a^2}e^{ax}cosbx - \frac{b^2}{a^2}\,I + c$$

=> $I + \frac{b^2}{a^2} I = \frac{e^{ax} sinbx}{a} - \frac{b}{a^2} e^{ax} cosbx + c$

=> $\left(\frac{a^2+b^2}{a^2}\right) I = e^{ax}\left[\frac{sinbx}{a} - \frac{b}{a^2} cosbx\right] + C$

=> $\left(\frac{a^2+b^2}{a^2}\right) I = e^{ax}\left[\frac{asinbx - bcosbx}{a^2}\right] + c$

=> I = $\frac{e^{ax}}{a^2+b^2}[a\ sinbx - b\ cosbx] + c$

∴ $$\int e^{ax}\ sinbxdx = \frac{e^{ax}}{a^2+b^2}(a\ sinbx - bcosbx) + c$$

Example-25

Integrate $\int \boldsymbol{e^{ax}\ cosbx\ dx}$

Ans:- By adopting the same technique as we have done in example-24 We get

$$\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2}(a\ cosbx + b\ sinbx) + c$$

Example-26 : - Evaluate $\int e^{2x}\ sin3x\ dx$ **(2017-S)**

Ans:- $\int e^{2x}\ sin3x\ dx$

(Proceeding in the same manner as we have done in example-24 with a=2 and b= 3)

= $\frac{e^{2x}}{2^2+3^2}[2\ sin3x - 3cos3x] + c$

= $\frac{e^{2x}}{4+9}[2\ sin3x - 3cos3x] + c$

= $\frac{e^{2x}}{13}[2\ sin3x - 3cos3x] + c$ (Ans)

Example-26 : - Evaluate $\int e^{3x}\ cos2x\ dx$ **(2016-S)**

Ans:- $\int e^{3x}\ cos2x\ dx$ (Putting a=3 and b=2 in the result obtained by Example-25)

= $\frac{e^{3x}}{3^2+2^2}[3cos2x + 2sin2x] + c$ *(Note- In exam you have to proceed as example-24)*

= $\frac{e^{3x}}{9+4}[3cos2x + 2sin2x] + c$

= $\frac{e^{3x}}{13}[3cos2x + 2sin2x] + c$ (Ans)

Exercise

1.Evaluate the following Integrals (2 marks questions)

i) $\int \frac{1}{x\sqrt{x}}dx$

ii) $\int (x^{\frac{4}{7}} + \frac{1}{x^{1/3}})\, dx$

iii) $\int \frac{1-sin^3x}{sin^2x}dx$

$iv)\ \int \frac{cos2x}{sin^2xcos^2x}dx$

$v)\ \int \sqrt{1+sin2x}dx$

$vi)\ \int \frac{e^{2x}\ +1}{e^x}dx$

$vii)\ \int e^{2lnx}dx$

$viii)\ \int (\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}})\, dx$

$ix)\ \int \frac{x^2+\sqrt{x^2-1}}{x^3\sqrt{x^2-1}}dx$

$x)\ \int \frac{1-cos2x}{1+cos2x}dx$

xi) $\int \frac{dx}{1+sinx}$

xii) $\int (x^{e+}e^x + e^e)dx$

xiii) $\int (x^2+\sqrt{x})^2dx$

2.Evaluate the following (2 marks questions)

$i)\ \int \frac{x^2dx}{(1+x^3)2}$

$ii)\ \int sec^2\,(3x+5)dx$

$iii)\ \int \frac{(tan^{-1}x)^3}{1+x^2}dx$

$iv)\ \int \frac{sec^2\sqrt{x}}{\sqrt{x}}dx$

$v)\ \int tan^3x sec^2x\, dx$

$vi)\ \int \sqrt{1 - sinx}\ cosxdx$

$vii)\ \int x\sqrt{x^2+3}\ dx$

$viii)\ \int \frac{dx}{2-3x}$ **(2017-W)**

$ix)\ \int \frac{xdx}{\sqrt{x^2-a^2}}$

$x)\ \int \frac{e^x}{(e^x-2)^2}\ dx$

$xi)\ \int e^{x^3}x^2\ dx$

$xii)\ \int e^{cos^2x} sin2x\, dx$

$xiii)\ \int 2x\cot(x^2+3)\, dx$

$xiv)\ \int e^x tane^x dx$

$xv)\ \int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

$xvi)\ \int 3^x e^{2x} dx$

$xvii)\ \int \frac{sinx}{\sin(x+\propto)}dx$

$xviii) \int \frac{tanx + tan \propto}{tanx - tan \propto}\ dx$

$xix)\ \int sec^3x\, .\, tanx\, dx$ **(2016-S)**

$xx)\ \int \frac{e^{tan^{-1}x}}{1+x^2}\ dx$ **(2014-S)**

$xxi)\ \int sin^{20}x\ cos^3x\ dx$ **(2017-W)**

Question with long answers (5 and 10 marks)

(10 marks questions are indicated in right side of the question.)

3.Evaluate the following:-

$i)\ \int sin4x\, cos3x\, dx$

$ii)\ \int cos5x\, cos2x\, dx$

$iii)\ \int sin6x\, sin3x\, dx$

$iv)\ \int sin\frac{3x}{4}\, cos\frac{x}{2}dx$

$v)\ \int cos2x\, cos\frac{x}{2}dx$

$vi)\ \int sin^5x\, dx$ (10 marks)

$vii)\ \int cos^7x\, dx$ (10 marks)

$viii)\ \int sin^6x\, dx$ (10 marks)

$ix)\ \int cos^5x\, sin^3x\, dx$

$x)\ \int \frac{sin^3x}{cos^6x}dx$

$xi)\ \int sin^4x.cos^4x\, dx$

$xii)\ \int tan^5\,\theta.sec^4\theta\, d\theta$

$xiii)\ \int tan^5\theta\, d\theta$

$xiv)\ \int \frac{sin4x - sin2x}{cosx}\, dx$

4. Integrate the following :-

$i)\ \int \frac{dx}{\sqrt{11-4x^2}}$

ii) $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}dx$

iii) $\int \frac{dx}{x\sqrt{25-(lnx)^2}}$

iv) $\int \frac{cos\theta}{\sqrt{4-sin^2\theta}}d\theta$

v) $\int \frac{cos\theta d\theta}{\sqrt{4sin^2\theta+1}}$

vi) $\int \frac{dx}{\sqrt{5-x^2-4x}}$

vii) $\int \frac{x+3}{\sqrt{5-x^2-4x}}dx$

viii) $\int \frac{dx}{3x^2+7}$

ix) $\int \frac{e^{4x}}{e^{8x}+4}dx$

x) $\int \frac{\sec\theta\, tan\theta}{sec^2\theta+4}$

xi) $\int \frac{x^9}{x^{20}+4}dx$

xii) $\int \frac{dx}{x^2+6x+13}$

xiii) $\int \frac{dx}{\sqrt{e^{4x}-5}}$

xiv) $\int \frac{dx}{xlnx\sqrt{(lnx)^2-4}}$

xv) $\int \frac{dx}{\sqrt{4x^2-6}}$

xvi) $\int \frac{dx}{\sqrt{x^2+8x}}$

xvii) $\int \frac{x+7}{\sqrt{x^2+8x}}dx$

$xviii)\ \int \frac{1}{4cos^2x+9sin^2x}dx$ **(2014-S)**

$xix)\ \int \frac{1}{x\sqrt{(logx)^2-8}}dx$ **(2015-S)**

$xx)\ \int \frac{dx}{\sqrt{25-9x^2}}$ **(2015-S)**

$xxi)\ \int \frac{dx}{7-6x-x^2}$

5. *Evaluate the following*

(i) $\int (1+x)e^x dx$
(ii) $\int x^3 e^x dx$
(iii) $\int x\ sinx\ dx$
(iv) $\int x^2 sinax\ dx$
(v) $\int xcos^2 xdx$
(vi) $\int 2x\ cos3x\ cos2x\ dx$ (10 marks)
(vii) $\int 2x^3 cosx^2 dx$
(viii) $\int x^7 lnx\ dx$
(ix) $\int (lnx)^3 dx$
(x) $\int \frac{lnx}{x^5} dx$
(xi) $\int \sec^{-1} x\ dx$
(xii) $\int x\ \sin^{-1} x\ dx$
(xiii) $\int e^x cos^2 xdx$ (10 marks)
(xiv) $\int e^{2x} cos5xdx$
(xv) $\int \sqrt{7x^2+2}dx$
(xvi) $\int e^x (tanx + lnsecx)dx$
(xvii) $\int [\frac{1}{lnx} - \frac{1}{(lnx)^2}]\ dx$
(xviii) $\int \sin(lnx)\ dx$
(xix) $\int \frac{xe^x}{(1+x)^2} dx$
(xx) $\int \sqrt{x^2-8}\ dx$
(xxi) $\int \sqrt{9-x^2}dx$
(xxii) $\int e^{2x} sinx\ dx$ **(2016-S, 2017-W)**
(xxiii) $\int e^x sinx\ dx$ **(2019-W)**

Answer

1) i) $-2x^{1/2}$+c ii) $\frac{7}{11}x^{\frac{11}{7}} + \frac{3}{2}x^{\frac{2}{3}} + c$ iii) – cotx + cosx + c

iv) –cotx – tanx +c v) sinx- cosx +c vi) $e^x - e^{-x} + c$

vii) $\frac{x^3}{3} + c$ viii) $sin^{-1}x + c$ ix) $sec^{-1}x - \frac{1}{2x^2} + c$

x) tanx –x +c xi) $tanx - secx + c$ xii) $\frac{x^{e+1}}{e+1} + e^x + e^e x + c$

xiii) $\frac{x^5}{5} + \frac{4}{7}x^{\frac{7}{2}} + \frac{x^2}{2} + c$

2) i) $\frac{-1}{3(1+x^3)}$+c ii) $\frac{1}{3}\tan(3x+5) + c$ iii) $\frac{(tan^{-1}x)^4}{4} + c$

iv) $2tan\sqrt{x} + c$ v) $\frac{tan^4x}{4} + c$ vi) $-\frac{2}{3}(1 - sinx)^{3/2}$+c

vii) $\frac{1}{3}(x^2+3)^{\frac{3}{2}} + c$ viii) $-\frac{1}{3}\log(2-3x) + c$ ix) $\sqrt{x^2 - a^2} + c$

x) $-\frac{1}{e^x - 2} + c$ xi) $\frac{1}{3}e^{x^3} + c$ xii) $-e^{cos^2x} + c$

xii) $ln\left|\sin(x^2+3)\right| + c$ xiv) $ln\left|\sec e^x\right| + c$ xv) $\ln(e^x - e^{-x}) + c$

xvi) $\frac{3^x e^{2x}}{2 + ln3}$+c xvii) $xcos\alpha - sin\alpha\, ln\left|\sin(x+\alpha)\right| + c$

xviii) $xcos2\alpha + sin2\alpha\, ln\left|\sin(x-\alpha)\right| + c$ xix) $\frac{sec^3x}{3} + c$

xx) $e^{tan^{-1}x} + c$ xxi) $\frac{sin^{21}x}{21} - \frac{sin^{23}x}{23} + c$

3) i) $\frac{-1}{14}cos7x - \frac{1}{2}cosx + c$ ii) $\frac{1}{14}sin7x + \frac{1}{6}sin3x + c$ iii) $\frac{1}{6}sin3x - \frac{1}{18}sin9x + c$

iv) $\frac{-2}{5}cos\frac{5}{4}x - 2\,cos\frac{x}{4} + c$ v) $\frac{1}{5}sin\frac{5}{2}x + \frac{1}{3}sin\frac{3x}{2} + c$ vii) $-cosx + \frac{2}{3}cos^3x - \frac{1}{5}cos^5x + c$

vii) $sinx - sin^3x + \frac{3}{5}sin^5x - \frac{1}{7}sin^7x + c$ viii) $\frac{1}{192}(60x - 45sin2x + 9sin4x - sin6x) + c$

ix) $\frac{1}{8}cos^8x - \frac{1}{6}cos^6x + c$ x) $\frac{1}{5}sec^5x - \frac{1}{3}sec^3x + c$ xi) $\frac{1}{128}\left(3x - sin4x + \frac{1}{8}sin8x\right) + c$

xii) $\frac{1}{6}tan^6\theta + \frac{1}{8}tan^8\theta + c$ xiii) $\frac{1}{4}tan^4\theta - \frac{1}{2}tan^2\theta + ln\left|sec\theta\right| + c$

xiv) $4cosx - \frac{2}{3}\,cos3x + c$

4) i) $\frac{1}{2}sin^{-1}\frac{2x}{\sqrt{11}} + c$ ii) $\frac{1}{3}sin^{-1}\frac{e^{3x}}{2} + c$ iii) $sin^{-1}\frac{lnx}{5} + c$

iv) $sin^{-1}(\frac{sin\theta}{2}) + c$ v) $\frac{1}{2}ln\left|sin\theta + \sqrt{sin^2\theta + \frac{1}{4}}\right| + c$ vi) $sin^{-1}\frac{x+2}{3} + c$

vii) $sin^{-1}\frac{x+2}{3} - \sqrt{5 - x^2 - 4x} + c$ viii) $\frac{1}{\sqrt{21}}tan^{-1}\frac{\sqrt{3}x}{\sqrt{7}}$+c ix) $\frac{1}{8}tan^{-1}\left(\frac{e^{4x}}{2}\right) + c$

x) $\frac{1}{2}tan^{-1}\left(\frac{sec\theta}{2}\right) + c$ xi) $\frac{1}{20}tan^{-1}\left(\frac{x^{10}}{2}\right) + c$ xii) $\frac{1}{2}tan^{-1}\frac{x+3}{2} + c$

xiii) $\frac{1}{2\sqrt{5}}sec^{-1}\left(\frac{e^{2x}}{\sqrt{5}}\right) + c$ xiv) $\frac{1}{2}sec^{-1}\left(\frac{lnx}{2}\right) + c$ xv) $\frac{1}{2}ln\left|2x + \sqrt{4x^2 - 6}\right| + c$

xvi) $ln\left|x + 4 + \sqrt{x^2 + 8x}\right| + c$ xvii) $\sqrt{x^2 + 8x} + 3\,ln\left|x + 4 + \sqrt{x^2 + 8x}\right| + c$

xviii) $\frac{1}{6}tan^{-1}(\frac{3tanx}{2})$+c xix) $\log\left|logx + \sqrt{(logx)^2 - 8}\right| + c$

xx) $\frac{1}{3}sin^{-1}\frac{3x}{5}$ + c xxi) $\frac{1}{8}ln\left|\frac{7+x}{1-x}\right| + c$

5) i) $xe^x + c$ ii) $e^x(x^3 - 3x^2 + 6x - 6) + c$ iii) $sinx - xcosx + c$

iv) $\frac{1}{a^3}[(2 - a^2x^2)cosax + 2axsinax] + c$ v) $\frac{1}{8}(2x^2 + 2xsin2x + cos2x) + c$

vi) $cosx + \frac{1}{25}cos5x + x\left(sinx + \frac{1}{5}sin5x\right) + c$ vii) $x^2sinx^2 + cosx^2 + c$

viii) $\frac{x^8}{64}(8lnx - 1)$ ix) $x[(lnx)^3 - 3(lnx)^2 + 6lnx - 6] + c$

x) $-\frac{1}{16x^4}(1 + 4lnx) + c$ xi) $xsec^{-1}x - ln\left|x + \sqrt{x^2 - 1}\right| + c$

xii) $\left(\frac{x^2}{2} - \frac{1}{4}\right)sin^{-1}x + \frac{1}{4}x\sqrt{1 - x^2} + c$ xiiii) $\frac{e^x}{10}(5 + cos2x + 2sin2x) + c$

xiv) $\frac{e^{2x}}{29}(2cos5x + 5sin5x) + c$ xv) $\frac{x}{2}\sqrt{7x^2 + 2} + \frac{1}{\sqrt{7}}ln\left|\sqrt{7}x + \sqrt{7x^2 + 2}\right| + c$

xvi) $e^x\ln(secx) + c$ xvii) $\frac{x}{lnx} + c$ xviii) $\frac{x}{2}[\sin(lnx) - \cos(lnx)] + c$

xix) $\frac{e^x}{x+1} + c$ xx) $\frac{x}{2}\sqrt{x^2 - 8} - 4ln\left|x + \sqrt{x^2 - 8}\right| + c$

xxi) $\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}sin^{-1}\frac{x}{3} + c$ xxii) $\frac{e^{2x}}{5}(2sinx - cosx) + c$ xxiii) $\frac{e^x}{2}(sinx - cosx) + c$

DEFINITE INTEGRAL

Introduction

It was stated earlier that integral can be considered as process of summation. In such case the integral is called definite integral.

Objective

After completion of the topic you will be able to

1. Define and interpret geometrically the definite integral as a limit of sum.
2. State fundamental theorem of integral calculus.
3. State properties of definite integral.
4. Find the definite integral of some functions using properties.
5. Apply definite integral to find the area under a curve

Expected Background knowledge

1. Functional value of a function at a point.
2. Integration.

Definite Integral

Integration can be considered as a process of summation. In this case the integral is called as definite integrals.

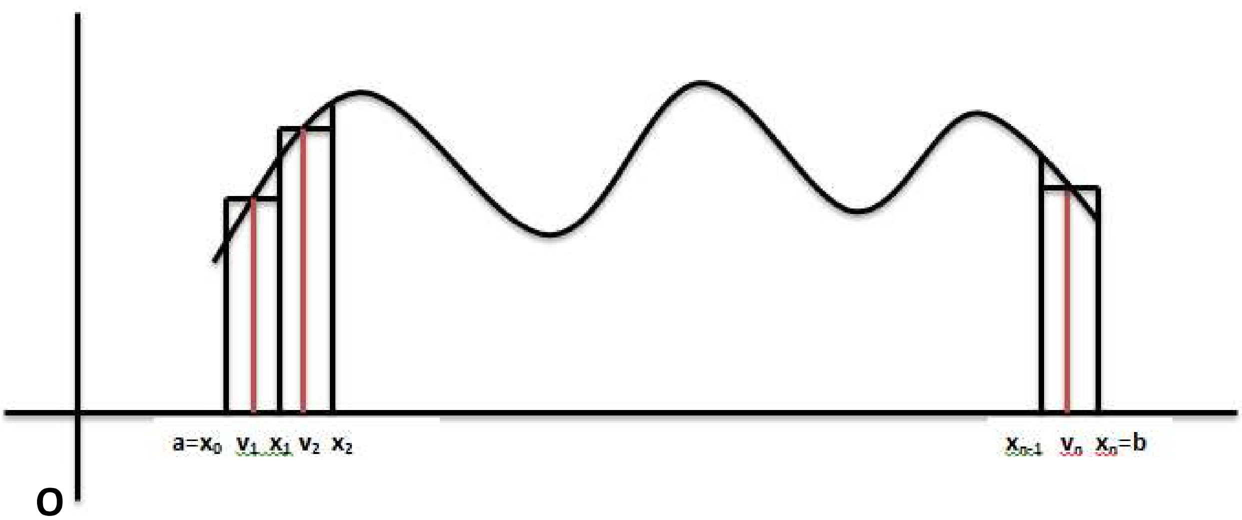

Fig-1

Definition:-

Let f(x) be a continuous function in [a,b] as shown in Fig-1 . Divide [a,b] into n sub-intervals of length h_1, h_2, ………….. h_ni.e. $h_1 = x_1 - x_0$, $h_2 = x_2 - x_1$, …………, $h_n = x_n - x_{n-1}$

Let v_r be any point in $[x_{r-1}, x_r]$ i.e. $v_1 \in [x_0, x_1]$, $v_2 \in [x_1, x_2]$,………….. , $v_n \in [x_{n-1}, x_n]$.

Then the sum of area of the rectangles (as shown in fig) when n→∞ is defined as the definite integral of f(x) from a to b, denoted by $\int_a^b \boldsymbol{f(x)dx}$

$$Here, \quad a = lowerlimitofintegration$$
$$b = upperlimitofintegration$$

Mathematically,

$$\int_a^b f(x)dx = \lim_{n\to\infty}[h_1f(v_1) + h_2f(v_2) + \dots\dots + h_nf(V_n)]$$

Fundamental Theorem of Integral Calculus

If f(x) is a continuous function in [a,b] and $\int f(x)dx = \emptyset(x) + c$, them

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a)$$

Note :- No arbitrary constants are used in definite integral.

Example:

1. Find $\int_1^2 x^3\, dx$

Ans.

First find $\int x^3\, dx = \frac{x^4}{4} + c$

Here, f(x) = x^3 , Φ (x) = $\frac{x^4}{4}$

By fundamental theorem

$$\int_1^2 x^3\, dx = \Phi(2) - \Phi(1)$$
$$= \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

2. Find $\int_0^1 \frac{dx}{1+x^2}$

Ans.

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1$$
$$= \tan^{-1} 1 - \tan^{-1} 0$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

3. Find $\int_2^3 2xe^{x^2}$dx

Ans.

$\int_2^3 2xe^{x^2}$dx

{Let x^2 = u => 2xdx = du, when x = 2, u = x^2 = 4,

When x = 3, u = x^2 = 9,

So, lower limit changes to 4 and upper limit changes to 9}

= $\int_4^9 e^u du$

= $[e^u]_4^9$ = e^9 - e^4 (Ans)

Properties of Definite Integral

1. $\int_a^b f(x)dx$ **=** $\int_a^b f(t)dt$

Explanation

Definite integral is independent of variable.

e.g. $\int_2^3 x^2dx$ = $\int_2^3 u^2\,du$ = $\int_2^3 t^2dt$

2. $\int_a^b f(x)dx$ **= -** $\int_b^a f(x)dx$

Explanation

If limits of definite integrals are interchanged then the value changes to its negative.

e.g. $\int_2^3 xdx$ = - $\int_3^2 xdx$

3. $\int_a^b f(x)dx$ **=** $\int_a^c f(x)dx$ **+** $\int_c^b f(x)dx$ **where, a<c<b**.

Explanation

If we integrate f(x) in [a,b] and c ∈ [a,b] such that a<c<b, then the above integral is same if we integrate f(x) in [a,c] and [c,b] and then add them.

e.g. $\int_2^6 xdx$ = $\int_2^4 xdx$ + $\int_4^6 xdx$

verification

$\int_2^6 xdx$ = $[\frac{x^2}{2}]_2^6$

= $\frac{6^2}{2}$ - $\frac{2^2}{2}$ = $\frac{36}{2}$ - $\frac{4}{2}$ = 18-2 = 16 -------------(1)

$\int_2^4 xdx$ + $\int_4^6 xdx$ = $[\frac{x^2}{2}]_2^4$+ $[\frac{x^2}{2}]_4^6$ = $[\frac{4^2}{2} - \frac{2^2}{2}]$ + $[\frac{6^2}{2} - \frac{4^2}{2}]$

= $[\frac{16}{2} - \frac{4}{2}]$ + $[\frac{36}{2} - \frac{16}{2}]$ = (8-2) + (18-8)

= 6 + 10 = 16 ---------- (2)

From (1) and (2) we have,

$\int_2^6 xdx$ = $\int_2^4 xdx$ + $\int_4^6 xdx$ (verified)

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

e.g. $\int_0^{\frac{\pi}{2}} sinxdx = \int_0^{\frac{\pi}{2}} \sin(\frac{\pi}{2} - x)\, dx$

verification

$\int_0^{\frac{\pi}{2}} sinxdx$ = [- cos x]$_0^{\frac{\pi}{2}}$

= - [cos $\frac{\pi}{2} - cos\ 0$] = - [0 - 1] = 1 ----------------------- (1)

$\int_0^{\frac{\pi}{2}} sin(\frac{\pi}{2} - x)$ dx = $\int_0^{\frac{\pi}{2}} \cos x dx$

= [sin x]$_0^{\frac{\pi}{2}}$ = sin $\frac{\pi}{2}$ - sin 0 = 1 – 0 = 1 ------------------------ (2)

From (1) and (2)

$\int_0^{\frac{\pi}{2}} sinxdx = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2} - x\right) dx$ (verified)

5. (i) If f(x) is an even function, then

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(ii) If f(x) is an odd function, then

$\int_{-a}^{a} f(x)dx = 0$

Example: - By this formula without integration we can find the integral for

f(x) is an odd function if

f (-x)= -f(x)

Sinx , x ,x^3 ……… are examples of odd functions.

f(x) is an even function if

f (-x) = f(x)

Cosx , x^2 , x^4 ……… are examples of even functions .

Example:-

$\int_{-2}^{2} x^2 dx = 2\int_0^2 x^2 dx$

{f(x) = $x^2$is an even function as, f(-x) = $(-x)^2$ = x^2 . So, $f(-x) = f(x)$}

Similarly,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sinx dx = 0$$

Reason

f(x) = sinx => f(-x) = sin (-x) = -sinx

So, f(-x) = -f(x)

⇨ f(x) is an odd function.

6. (i) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ **if f(2a - x) = f(x)**

(ii) $\int_0^{2a} f(x)dx = 0$ **if f(2a - x) = -f(x).**

7. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Problems

Q1. Find $\int_{-2}^{1}|x|dx$

Ans.

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

|x| changes its definition at '0', so divide the integral into two parts (-2,0) and (0,1).

Now, $\int_{-2}^{1}|x|dx$

= $\int_{-2}^{0}|x|dx$ + $\int_{0}^{1}|x|dx${Property (3) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx$}

= $\int_{-2}^{0} -xdx$ + $\int_0^1 xdx${when, -2 < x < 0 i.e. x < 0 then, |x| = -x }

= - $[\frac{x^2}{2}]_{-2}^{0}$ + $[\frac{x^2}{2}]_0^1${when, 0 < x < 1 i.e. 0 < x then, |x| = x }

= - $[\frac{0^2}{2} - \frac{(-2^2)}{2}]$ + $[\frac{1^2}{2} - \frac{0^2}{2}]$

= - [0 - $\frac{4}{2}$] + [$\frac{1}{2}$ – 0] = 2 + $\frac{1}{2}$ = $\frac{3}{2}$

Q2.$\int_{-6}^{6}|x+2|$ dx = ?

Ans.

$\int_{-6}^{6}|x+2|$ dx = $\int_{-4}^{8}|u|du$ {Let u = x + 2 => du = dx, when, x = -6, u = -6 +2 = -4}

{when, x = 6, u = 6 +2 = 8}

= $\int_{-4}^{0}|u|$ du + $\int_0^8|u|du${property (3)}

= $\int_{-4}^{0} -udu$ + $\int_0^8 udu$ { when -4<u<0 then IuI = -u and when 0<u<8 ,then IuI = u}

= - $[\frac{u^2}{2}]_{-4}^{0}$ + $[\frac{u^2}{2}]_0^8$

$= -\frac{1}{2}[u^2]_{-4}^{0} + \frac{1}{2}[u^2]_0^8 = -\frac{1}{2}[0^2 - (-4)^2] + \frac{1}{2}[8^2 - 0]$

$= -\frac{1}{2}(-16) + \frac{1}{2}(64)$

= 8 + 32 = 40(Ans)

Q3. Find $\int_1^3 [x]dx$

Ans.

[x] is a function which changes it value at every integral point. So, we have to break the range into different integral ranges i.e. (1,3) can be breaked into (1,2) and (2,3)

$\int_1^3 [x]dx = \int_1^2 [x]dx + \int_2^3 [x]dx$ {applying property (3) i.e. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$}

$= \int_1^2 1dx + \int_2^3 2dx$ {when 1 < x< 2 then [x] = 1, when 2 <x < 3, then [x] = 2 }

$= [x]_1^2 + [2x]_2^3$

= (2 - 1) + 2 (3 - 2) = 1 + (2 X 1) = 1 + 2 = 3

Q4. Evaluate $\int_0^{\frac{3}{2}} [2x]dx$

Ans.

$\int_0^{\frac{3}{2}} [2x]dx = \int_0^3 [u]\frac{du}{2}$

{Put u = 2x, du = 2dx => dx = $\frac{du}{2}$ when, x = 0, u = 2x = 0 when x = $\frac{3}{2}$, u = 2x = 3}

$= \frac{1}{2}[\int_0^1 [u]du + \int_1^2 [u]du + \int_2^3 [u]du]$

$= \frac{1}{2}[[0]_0^1 + [u]_1^2 + 2[u]_2^3] = \frac{1}{2}[0 + (2 - 1) + 2(3 - 2)]$

$= \frac{1}{2}[0 + 1 + 2] = \frac{3}{2}$(Ans)

Q5. Find $\int_{-1}^{1} \{|x| + [x]\} dx$

Ans.

$\int_{-1}^{1} (|x| + [x])dx = \int_{-1}^{1} |x| dx + \int_{-1}^{1} [x] dx$

$= \int_{-1}^{0} |x| dx + \int_0^1 |x| dx + \int_{-1}^{0} [x]dx + \int_0^1 [x]dx$

$= \int_{-1}^{0} -x\, dx + \int_0^1 x\, dx + \int_{-1}^{0} (-1)\, dx + \int_0^1 0\, dx$ {By def^n of |x| & [x]}

$= -[\frac{x^2}{2}]_{-1}^{0} + [\frac{x^2}{2}]_0^1 - [x]_{-1}^{0} + 0$

$= -\frac{1}{2}(0^2 - (-1)^2) + \frac{1}{2}(1^2 - 0) - (0 - (-1))$

$= -\frac{1}{2}(-1) + \frac{1}{2}.1 - (1)$

$= \frac{1}{2} + \frac{1}{2} - 1 = 0$ (Ans)

Q6. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx} + \sqrt{cosx}} dx$ **{ 2016-S, 2017-W}**

Ans.

In this type of problems we generally use property (4). And this type of problem can be solved by following technique.

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx} + \sqrt{cosx}} dx$ -------------------- (1)

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}$ dx

{In above x is replaced by $\frac{\pi}{2} - x$. As by property(4) there is no change in integral value}

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{cosx}+\sqrt{sinx}}$ dx

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}}$ dx ------------------ (2)

Now equation (1) +(2)

⇨ $I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}}dx$

⇨ $2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}+\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}}\,dx$

⇨ $2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$

⇨ $2I = (\frac{\pi}{2} - 0) = \frac{\pi}{2}$

⇨ $I = \frac{\pi}{4}$

Hence, $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}dx = \frac{\pi}{4}$

Q7. Evaluate $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)d\theta$ **{2016-S}**

Ans.

Let $I = \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan(\frac{\pi}{4} - \theta)\right) d\theta$ {As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$}

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{\tan\frac{\pi}{4} - tan\theta}{1+tan\frac{\pi}{4}.\tan\theta}\right) d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1-tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+tan\theta+1-tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\frac{\pi}{4}} \log(\frac{2}{1+tan\theta})\, d\theta$

$= \int_0^{\frac{\pi}{4}} (log2 - \log(1 + tan\theta))d\theta$ {$\log(\frac{a}{b}) = \log a - \log b$}

$= \int_0^{\frac{\pi}{4}} \log 2\, d\theta$ - $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$

⇨ $I = \int_0^{\frac{\pi}{4}} \log 2\, d\theta$ - $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta$

⇨ $I = \log 2 \int_0^{\frac{\pi}{4}} d\theta$ - I {As $I = \int_0^{\frac{\pi}{4}} \log(1 + tan\theta)$}

⇨ $2I = \log 2\,[\theta]_0^{\frac{\pi}{4}} = \log 2\,(\frac{\pi}{4} - 0)$

⇨ $2I = \frac{\pi}{4}\log 2$

⇨ $I = \frac{\pi}{8}\log 2$

Hence, $\int_0^{\frac{\pi}{4}} \log(1 + tan\theta)\, d\theta = \frac{\pi}{8}\log 2$ (Ans)

Q8. Evaluate $\int_0^1 \frac{x}{x+1} dx$

Ans.

$\int_0^1 \frac{x}{x+1} dx = \int_0^1 \frac{x+1-1}{x+1} dx$

$= \int_0^1 (\frac{x+1}{x+1} - \frac{1}{x+1}) dx$

$= \int_0^1 (1 - \frac{1}{x+1})\, dx$

$= [x - \ln(x+1)]_0^1$

$= [(1 - \ln(1+1)) - (0 - \ln(0+1))]$

$= 1 - \ln 2 - 0 + \ln 1$

$= 1 - \ln 2 - 0 + 0$

$= 1 - \ln 2$

Q9. $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Ans.

$\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^x . e^x + 1}{e^x}}$

$= \int_0^1 \frac{e^x dx}{e^{x+x}+1} = \int_0^1 \frac{e^x}{e^{2x}+1} dx$

$= \int_0^1 \frac{e^x}{(e^x)^2+1} dx$

{ Let $t = e^x$, => $dt = e^x dx$

when, $x = 0$, $t = e^x = e^0 = 1$

when $x = 1$, $t = e^1 = e$ }

$= \int_1^e \frac{dt}{t^2+1} = [tan^{-1}t]_1^e$

$= tan^{-1}e - tan^{-1}1$

$= tan^{-1}e - \frac{\pi}{4}$ (Ans)

Q10. $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = ?$

Ans.

$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^{\sqrt{2}}$

$= \sec^{-1}\sqrt{2} - \sec^{-1} 1$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ (Ans)

Q11. Find $\int_0^4 \frac{1}{x+\sqrt{x}} dx$

Ans.

$$\int_0^4 \frac{1}{x+\sqrt{x}} dx$$

Let $\sqrt{x} = \text{t}$, Then, $\frac{1}{2\sqrt{x}} dx = \text{dt}$

⇨ $\text{dx} = 2\sqrt{x}\, \text{dt} = 2\text{t dt}$

When $\text{x} = 0, \text{t} = \sqrt{x} = \sqrt{0} = 0$

When $\text{x} = 4, \text{t} = \sqrt{4} = 2$

Now,

$$\int_0^4 \frac{1}{x+\sqrt{x}} dx = \int_0^2 \frac{2tdt}{t^2+t}$$

$$= 2\int_0^2 \frac{tdt}{t\,(1+t)} = 2\int_0^2 \frac{dt}{1+t)} = 2\,[\ln(1+\text{t})]_0^2$$

= 2 (ln3 – ln1) = 2 (ln3 - 0) = 2 ln3 (Ans)

Q12. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx} dx$

Ans.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot(\frac{\pi}{2}-x)} dx \qquad \{\text{Property 4}\}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+tanx} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{cotx}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\frac{cotx+1}{cotx}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{cotx}{1+cotx} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{cotx+1-1}{1+cotx} dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \frac{1}{1+cotx}) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \text{ - } \int_0^{\frac{\pi}{2}} \frac{1}{1+cotx} dx \text{ ---------------------- } (1)$$

Let $\text{I} = \int_0^{\frac{\pi}{2}} \frac{1}{1+cotx} dx$

Then from (1)

$$\text{I} = \int_0^{\frac{\pi}{2}} dx - \text{I}$$

⇨ $2\text{I} = \int_0^{\frac{\pi}{2}} dx = [\text{x}]_0^{\frac{\pi}{2}} = (\frac{\pi}{2} \text{ - } 0) = \frac{\pi}{2}$

⇨ $\text{I} = \frac{\pi}{4}$

Hence, $\int_0^{\frac{\pi}{2}} \frac{1}{1+cotx} dx = \frac{\pi}{4}$ (Ans)

Q13. Prove that $\int_0^{\frac{\pi}{2}} \log(sinx) dx = \int_0^{\frac{\pi}{2}} \log(cosx) dx = -\frac{\pi}{2} \log 2$ { 2018-S}

Ans.

Let $I = \int_0^{\frac{\pi}{2}} \log(sinx)dx$ --------------------- (1)

$= \int_0^{\frac{\pi}{2}} \log(\sin(\frac{\pi}{2} - x))dx$ $\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\}$

$= \int_0^{\frac{\pi}{2}} \log(cosx)dx$ -------------------- (2)

Add (1) and (2)

$I + I = \int_0^{\frac{\pi}{2}} \log(sinx)dx + \int_0^{\frac{\pi}{2}} \log(cosx)dx$

$= \int_0^{\frac{\pi}{2}} \{\log(sinx) + \log(cosx)\}dx$

$= \int_0^{\frac{\pi}{2}} \log(sinx.cosx)dx$ $\{\log a + \log b = \log ab\}$

$= \int_0^{\frac{\pi}{2}} \log(\frac{2sinx.cosx}{2})dx$

$= \int_0^{\frac{\pi}{2}} \log(\frac{sin2x}{2})dx$

$= \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)dx$

$2I = \int_0^{\frac{\pi}{2}} \log sin2x\, dx$ - $\int_0^{\frac{\pi}{2}} \log 2\, dx$ ------------------- (3)

Now, $\int_0^{\frac{\pi}{2}} \log sin2x\, dx$ {Put $t = 2x$ we have $dt = 2dx$ when $x = 0, t = 0$ when $x = \frac{\pi}{2}, t = \pi$}

$= \int_0^{\pi} \log sint \frac{dt}{2}$

$= \frac{1}{2}\int_0^{\pi} \log sintdt$ { Here $f(t) = \log \sin(t)$, Then $f(\pi - t) = \log \sin(\pi - t) = \log \sin(t) = f(t)$}

$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log(sint)dt$ {By property-6 i.e.$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$,Where f(2a-x) = f(x)}

$= \int_0^{\frac{\pi}{2}} \log(sint)dt$ { We have $\int_0^{\pi} \log sintdt = 2\int_0^{\frac{\pi}{2}} \log(sint)dt$ }

$= \int_0^{\frac{\pi}{2}} \log(sinx)dx$ {Property (1) $\int_a^b f(x)dx = \int_a^b f(t)dt$}

$= I$ (from (1))

⇨ $I = \int_0^{\frac{\pi}{2}} \log sin2xdx$ --------------------- (4)

From (3) and (4)

⇨ $2I = I - \int_0^{\frac{\pi}{2}} \log 2\, dx$

⇨ $2I = I - \log 2\, [x]_0^{\frac{\pi}{2}}$

⇨ $I = -\log 2\, (\frac{\pi}{2} - 0)$

Hence, $\int_0^{\frac{\pi}{2}} \log(sinx)dx = \int_0^{\frac{\pi}{2}} \log cosx\, dx$

$= -\frac{\pi}{2} \log 2$ (Proved)

Exercise

Evaluate the integrals (2 marks and 5 marks questions)

(Questions with 2 marks is marked on right of the questions)

1) $\int_0^2 [x^2]\, dx$

2) $\int_0^{\frac{\pi}{2}} \frac{cosx}{sinx+cosx} dx$ **(2015-S)**

3) $\int_0^{\frac{\pi}{2}} \frac{1}{1+tanx} dx$

4) $\int_0^1 \frac{\log(1+x)}{(1+x^2)} dx$

5) $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}} dx$

6) $\int_0^\pi \frac{xdx}{1+sinx}$

7) $\int_0^{\frac{\pi}{2}} \log(tanx) dx$ **(2017-S) (2014-S)**

8) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx}$

9) $\int_0^{\frac{\pi}{2}} sin2x \log(tanx)\, dx$

10) $\int_0^\pi \frac{xsinx}{1+cos^2x} dx$

11) $\int_0^{\frac{\pi}{2}} \frac{sinx}{sinx+cosx} dx$ **(2017-W)**

12) $\int_{-1}^1 [x] dx$ **(2018-S)** (2marks)

13) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} dx$ **(2016-S) (2017-S)**

14) $\int_0^1 \frac{dx}{1+x^2} dx$ (2marks)

15) $\int_0^3 [x] dx$ **(2017-W)**

ANSWERS

1. $5-\sqrt{3} - \sqrt{2}$
2. $\frac{\pi}{4}$
3. $\frac{\pi}{4}$
4. $\frac{\pi}{8}$ log 2 {Hints Put x = tanθ}
5. $\log(\frac{5+3\sqrt{3}}{1+\sqrt{3}})$
6. π
7. 0
8. 1
9. 0

10. $\frac{\pi^2}{4}$ 11. $\frac{\pi}{4}$ 12. -1

13. $\frac{\pi}{2}$ 14. $\frac{\pi}{4}$ 15. 3

Area under plane curve

In our previous study we know that the definite integral represents the area under the curve.

Area enclosed by curve and X- axis

Area enclosed by a curve y = f(x) , X-axis , x = a and x =b is given by

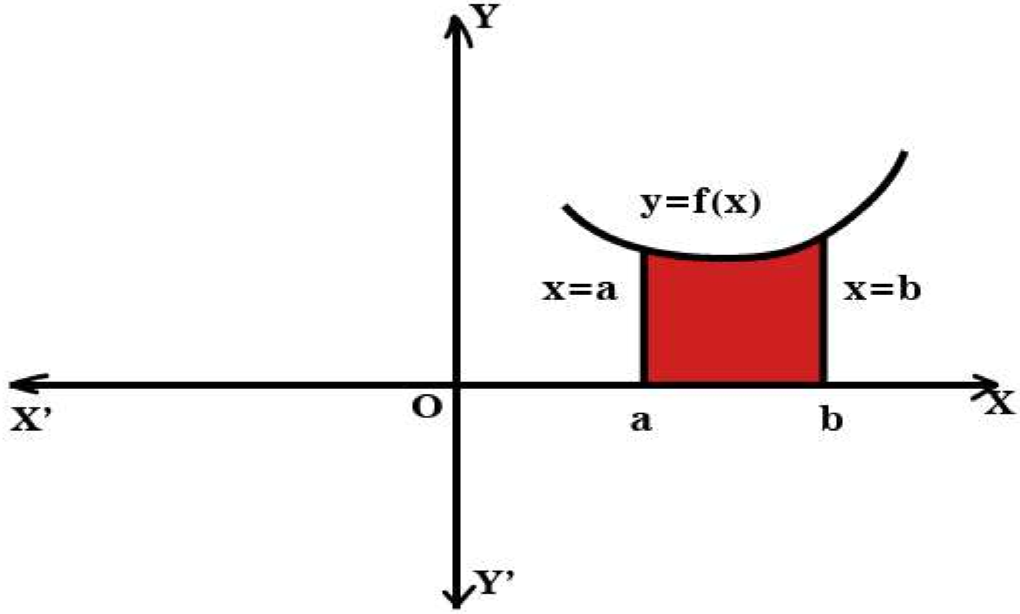

Fig-2

Area = $\int_a^b f(x)dx$

Example – 1

Find the area bounded y = e^x , X-axis x = 4 and x = 2

Ans.

Here y = e^x is the curve

Area of the curve bounded by X-axis , x = 4 and x = 2 is

Area = $\int_2^4 y\,dx$ = $\int_2^4 e^x dx$

= $[e^x]_2^4$ = $e^4 - e^2$ (Ans)

Example – 2

Find the area enclosed by y = 9 -x^2 , y = 0 , x = 0 and x = 2.

Ans.

Area = $\int_0^2 y\,dx$ = $\int_0^2 (9 - x^2)dx$

= $[9x - \frac{x^3}{3}]_0^2$ = [(9X2) - $\frac{2^3}{3}$ - (0-0)]

= 18 - $\frac{8}{3}$ = $\frac{54-8}{3}$ = $\frac{46}{3}$ (Ans)

Area of a circle with centre at origin

As shown in figure-3, the circle with centre at origin is divided into four equal parts by the co-ordinate axes

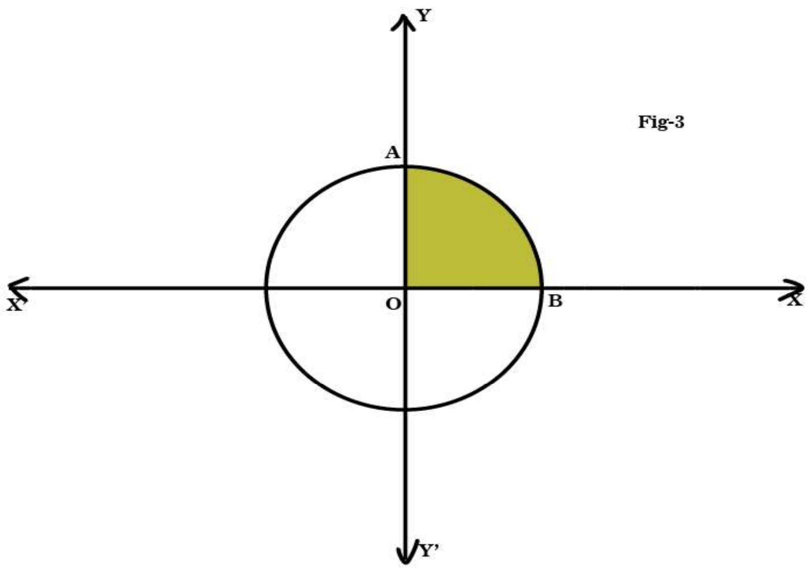

Fig-3

Hence area of the circle = 4 X area OAB

Example – 3

Find the area of the circle $x^2 + y^2 = a^2$ **(2015-S)**

Ans.

Area of circle = 4 X area OAB (see fig-4)

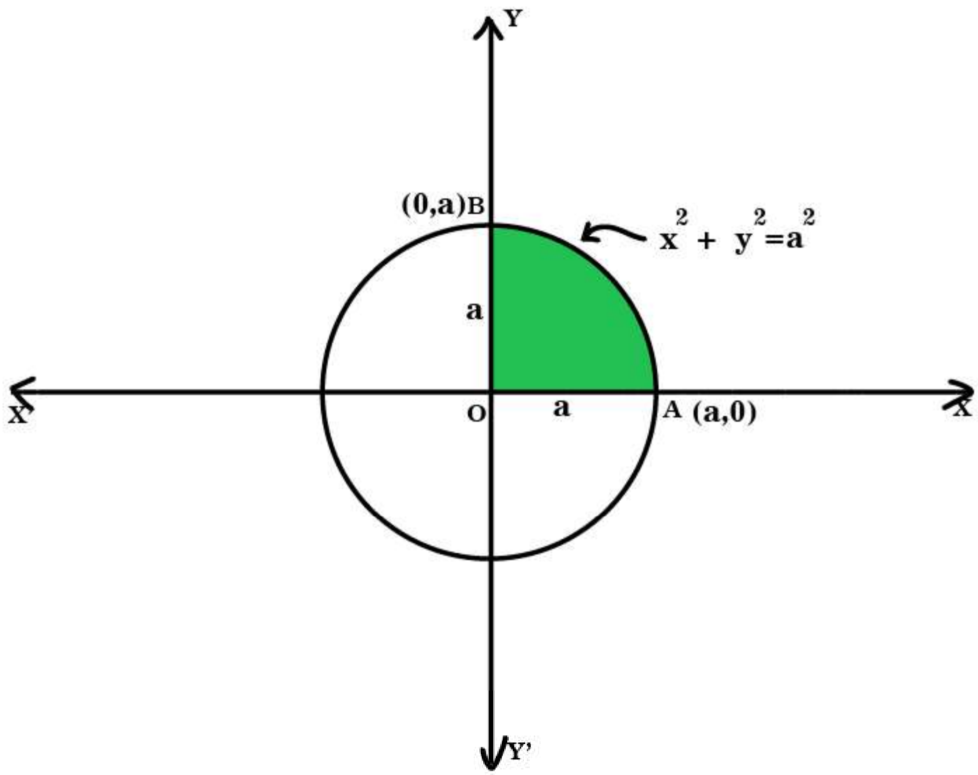

Fig-4

Now equation of circle is $x^2 + y^2 = a^2$

$\Rightarrow$ y = $\sqrt{a^2 - x^2}$ (for portion OAB)

(Actually y = $\pm\sqrt{a^2 - x^2}$) , but in 1st quadrant y is +ve)

$\Rightarrow$ y = $\sqrt{a^2 - x^2}$

Now the portion OAB is bounded by y – axis i.e. x = 0 ,

X axis and y = $\sqrt{a^2 - x^2}$

In the given region x varies from 0 to a ; as it is clear from figure the point A is (a,0)

(A = (a,0) because $x^2 + y^2 = a^2$ has radius a)

Now Area of OAB = $\int_0^a y\,dx$

$= \int_0^a \sqrt{a^2 - x^2}\,dx$

$= [\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}]_0^a$

$$= [\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - (0 + \frac{a^2}{2}\sin^{-1}0)]$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 + 0 \quad = \frac{a^2}{2}\sin^{-1}1 = \frac{a^2}{2} . \frac{\pi}{2} = \frac{1}{4}\pi a^2$$

Hence area of circle is = 4 X area of OAB

$$= 4 \text{ X } \frac{1}{4}\pi a^2 = \pi a^2 \text{ sq units}$$

Example – 2

Find the area bounded by the curve $x^2 + y^2 = 9$**(2017-S)**

Ans: - Area of the curve $x^2 + y^2 = 9$ i.e. circle = 4 X Area OAB { from fig-5}

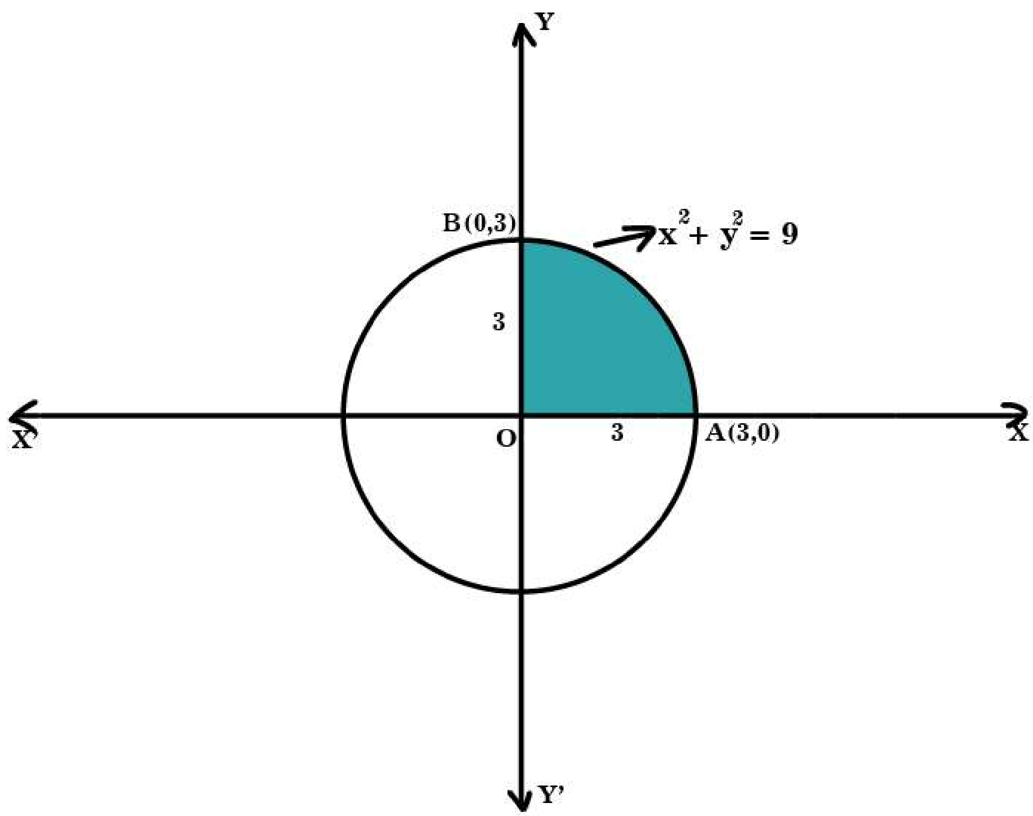

Fig-5

As $x^2 + y^2 = 9$ has radius 3

So, A is at (3,0)

Area of OAB is the area bounded by curve AB , Y axis and X axis.

Now Curve AB is $x^2 + y^2 = 9 \Rightarrow$ y = $\sqrt{9 - x^2}$

(as in 1st quadrant y is +ve)

In the region OAB x varies from 0 to 3.

Now area of OAB = $\int_0^3 y\,dx$ = $\int_0^3 \sqrt{9 - x^2}\,dx$

= $[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}]_0^3$

= $[\frac{3}{2}\sqrt{9 - 9} + \frac{9}{2}\sin^{-1}(\frac{x}{3})] - [0+\frac{9}{2}\sin^{-1} 0]$

= 0 + $\frac{9}{2}\sin^{-1} 1$ - 0

= $\frac{9}{2}$ X $\frac{\pi}{2}$ = $\frac{9\pi}{4}$

∴ Area bounded by the curve $x^2 + y^2 = 9$ is = 4 X Area of OAB

= 4 X $\frac{9\pi}{4}$ = $9\pi\ sq\ units$ (Ans)

Exercise

Q.1 Find the area bounded by the curve xy = c^2 , y = 0 , x = 2 and x = 3. (2 marks)

Q.2 Find the area bounded by the curve $x^2 + y^2 = 4$**.** **(2015-S)** (10 marks)

Q.3 Find the area of the circle $x^2 + y^2 = 16$. (10 marks)

Ans . (1)$c^2 \log\frac{3}{2}$

(2) 4π sq units

(3) 16 π sq units

Differential Equation

Introduction

After the discovery of calculus, Newton and Leibnitz studied differential equation in connection with problem of Physics especially in theory of bending beams, oscillation of mechanical system etc. The study of differential equation is a wide field in pure and applied mathematics, physics and engineering.

Objectives

1. Define differential equation.
2. Determine order and degree of differential equation.
3. Form differential equation from a given solution.
4. Solve differential equation by using different techniques.

Definition of Differential Equation

A differential equation is an equation involving dependent variables, independent variables and derivatives of dependent variables with respect to one or more independent variables.

Here x is an independent variable, y is dependent variable and $\frac{dy}{dx}$ is the derivative of the dependent variable w.r.t. the independent variable.

Examples: - $i)$ $\frac{dy}{dx} + xy = x^2$

$ii)$ $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + (\frac{dy}{dx})^4 + y = sinx$

$iii)$ $\frac{dy}{dx} + sinx = cosx.$

Differential equations are of two types as follows: -

Ordinary Differential Equation

An ordinary differential equation is an equation involving one dependent variables, one independent variable and derivatives of dependent variablewith respect to independent variable.

Mathematically F $(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ...) = 0$

Examples :- $i)$ $\frac{dy}{dx} + y = x^2$

$ii)$ $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + (\frac{dy}{dx})^4 + y = \sin x$

$$iii)\ \frac{dy}{dx} + \text{ytanx} = \text{secx}$$

Partial Differential Equation

A partial differential equation is an equation involving dependent variables, independent variables and partial derivatives of dependent variable with respect to independent variables.

Examples:- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

In this chapter we only discuss about the Ordinary differential equation.

Order and Degree of Differential equation

Order

The order of the differential equation is the highest order of the derivatives occurring in it i.e. order of a differential equation is 'n' if the order of the highest order derivative term present in the equation is n.

Example1: - $\frac{dy}{dx} + y = 2x$

The highest order derivative term in the equation is $\frac{dy}{dx}$, which has order 1.

∴order of the differential equation is 1.

Example-2 : - $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + (\frac{dy}{dx})^4 + y = x$

The highest order derivative term is $\frac{d^4y}{dx^4}$, having order 4.

Hence the above differential equation has order 4.

Degree

A differential equation is said to be of degree 'n' , if the power i.e. highest exponent of the highest order derivative in the equation is 'n' after the equation has been freed from fractions and radicals as far as derivatives are concerned.

Before finding degree of a differential equation, first we have to eliminate those derivative terms present in fraction form i.e. in the denominator and derivatives with radicals i.e $\sqrt{\frac{dy}{dx}}, \sqrt[3]{\frac{dy}{dx}}, \sqrt[4]{\frac{dy}{dx}}$ terms.

Example:- Find the order and degree of following ordinary differential equations.

$\boldsymbol{i})$ $\frac{d^2y}{dx^2}$ = 3$(\frac{dy}{dx})^4$ + $x$$\boldsymbol{ii})$ $(\frac{d^4y}{dx^4})^3$ + $\frac{d^3y}{dx^3}$ + 3$(\frac{dy}{dx})^4$ + cosx =0

$\boldsymbol{iii})$ $\frac{d^2y}{dx^2}$ = $\sqrt{1 + (\frac{dy}{dx})^2}$ **(2017-W)**

$\boldsymbol{iv})$ $[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}}$ = m $\frac{d^2y}{dx^2}$ **(2016-S)**

v) $(\frac{dy}{dx})^2 + \frac{1}{\frac{dy}{dx}}$ = 2

vi) $\frac{\sqrt{1+(\frac{dy}{dx})^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$

Ans: - ***i)*** $\frac{d^2y}{dx^2}$ = 3$(\frac{dy}{dx})^4$ +x

Here $\frac{d^2y}{dx^2}$ is the highest order derivative term.

Hence order of the differential equation is 2.

Again equation does not contain any derivative term in fractional form or with radical.

Power of the highest order derivative term $\frac{d^2y}{dx^2}$ is 1.

Hence degree of differential equation is 1.

ii) $(\frac{d^4y}{dx^4})^3$ + $\frac{d^3y}{dx^3}$ + 3$(\frac{dy}{dx})^4$ + cosx =0

From above it is clear that $\frac{d^4y}{dx^4}$ is the highest order derivative term with power 3.

Hence order = 4 and degree = 3

iii) $\frac{d^2y}{dx^2}$ = $\sqrt{1 + (\frac{dy}{dx})^2}$

{As the above equation contain square root, so first we have to remove square root .}

Squaring both sides we have, $(\frac{d^2y}{dx^2})^2 = 1 + (\frac{dy}{dx})^2$

Now $\frac{d^2y}{dx^2}$ is the highest order term with power 2.

∴ order = 2 and degree = 2 .

iv) $[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}} = m\,\frac{d^2y}{dx^2}$

As power of the left hand side derivative term is $\frac{3}{2}$, so we have to eliminate the fractional power.

Now squaring both sides we have,

$[1 + (\frac{dy}{dx})^2]^3 = (m\,\frac{d^2y}{dx^2})^2$

The power of highest order derivative term $\frac{d^2y}{dx^2}$ is 2.

Hence order = 2 and degree = 2 .

v) $(\frac{dy}{dx})^2 + \frac{1}{\frac{dy}{dx}} = 2$

As $\frac{dy}{dx}$ is present in the denominator of 2nd term in L.H.S. , so we have to remove it first.

Multiplying both side by $\frac{dy}{dx}$ we have,

$(\frac{dy}{dx})^3 + 1 = 2\,\frac{dy}{dx}$

Now the only derivative term $\frac{dy}{dx}$ has power 3.

Hence order = 1 and degree = 3.

vi) $\frac{\sqrt{1+(\frac{dy}{dx})^2}}{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{d^2y}{dx^2}}$

The equation contain both fractional form as well as radicals, so we have to remove it.

1st multiplying $\frac{d^2y}{dx^2}$ on both sides we have,

$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt[3]{\frac{d^2y}{dx^2}}\frac{d^2y}{dx^2}$

Now squaring both sides we have,

$1 + (\frac{dy}{dx})^2 = (\frac{d^2y}{dx^2})^{\frac{2}{3}}(\frac{d^2y}{dx^2})^2$

Again taking cube of both sides we have

$(1 + (\frac{dy}{dx})^2)^3 = (\frac{d^2y}{dx^2})^2(\frac{d^2y}{dx^2})^6 = (\frac{d^2y}{dx^2})^8$

From above the highest order derivative term $\frac{d^2y}{dx^2}$ has power 8.

Hence order = 2 and degree = 8.

Linear and Non-linear Differential Equation

A differential equation is said to be linear if it satisfies following conditions.

i) Every dependent variable and its derivatives have power '1'.

ii) The equation has neither terms having multiplication of dependent variable with its derivatives nor multiplication of two derivative terms.

Otherwise the equation is said to be non linear.

Examples:- i) $\frac{dy}{dx} + xy = x^2$

ii) $\frac{d^3y}{dx^3} + x^4\frac{d^2y}{dx^2} + y = \sin x$

iii) $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 4x$

iv) $(\frac{dy}{dx})^3 + 1 = 2\frac{dy}{dx}$

v) $\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + y = 4x^2$

Among the above examples (i) and (ii) satisfy all the condition of linear equation. So the 1st two equations represent linear equations.

The (iii) is non linear because of the term $y\frac{d^2y}{dx^2}$ which is a multiplication of dependent variable y and derivative term $\frac{d^2y}{dx^2}$.

The example (iv) is not linear due to the 1st term which contain $\frac{dy}{dx}$ with power 3 violating the 1st condition of linearity.

The example (v) is not linear due to 2nd term which does satisfy the 2nd linearity property.

Solutionof a differential equation:-

The relationship between the variables of a differential equation satisfying the differential equation is called a Solution of the differential equation i.e y = f(x) or F(x,y)=0 represent a Solution of the ordinary differential equation F $(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},...,\frac{d^ny}{dx^n})$ = 0 of order n if it satisfy it. There are two types of Solutions i) General Solution ii) particular Solution

General Solution

The Solution of a differential equation containing as many arbitrary constants as the order of the differential equation is called as the general Solution.

Example- y= Acosx+ B sinx is a general Solution of differential equation $\frac{d^2y}{dx^2}$ +y = 0

Particular Solution

The Solution obtained by giving particular values to the arbitrary constants in the general solution is called particular solution

Example: - y= 3 cosx+ 2 sinx is a particular Solution of differential equation $\frac{d^2y}{dx^2}$ +y = 0.

Differential equation of first order and first degree

A differential equation of 1^{st} order and 1^{st} degree involves x,y and $\frac{dy}{dx}$.

Mathematically it is written as $\frac{dy}{dx} = f(x,y)$ or $F(x, y, \frac{dy}{dx}) = 0$

Solution of Differential equation of first order and first degree

The Solution of 1^{st} order and 1^{st} degree differential equation is obtained by following methods if they are in some standard forms as i) Variable separable form ii) Linear differential equation form.

Variable Separable form

If the differential equation is expressed in the form,

$f(x)dy + g(y)dx = 0$, then we say it variable separable form and this can be solved by integrating the terms separately as follows.

Solution is given by $\int \frac{dy}{g(y)} = -\int \frac{dx}{f(x)}$

=>$\log|g(y)| + \log|f(x)| = \log c$

=>$g(y)f(x) = c$

Where $g(y)$ and $f(x)$ are functions of y and x respectively, is called a variable and separable type equation.

Example1: - Solve $\frac{dy}{dx} = x^2 + 2x + 5$

Ans: - $\frac{dy}{dx} = x^2 + 2x + 5$

=> dy = $(x^2 + 2x + 5)dx$

Integrating both sides we have,

=>$\int$ dy = $\int (x^2 + 2x + 5)\, dx$

=> y = $\frac{x^3}{3} + \frac{2x^2}{2}$ + 5x +C= $\frac{x^3}{3} + x^2 + 5x + c$ (Ans)

Example-2: - Solve $\frac{dy}{dx} = \frac{2y}{x^2+1}$.

Ans: - $\frac{dy}{dx} = \frac{2y}{x^2+1}$

=>$\frac{dy}{2y} = \frac{dx}{x^2+1}$

=>$\int \frac{dy}{2y} = \int \frac{dx}{x^2+1}$

=>$\frac{1}{2}\log_e y = tan^{-1}x$ + C

=>$\log_e y = 2tan^{-1}x$ + K { 2C = K is a constant as C is constant}

Example-3: - Solve $\frac{dy}{dx}$ = x cosx

Ans: - $\frac{dy}{dx}$ = x cosx =>$dy = xcosxdx$

Integrating both sides we have, =>$\int dy = \int xcosx\, dx$

=> y = $x \int cosx\, dx - \int \{\frac{d(x)}{dx} \int cosxdx\} dx$ {integrating by parts}

=> y = x sinx – $\int 1.sinxdx$ = x sinx +cosx + C (Ans)

Example-4: - Solve $\frac{dy}{dx} = \sqrt{1-y^2}$

Ans:-$\frac{dy}{dx} = \sqrt{1-y^2}$

=> $\frac{dy}{\sqrt{1-y^2}}$ = dx

=> $\int \frac{dy}{\sqrt{1-y^2}} = \int dx$ { Integrating both sides}

=> $sin^{-1}y = x + c$. (Ans)

Example5: - Solve $sec^2x\, tany\, dx + sec^2y\, tanx\, dy = 0$

Ans: -$sec^2x\, tany\, dx + sec^2y\, tanx\, dy = 0$

=>$sec^2y\, tanx\, dy$ = - $sec^2x\, tany\, dx$

=>$\frac{sec^2y\, dy}{tany}$ = -$\frac{sec^2x\, dx}{tanx}$

=>$\int \frac{sec^2y\, dy}{tany} = -\int \frac{sec^2x\, dx}{tanx}$

Let u = tany => du = $sec^2y\, dy$ and let v= tanx => dv = $sec^2x\, dx$

=>$\int \frac{du}{u} = -\int \frac{dv}{v}$

=> lnu = - lnv + lnC => lnu + lnv = lnC

=> lnuv = lnC => uv = C => tany tanx = C (Ans)

Example-6: - Solve $x\, cos^2y\, dx = y\, cos^2x\, dy$

Ans:-$x\, cos^2y\, dx = y\, cos^2x\, dy$

=>$\frac{ydy}{cos^2y} = \frac{xdx}{cos^2x}$

=>$ysec^2y\, dy = xsec^2x\, dx$

Integrating both sides,

=>$\int ysec^2y\, dy = \int xsec^2x\, dx$

=>$y\int sec^2y\, dy - \int\{\frac{d(y)}{dy}.\int sec^2y\, dy\}dy = \int xsec^2x\, dx - \int\{\frac{d(x)}{dx}.\int sec^2x\, dx\}dx$

=> y tany – $\int 1.tany\, dy$ = x tanx - $\int 1.tanx\, dx$

=> y tany – log |secy| = x tanx – log |secx|+ C

=> y tany – log |secy| -x tanx+ log |secx| = C (Ans)

Example-7: - Solve $(1+y^2)dx + (1+x^2)\, dy = 0$ **(2015-S)**

Ans :- $(1+y^2)dx + (1+x^2)\, dy = 0$

=> $(1+x^2)\, dy$ = - $(1+y^2)dx$

=>$\frac{dy}{1+y^2}$ = - $\frac{dx}{1+x^2}$ =>$\int \frac{dy}{1+y^2} = -\int \frac{dx}{1+x^2}$

=>$tan^{-1}y = -tan^{-1}x + tan^{-1}C$ { as $tan^{-1}C$ can be taken as a constant}

$$=> tan^{-1}y + tan^{-1}x = tan^{-1}C$$

$$=> tan^{-1}\frac{y+x}{1-yx} = tan^{-1}C$$

$$=> \frac{y+x}{1-yx} = C \quad \text{(Ans)}$$

Example-8:- Solve $\frac{dy}{dx}$ = sin(x+y)

Ans :- $\frac{dy}{dx}$ = sin(x+y) { Let x+y = z differentiating w.r.t. x , 1+ $\frac{dy}{dx} = \frac{dz}{dx}$ }

$$=> \frac{dz}{dx} - 1 = \sin z$$

$$=> \frac{dz}{dx} = 1 + \sin z$$

$$=> \frac{dz}{1+\sin z} = dx$$

Integrating both sides we have,

$$=> \int \frac{dz}{1+\sin z} = \int dx$$

$$=> \int \frac{(1-sinz)dz}{(1-\sin z)(1+\sin z)} = \int dx$$

$$=> \int \frac{(1-sinz)dz}{1-sin^2z} = \int dx$$

$$=> \int \frac{(1-sinz)dz}{cos^2z} = \int dx$$

$$=> \int (sec^2z - \frac{sinz}{cosz}\frac{1}{cosz})dz = \int dx$$

$$=> \int (sec^2z - tanz\, secz)dz = \int dx$$

$$=> \tan z - \sec z = x + C$$

$$=> \tan(x+y) - \sec(x+y) - x = C$$

Example-9: - Find the particular solution of $\frac{dy}{dx} = cos^2y$, y = $\frac{\pi}{4}$ when x = 0.

Ans: - $\frac{dy}{dx} = cos^2y$

$$=> \frac{dy}{cos^2y} = dx \qquad => sec^2y\, dy = dx$$

Integrating both sides we have,

$$=> \int sec^2y\, dy = \int dx$$

=> $\tan y = x + C$------------(1) (general Solution)

Now putting $x=o$ and y = $\frac{\pi}{4}$ in equation (1) we have,

=> $\tan\frac{\pi}{4}$ = 0+ C => C = 1---------------------------(2)

From (1) and (2) we have,

$\tan y = x+1$ (Ans)

Example-10:- Find the particular solution of $(1+x)y\,dx + (1\text{-}y)x\,dy = 0$, Given $y=2$ at $x=1$.

Ans:-$(1+x)y\,dx + (1\text{-}y)x\,dy = 0$

=> $(1\text{-}y)x\,dy =$ - $(1+x)y\,dx$

=> $\frac{1-y}{y}\,dy = -\frac{(1+x)}{x}\,dx$

=> $\left(\frac{1}{y} - 1\right)dy = -\left(\frac{1}{x} + 1\right)dx$

=>$\int\left(\frac{1}{y} - 1\right)dy = -\int\left(\frac{1}{x} + 1\right)dx$

=> $\log y - y =$ - $(\log x + x) + C$ (general solution)--------------(1)

Putting x=1 and y =2 in Equation(1) we have,

=> log2 – 2 = - (log1 +1) +C

=> log2 – 2 = - (0 +1) +C =-1 + C

=> C = log2 -1----------------------(2)

From (1) and (2) we have,

$\log y - y =$ - $\log x - x + \log 2$ -1

=> $\log y - y + \log x + x = \log 2$ -1 (Ans)

Linear Differential Equation

A differential equation is said to be linear, if the dependent variable and its derivative occurring in the equation are of first degree only and are not multiplied together.

Example: -i)$\frac{dy}{dx} + y = sinx$

ii)$\frac{dy}{dx} + ytanx = secx$ etc.

General form of linear differential equation

The general form of linear differential equation is given by,

$\frac{dy}{dx} + Py = Q$ is linear in y and $\frac{dy}{dx}$.

Where P and Q are the functions of x only or constants.

This type of differential equation are solved when they are multiplied by a factor, which is called integrating factor (I.F.).

I.F. = $e^{\int P\,dx}$

Then the solution is given by y (I.F) = $\int Q.(I.F.)dx + C$.

If equation is given in the form

$\frac{dx}{dy} + Px = Q$ **,** where P and Q are functions of y only or constants and is linear in x and $\frac{dx}{dy}$, then

I.F. = $e^{\int P\,dy}$

Then the solution is given by x (I.F) = $\int Q.(I.F.)dy + C$.

This can be better understood by following examples.

Example-11: - Solve $(1+x^2)\frac{dy}{dx} + 2xy = x^3$ **(2014-S, 2016-S, 2017-W).**

Ans: -$(1+x^2)\frac{dy}{dx} + 2xy = x^3$

=>$\frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{x^3}{1+x^2}$

By comparing with the general form of linear differential equation $\frac{dy}{dx} + Py = Q$**.**

Here P = $\frac{2x}{(1+x^2)}$& Q = $\frac{x^3}{1+x^2}$

Now integrating factor I.F. = $e^{\int P\,dx}$ = $e^{\int \frac{2x}{(1+x^2)}dx}$ (putting $1 + x^2 = t$ =>$2xdx = dt$)

= $e^{\int \frac{dt}{t}}$ = e^{lnt}= t = $1 + x^2$

Solution is given by

$y \times I.F. = \int Q.(I.F.)dx + C$ = $\int \frac{x^3}{1+x^2}(1+x^2)dx + C$ = $\int x^3 dx + c$

$\Rightarrow \quad y(1+x^2) = \frac{x^4}{4} + c$

$\therefore y = \frac{x^4}{4(1+x^2)} + \frac{c}{1+x^2}$

Example-12: - Solve $\frac{dy}{dx} + y = e^{-x}$

Ans: - By comparing the given equation with general form of linear differential equation we have, P= 1 and Q = e^{-x}

I.F. = $e^{\int P\,dx}$ = $e^{\int 1.dx}$ = e^{x}

Solution is $y.(I.F.) = \int Q.(I.F.)dx\ + C$

$= \int e^{-x}e^{x}dx + c$

=> ye^{x} $= \int 1.dx + c = x + c$

=> y $= xe^{-x} + ce^{-x}$

Example-13: - Solve $(1-x^2)\frac{dy}{dx} - xy = 1$**(2017-S)**

Ans:- $(1-x^2)\frac{dy}{dx} - xy = 1$

=>$\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$

Comparing with general form we have,

P = $-\frac{x}{1-x^2}$ and Q = $\frac{1}{1-x^2}$

Now I.F. = $e^{\int P\,dx}$ = $e^{\int -\frac{x}{1-x^2}dx}$ (Put $1-x^2 = t \Rightarrow\ -2xdx = dt \Rightarrow -xdx = \frac{dt}{2}$)

$= e^{\int\frac{dt}{2t}} = e^{\frac{1}{2}lnt} = e^{ln\sqrt{t}} = \sqrt{t}$

$= \sqrt{1-x^2}$

Solution is $y.(I.F) = \int Q.(I.F.)dx\ + C$

=>$y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}dx\ + C = \int \frac{dx}{\sqrt{1-x^2}} + c$

$= sin^{-1}x + c$

Hence solution of the differential equation is given by

$y = \frac{sin^{-1}x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$ (Ans)

Example-14 ; - Solve $\frac{dy}{dx} + y\, cotx = cosx.$

Ans:- By comparing the given equation with general form,

P = $cotx \quad and\ Q = cosx$

I.F. = $e^{\int P\,dx}$ = $e^{\int cotx\,dx}$ = e^{lnsinx} = $sinx$

solution is given by $y.(I.F.) = \int Q.(I.F.)dx + C$

=>$y.sinx$ = $\int cosx\, sinx\, dx + c$ (put $sinx = t => cosxdx = dt$)

= $\int tdt + c$ = $\frac{t^2}{2} + c$ = $\frac{sin^2x}{2} + c$

Hence $y = \frac{sinx}{2} + \frac{c}{sinx}$ (Ans)

Example-15 ; - Solve $\frac{dy}{dx} + y\, secx = tanx.$ **(2017-W).**

Ans: - Comparing the given equation with general from of linear equation.

P = secx and Q = tanx

I.F. = $e^{\int Pdx}$ = $e^{\int secx\,dx}$

= $e^{\ln(secx+tanx)}$

= $secx + tanx$

The solution is given by, $y.I.F. = \int Q.(I.F.)dx + C$

⇨ $y(secx + tanx)$ = $\int tanx.(secx + tanx)dx + C$

= $\int (secxtanx + tan^2x)dx + C$

= $\int (secxtanx + sec^2x - 1)dx + C$

= $secx + tanx - x + c$

Hence $y = 1 - \frac{x}{secx+tanx} + \frac{c}{secx+tanx}$ (Ans)

Example16: - Solve $(x + y + 1)\frac{dy}{dx} = 1.$

Ans:- $(x + y + 1)\frac{dy}{dx} = 1$

=> $(x + y + 1)dy = dx$

=> $(x + y + 1) = \frac{dx}{dy}$

=> $\frac{dx}{dy} = (x + y + 1)$

=> $\frac{dx}{dy} - x = y + 1$

Now comparing with the general form $\frac{dx}{dy} + Px = Q$, we have,

P = -1 & Q = y+1

Now I.F. = $e^{\int Pdy}$ = $e^{\int -1dy}$ = e^{-y}.

The solution is given by

$x\,(I.F.) = \int Q.(I.F.)dy + c$

⇨ $xe^{-y} = \int (y+1).e^{-y} + c$

= $(y+1)\int e^{-y}dy - \int (\int e^{-y}dy)\left(\frac{d}{dy}(y+1)\right)dy + c$

= $(y+1)(-e^{-y}) - \int -e^{-y}.1.dy + c$

= $-e^{-y}(y+1) + (-e^{-y}) + c$

= $-e^{-y}(y+1+1) + c$ = $-e^{-y}(y+2) + c$

Hence $x = -y - 2 + ce^{y}$ (Ans)

Exercise

Question with short answers (2 marks)

1. Find the order and degree of the differential equations.

i) $a\frac{d^2y}{dx^2} = [1 + (\frac{dy}{dx})^3]^{3/2}$ **(2016-S)**

ii) $\frac{d^2y}{dx^2} = (\frac{dy}{dx})^{2/3}$ **(2019-W)**

iii) $\frac{d^3y}{dx^3} = \sqrt{x + (\frac{dy}{dx})^5}$ **(2015-S,2017-W)**

iv) $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1+\frac{dy}{dx}}}$ **(2014-S)**

v) $2\frac{d^2y}{dx^2} + 3\sqrt{1 - (\frac{dy}{dx})^2 - y} = 0.$ **(2017-S)**

vi) $(\frac{dy}{dx})^2 + y^3 = \frac{d^3y}{dx^3}$

2) Solve the following

i) $\frac{dy}{dx} = e^{x+y}$ **(2019-W)**

ii) $\frac{dy}{dx} = \frac{2y}{x^2+1}$

iii) $\frac{dy}{dx} = tany$

iv) $cos^2x\, dy = cos^2y dx$

v) $ydx = xdy$

Questions with long answers (5 marks)

3) Solve the following

i) $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ **(2019-W)**

ii) $\frac{dy}{dx} + ytanx = secx$ **(2015-S)**

iii) $x(1 + y^2)dx - y(1 + x^2)dy = 0$ **(2015-S)**

iv) $\frac{dy}{dx} - y = e^x$

v) $(x^2-1)\frac{dy}{dx}+2xy=1$.**(2016-S)**

vi) $\frac{dy}{dz}=\frac{\sqrt{1-y^2}}{\sqrt{1-z^2}}$ **(2014-S)**

vii) $\frac{dy}{dx}+ycotx=4xcosecx$ **(2017-S)**

viii)$(x+2y^3)\frac{dy}{dx}=y$.

ix) $(e^y+1)cosxdx+e^y sinxdy=0$.

x) $sinx\ siny\ dy=cosx\ cosy\ dx$

xi) $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

Questions with long answers (10 marks)

4) Solve the following.

i) $e^x\sqrt{1-y^2}\ dx+\frac{y}{x}dy=0$ **(2014-S)**

ii) $cos^2x\frac{dy}{dx}+y=tanx$. **(2017-S)**

iii) $\cos(x+y)\ dy=dx$.

Answers

1) i) 2,2 ii)2,3 iii) 3,2 iv) 2,2 v) 2,2 vi) 3,1

2) i) $-e^y=e^x+c$ ii) $lny=2tan^{-1}x+c$ iii) $\log(siny)=x+c$

iv) tany-tanx = c v) $\frac{y}{x}=c$

3) i) $y(1+x^2)=\frac{4x^3}{3}+c$ ii) $ysecx=tanx+c$ iii) $\frac{1+y^2}{1+x^2}=c$

iv) $y=xe^x+ce^x$ v) $y(x^2-1)=x+c$ vi) $sin^{-1}(y\sqrt{1-z^2}-z\sqrt{1-y^2})=c$

vii)$ysinx=2x^2+c$ viii) $x=y^3+cy$ ix) $sinx(e^y+1)=c$

x) $secy\ cosecx=c$ xi) $cx=ye^{y-x}$

4) i) $\sqrt{1-y^2}=e^x(x-1)+c$ ii) $ye^{tanx}=e^{tanx}(tanx-1)+c$

iii) $y=\tan\left(\frac{x+y}{2}\right)+c$

Multiple Choice Questions

Q.1 The value of α for which the vectors $\vec{a}$ = 3 $\hat{\imath}$ + 2$\hat{\jmath}$ + 9$\hat{k}$ and $\vec{b}$ = $\hat{\imath}$ + α $\hat{\jmath}$ + 3$\hat{k}$ are perpendicular is

a) 10 b) -10 c) 15 d) -15

Q2. If the vectors $\vec{a}$ = α $\hat{\imath}$ + 3$\hat{\jmath}$ - 6$\hat{k}$ and $\vec{b}$ = $\hat{\imath}$ - $\hat{\jmath}$ + 2$\hat{k}$ are parallel then the value of α is

a) -3 b) 3 c)-4 d) 4

Q3. A unit vector in the direction ($\vec{a}$+ $\vec{b}$) where $\vec{a}$ = $\hat{\imath}$ + 2$\hat{\jmath}$ - $\hat{k}$ and $\vec{b}$ = = $\hat{\imath}$ - 2$\hat{\jmath}$ + 3 $\hat{k}$ is equal to

a)$\frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{k}$ b)$\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{k}$ c) $\frac{2}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{k}$ d) None of these

Q4. If points A and B have the following co-ordinates A(3,0,2) , B(-2,1,4) , then the vector AB is

a)5$\hat{\imath}$ + $\hat{\jmath}$ + 2$\hat{k}$ b) -5$\hat{\imath}$ -2$\hat{\jmath}$ + 2$\hat{k}$ c)-5$\hat{\imath}$ + $\hat{\jmath}$ + 2$\hat{k}$ d) -5$\hat{\imath}$ - $\hat{\jmath}$ - 2$\hat{k}$

Q5. A unit vector parallel to the sum of the vectors 3 $\hat{\imath}$ - 2$\hat{\jmath}$ + $\hat{k}$ and -2$\hat{\imath}$ + $\hat{\jmath}$ - 3$\hat{k}$ is

a)$\frac{1}{\sqrt{6}}\hat{\imath} - \frac{1}{\sqrt{6}}\hat{\jmath} - \frac{2}{\sqrt{6}}\hat{k}$ b) $\frac{1}{\sqrt{6}}\hat{\imath} + \frac{1}{\sqrt{6}}\hat{\jmath} - \frac{1}{\sqrt{6}}\hat{k}$ c) $\frac{1}{\sqrt{6}}\hat{\imath} - \frac{2}{\sqrt{6}}\hat{k}$ d) $\frac{1}{\sqrt{6}}\hat{\imath} + \frac{1}{\sqrt{6}}\hat{\jmath}$

Q6. The scalar projection of $\vec{a}$ = ($\hat{\imath}$ - 2$\hat{\jmath}$ + $\hat{k}$) on $\vec{b}$ = (4$\hat{\imath}$ - 4$\hat{\jmath}$ + 7$\hat{k}$) is

a) $\frac{19}{6}$ b) $\frac{19}{3}$ c) $\frac{19}{9}$ d) $\frac{19}{7}$

Q7. The unit vector along $\hat{\imath} + \hat{\jmath} + \hat{k}$ is

a) $\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$ b) $\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{3}$ c) $\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{5}}$ d) None of these

Q8. The value of ($\hat{\imath}$ + 2$\hat{\jmath}$ - $\hat{k}$) . ($\hat{\imath} + \hat{\jmath} + 2\hat{k}$) is

a) -1 b) 1 c) 2 d) -2

Q9. Two forces act on a particle at a point. If they are (4$\hat{\imath}$ + $\hat{\jmath}$ - 3$\hat{k}$) and (3$\hat{\imath}$ + $\hat{\jmath}$ - $\hat{k}$) , then their resultant is

a)7$\hat{\imath}$ + 2$\hat{\jmath}$ - 4$\hat{k}$ b) 7$\hat{\imath}$ - 2$\hat{\jmath}$ - 4$\hat{k}$ c) -7$\hat{\imath}$ + 2$\hat{\jmath}$ - 4$\hat{k}$ d) -7$\hat{\imath}$ - 2$\hat{\jmath}$ - 4$\hat{k}$

Q10. The magnitude of 5$\hat{\imath}$ + 3$\hat{\jmath}$ - 2$\hat{k}$ is

a)$\sqrt{35}$ b) $\sqrt{37}$ c) $\sqrt{38}$ d) $\sqrt{40}$

Q11. The value of $\lim_{x\to 0^+}\frac{|x|}{x}$ is

a)-1 b) 1 c) 0 d) None of these

Q12. The value of $\lim_{x\to\infty} \frac{2x}{3+4x}$ is

a)-1 b) 1 c) $\frac{1}{2}$ d) - $\frac{1}{2}$

Q13. The value of $\lim_{x\to b} \frac{x^2-b^2}{x-b}$ is

a) b b) $\frac{1}{2}b$ c) 2b d) –b

Q14. $\lim_{x\to 2} \frac{x^3-8}{x-2}$ is equal to

a) 4 b) 3 c) 12 d) 6

Q15. The value of $\lim_{x\to 1} \frac{\sqrt{x-1}}{x-1}$ is

a) 0 b) -1 c) 1 d) does not exists

Q16. The value of $\lim_{x\to \frac{5}{2}} [x]$ is

a) 2 b) 3 c) 2.5 d) None of these

Q17. The value of $\lim_{\theta\to 0} \frac{\sin 7\theta}{\sin 5\theta}$ is

a) $\frac{5}{7}$ b) $\frac{7}{5}$ c) 1 d) 0

Q18. If f(x) = $\frac{x^2-16}{x-4}$, x $\neq 4$ is continuous at x = 4, then the value of f(4) is

a) 8 b) 4 c) 10 d) 16

Q19. The value of k for which f(x) = $\frac{sin^2 Kx}{x^2}$, x $\neq$ *0*, f(0) = 1 is continuous at x = 0 is

a) ± 2 b) $\pm \frac{1}{2}$ c) 0 d) ± 1

Q20. The value of $\lim_{x\to 1} \frac{2x^2-3x+1}{x^2-1}$ is

a) $\frac{1}{2}$ b) $\frac{3}{2}$ c) 2 d)1

Q21. The slope of the curve $\mathrm{y} = \frac{5}{3}x^2$ at $\mathrm{x} = 2$

a)$\frac{10}{3}$ b) $\frac{20}{3}$ c) $\frac{5}{3}$ d) $\frac{25}{3}$

Q22. If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, then $\frac{dy}{dx}$ is

a) $\tan x$ b) $\cot x$ c) $sec^2 x$ d) - $cosec^2 x$

Q23. The derivative of x w.r.t tan x is

a) $sec^2 x$ b) $cos^2 x$ c) $-tan^2 x$ d) $-cot^2 x$

Q24. If $x = 4t$, $y = t^2$, then $\frac{dy}{dx}$ is

a) $\frac{2}{x}$ b) $\frac{t}{2}$ c) $\frac{2}{t}$ d) $\frac{x}{2}$

Q25. If $y = \sqrt{1 - \cos 2x}$, then $\frac{dy}{dx}$ is

a) $2\sin 2x$ b) $\sqrt{2}\cos x$ c) - $\sqrt{2}\sin x$ d) None of these

Q26. If $y = \sqrt{1 + sin2x}$, then $\frac{dy}{dx}$ is

a) $\cos x$ b) $\sin x$ c) 0 d) None of these

Q27. The derivative of $\sin x^0$ is

a) $\cos x^0$ b) $\frac{\pi}{180}\cos x^0$ c) $\pi \sec x^0$ d) None of these

Q28. If $y = \log(\tan x)$, then $\frac{dy}{dx}$ is

a) $\frac{1}{\sin 2x}$ b) $\frac{1}{\cos 2x}$ c) $\frac{2}{\sin 2x}$ d) $\frac{2}{\cos 2x}$

Q29. If $y = \sin^{-1} x + \cos^{-1} x$, then $\frac{dy}{dx}$ is

a) 1 b) -1 c) 0 d) 2

Q30. If $x^2 + y^2 = a^2$, then $\frac{dy}{dx}$ is

a) $2x$ b) 0 c) $\frac{x}{y}$ d) $-\frac{x}{y}$

Q31. If $y = \log_e x$ then y_2 is

a) $\frac{1}{x}$ b) $-\frac{1}{x^2}$ c) x d) $-x^2$

Q32. $\frac{d^2y}{dx^2}$ when $y = e^x \sin x$, is

a) $2e^x \cos x$ b) $2e^x \sin x$ c) $e^x \cos x$ d) $e^x \sin x$

Q33. If $y = \ln(\sin x)$, then y_2 is

a) cot x b) tan x c) sec^2x d) $-cosec^2x$

Q34. If $y = e^{\sin^{-1} x}$, then $(1 - x^2)\, y_2 - xy_1$ is equal to

a) y b) -y c) 0 d) None of these

Q35. If $y = \tan^{-1} x$, then $(1 + x^2)\, y_2 + 2xy_1$ is

a) $\frac{1}{1+x^2}$ b) $-\frac{1}{1+x^2}$ c) 1 d) 0

Q36. The function whose 2nd derivative is itself is

a) x b) log x c) e^x d) non of these

Q37. If $f(x,y) = e^{xy}$, then $y.\frac{\partial f}{\partial y} - x.\frac{\partial f}{\partial x}$ is

a) $2x\, e^{xy}$ b) $2ye^{xy}$ c) 0 d) None of these

Q38. If $z = \tan^{-1}(\frac{y}{x})$ then $\frac{\partial z}{\partial x}$ is

a) $\frac{x}{x^2+y^2}$ b) $-\frac{y}{x^2+y^2}$ c) $\frac{1}{x^2+y^2}$ d) $-\frac{1}{x^2+y^2}$

Q39. If $z = f(\frac{x}{y})$, then $\frac{\partial z}{\partial y}$ is

a) $-\frac{x}{y^2}f'(x/y)$ b) $\frac{x}{y^2}f'(x/y)$ c)) $-\frac{1}{y^2}f'(x/y)$ d) None of these

Q40. If $z = x^2y + xy^2$, then $\frac{\partial z}{\partial y}$ is

a) $2xy + y^2$ b) $x^2 + 2xy$ c) 2xy d) 4xy

Q41 . If $z = \sin(x/y)$ then $\frac{\partial z}{\partial x}$ is

a) $\frac{1}{y}\cos(x/y)$ b) $-\frac{x}{y}\cos(x/y)$ c) $\cos(x/y)$ d)) None of these

Q42. $\int sin^2\frac{x}{2}\, dx$ is equal to

a) $\frac{1}{2}[x - \cos x] + c$ b) $\frac{1}{2}[x - \sin x] + c$ c) $(x - \sin x) + c$ d) $(x - \cos x) + c$

Q43. Evaluation of $\int \sqrt{1 - \sin 2x}\; dx$ is

a) $(\sin x + \cos x) + c$ b) $\sin x - \cos x + c$ c) $-\sin x + \cos x + c$ d) None of these

Q44. $\int \frac{x^2}{x^2+1} dx$ is

a)$x + \tan^{-1} x + c$ b) $\tan^{-1} x + c$ c) $2\tan^{-1} x + c$ d) $x - \tan^{-1} x + c$

Q45. $\int \log e^x dx$ is

a)$\frac{x^2}{2} + c$ b) $2x^2 + c$ c) $x^2 + c$ d) $(x^2 + 1) + c$

Q46. The value of 'n' for which $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ is not true is

a)n = 1 b) n = -1 c) n = 0 d) None of these

Q47. $\int \sin\frac{x}{2} . \cos\frac{x}{2}\, dx$ is

a)$\frac{1}{2}\cos x + c$ b) $\frac{1}{2}\sin x + c$ c) $-\frac{1}{2}\cos x + c$ d) $-\frac{1}{2}\sin x + c$

Q48. The value of $\int e^{(\sin^{-1} x + \cos^{-1} x)}\, dx$ is

a)$e^{\frac{\pi}{2}}\, x + c$ b) x +c c) $e^{\frac{\pi}{2}} + c$ d) None of these

Q49. The value of $\int |x| dx$, when x < 0 is

a)$\frac{x^2}{2} + c$ b) $-\frac{x^2}{2} + c$ c) $x^2 + c$ d) $-x^2 + c$

Q50. $\int 2^{x+2} dx$ is

a) $2.\frac{2^x}{\log 2}$ + c b) $4.\frac{2^x}{\log 2}$ +c c) $8.\frac{2^x}{\log 2}$ + c d) None of these

Q51. The value of $\int sin^2 x\, d(\sin x)$ is

a) $\frac{sin^3 x}{3} + c$ b) $\frac{sin^2 x}{2} + c$ c) $\frac{cos^2 x}{2} + c$ d) $\frac{cos^3 x}{3} + c$

Q52. Evaluation of $\int_0^{\frac{\pi}{2}} \sin x\, dx$ is

a) -1 b) 1 c) 0 d) $\frac{1}{2}$

Q53. The value of $\int_1^2 x^3\, dx$ is

a) $\frac{17}{3}$ b) $\frac{15}{4}$ c) $\frac{17}{4}$ d) None of these

Q54. The value of $\int_{-3}^{4} |x| dx$ is

a) $\frac{25}{2}$ b) $\frac{7}{2}$ c) $\frac{9}{2}$ d) $\frac{23}{2}$

Q55. The value of $\int_1^3 [x]dx$ is

a) 1 b) 2 c) 3 d) 4

Q56. The value of $\int_0^1 \frac{dx}{1+x^2}$ is

a) $\frac{\pi}{4}$ b) 0 c) $\frac{\pi}{2}$ d) None of these

Q57. The value of $\int_0^4 \frac{dx}{\sqrt{x}}$ is

a) 6 b) 4 c) 8 d) 10

Q58. The value of $\int_0^1 sin^2 x\, dx + \int_0^1 cos^2\, x\, dx$ - $\int_0^1 dx$ is

a) 0 b) -1 c) 1 d) None of these

Q59. The area bounded by y = x, x = 0 & x = 1 is

a) 1 sq. Unit b) $\frac{1}{2}$ sq. Unit c) $\frac{1}{3}$ sq. Unit d) None of these

Q60. The area bounded by the curve $\mathrm{xy} = k^2$, the x-axis and x = 2, x = 3 is

a) $k^2 \log\frac{2}{3}$ sq. unit b) $k^2 \log 2$ sq. Unit c) $k^2 \log 3$ sq. Unit d) $k^2 \log\frac{3}{2}$ sq. Unit

Q61. The order and degree of the differential equation $(\frac{dy}{dx})^4 + y^5 = \frac{d^3y}{dx^3}$ is

a) 3 and 1 b) 3 and 4 c) 1 and 4 d) 1 and 3

Q62. The order and degree of the differential equation $\frac{d^2y}{dx^2} = \mathrm{k}[1 + (\frac{dy}{dx})^2\ ]$ is

a) 2,2 b) 2, 1 c) 1,2 d) 1,1

Q63. The order and degree of the differential equation $\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$

a) 2,2 b) 2, 1 c) 1,2 d) 1,1

Q64. The degree of the differential equation $\frac{dy}{dx} = \frac{3}{\frac{dy}{dx}}$ is

a) 3 b) 2 c) 1 d) None of these

Q65. The solution of $\frac{dy}{dx} = \frac{x}{y}$ is

a) $x^2 + y^2 = c$ b) $\frac{x^2}{2} + y^2 = c$ c) $y^2 - x^2 = c$ d) $-y^2 - x^2 = c$

Q66. The solution of $\sqrt{4+\frac{dy}{dx}} = 2$ is

a) $y = x + c$ b) $y = c$ c) $y + x = c$ d) $x = c$

Q67. The solution of $\frac{dy}{dx} = sec^2 x$ is

a) $y = 2\sec x + c$ b) $y = \cot x + c$ c) $y = \tan x + c$ d) $y = \text{cosec } x + c$

Q68. The integrating factor of the linear differential equation $\frac{dy}{dx} + (\sec x)y = \tan x$ is

a) sec x + tan x b) cosec x – cot x c) tan x + cot x d) None of these

Q69. The I.F of the linear differential equation $\frac{dy}{dx} + \frac{3}{x}.y = x$ is

a) x^2 b) x^3 c) x^4 d) x

Q70. The solution of the differential equation $\frac{dx}{dy} + \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} = 0$ is

a) $\sin^{-1} x + \sin^{-1} y = c$ b) $\cos^{-1} x - \cos^{-1} y = c$

c) $\sin^{-1} x - \sin^{-1} y = c$ d) None of these

Answers: -

Q1. (d)	Q2. (a)	Q3. (b)	Q4. (c)	Q5. (a)	Q6. (c)
Q7. (a)	Q8. (b)	Q9. (a)	Q10. (c)	Q11. (b)	Q12.(c)
Q13. (c)	Q14. (c)	Q15. (d)	Q16. (a)	Q17. (b)	Q18. (a)
Q19. (d)	Q20. (a)	Q21.(b)	Q22. (c)	Q23. (b)	Q24. (b)
Q25. (b)	Q26.(d)	Q27. (b)	Q28. (c)	Q29. (c)	Q30. (d)
Q31.(b)	Q32.(a)	Q33. (d)	Q34. (a)	Q35.(d)	Q36.(c)
Q37.(c)	Q38. (b)	Q39. (a)	Q40.(b)	Q41.(a)	Q42.(b)
Q43. (a)	Q44.(d)	Q45. (a)	Q46. (b)	Q47.(c)	Q48.(a)
Q49.(b)	Q50.(b)	Q51.(a)	Q52.(b)	Q53. (b)	Q54.(a)
Q55. (c)	Q56. (a)	Q57.(b)	Q58.(a)	Q59.(b)	Q60.(d)
Q61. (a)	Q62. (b)	Q63.(a)	Q64.(b)	Q65.(c)	Q66.(b)
Q67.(c)	Q68.(a)	Q69.(b)	Q70.(a)		