LEARNING MATERIAL

OF

ENGINEERING MATHEMATICS-II

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Unit-1 – Vector

Note:-

Q.1. **What is vector?**

> A physical quanity with specified magnitude as well as definite direction.

2. **How vector represented?**

> A vector generally represented by a directed line segment, say $\vec{AB}$.
> 'A' is called initial point and B is called the terminal point.

3. **Zero vector**

> A vector of zero magnitude is a zero vector. i.e. which has the same initial & terminal point,
> It is denoted as "$\vec{0}$".
> The direction of zero vector is indeterminate. ∴ i.e infinitely many direction
> zero vector is also called null vector.

4. **Unit Vector.**

> vector of unit magnitude
> $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y & Z-axis.
> unit vector along any vector $\vec{a}$ is denoted by $\hat{a}$. and $\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$

5. **Equal Vector**

Two vector are said to be equal if they have the same magnitude, direction & represent the same physical quanity.

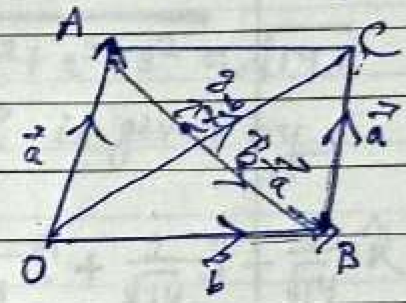
6. **Position vector of a point**

> Let 'O' be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$.

> If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then $\vec{AB} = \vec{b} - \vec{a}$.

7. **Addition of vectors**

If two vector $\vec{a}$ and $\vec{b}$ are represented by $\vec{OA}$ & $\vec{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\vec{OC}$, where OC is the diagonal of the parallelogram OACB



properties

> $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)

> $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associative)

> $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$

> $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

8. **multiplication of a vector by a scalar**

If $\vec{a}$ is a vector & m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times than $\vec{a}$, this multiplication is called scalar multiplication. If $\vec{a}$ & $\vec{b}$ are vector & m, n are scalar, then,

$m(\vec{a}) = (\vec{a})m = m\vec{a}$

$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$

$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

9. **representation of vector in component form**

Let 'O' be the origin and let $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along x, y, & z-axis respectively. Let the position vector of P be $\vec{r}$, then $\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$

x, y, z, are called scalar components of $\vec{r}$.

$x\hat{i}, y\hat{j}, z\hat{k}$ are called its vector components.

10. **Magnitude of vector**

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

Solved Example

• example-1

Find a unit vector in the direction of the vector $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

solution :- $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

Unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Example-2

Find a unit vector in the direction of $\vec{AB}$, where A (1, 2, 3) and B (4, 5, 6) are given points.

solution :-

we have position vector of $A = \hat{i} + 2\hat{j} + 3\hat{k}$

position vector of $B = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$$\therefore \vec{AB} = (\text{P.V. of } B) - (\text{P.V. of } A)$$

$$= (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$|\vec{AB}| = \sqrt{3^2+3^2+3^2} = \sqrt{27}$$

$\therefore$ Unit vector in the direction of $\vec{AB}$ = $\boxed{\frac{\vec{AB}}{|\vec{AB}|} = \hat{AB}}$

$$= \frac{3\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{27}} = \frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}} = \frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$$

Example. 9

If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, find a unit vector in the direction of $\vec{a} + \vec{b}$.

Solution :- Let $\vec{c} = \vec{a} + \vec{b}$, then,

$$\vec{c} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (2\hat{i} + 3\hat{j} + \hat{k})$$
$$= (3\hat{i} + 5\hat{j} - 2\hat{k})$$

$$|\vec{c}| = \sqrt{3^2 + 5^2 + (-2)^2} = \sqrt{38}$$

Hence, the required vector is given by

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{3\hat{i} + 5\hat{j} - 2\hat{k}}{\sqrt{38}} = \frac{3}{\sqrt{38}}\hat{i} + \frac{5}{\sqrt{38}}\hat{j} - \frac{2}{\sqrt{38}}\hat{k}$$

Assignment - 1

Q1. Write down the magnitude and a unit vector in the direction of the following vectors.

(i) $\vec{a} = \hat{i} + 2\hat{j} + 5\hat{k}$ (ii) $\vec{b} = 5\hat{i} - 4\hat{j} - 3\hat{k}$

Q2. Show that the point A, B, C with position vector $\vec{a} = (3\hat{i} - 9\hat{j} - 4\hat{k})$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ respectively form the vertices of a right-angled triangle.

Q3. Find the unit vector in the direction of the vector $\vec{a} + 2b - c$ if $\vec{a} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{b} = -3\hat{i} + 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \hat{k}$.

11. $\boxed{\text{Scalar Product of vectors}}$ (·Dot product)

> Let $\vec{a}$ and $\vec{b}$ be two vectors and let θ be the angle between them, then the scalar product, or dot product, of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\cdot\vec{b}$ is defined as $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$.

N.B.: the scalar product of two vectors is a scalar.

12. > $\boxed{\text{Angle between two vector in terms of scalar product}}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)}.$$

13. N.B.:- (i) If $\vec{a}$ and $\vec{b}$ are like vectors $\theta = 0$

$$\vec{a}\cdot\vec{b} = ab\cos 0 = ab \quad (\cos 0 = 1)$$

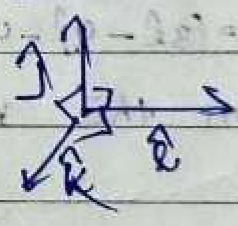
$$\boxed{\vec{a}\cdot\vec{a} = a^2}$$

(ii) If $\boxed{\vec{a} \perp \vec{b}}$, $\theta = 90$ $\Rightarrow \boxed{\vec{a}\cdot\vec{b} = 0}$

$$\vec{a}\cdot\vec{b} = \vec{a}\,\vec{b}\cos 90 = 0$$

(iii) $\boxed{\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0}$

$\boxed{\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1}$



14. > $\boxed{\text{Working Rule}}$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1\hat{i}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1\,\hat{i}\cdot\hat{i} + a_2b_2\,\hat{j}\cdot\hat{j} + a_3b_3\,\hat{k}\cdot\hat{k}$$

$$\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Examples

Q1. Find the angle between the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$?

Solution :- $|\vec{a}| = \sqrt{3^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$

Now $\vec{a} \cdot \vec{b} = 3 \cdot 1 + (-2)(-2) + (1)(3)$

$= 3 + 4 + 3 = 10$

$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{10}{\sqrt{14}\sqrt{14}}\right) = \cos^{-1}\left(\frac{10}{14}\right)$

$\Rightarrow \boxed{\theta = \cos^{-1}\frac{5}{7}}$

Q2. Find the value of λ for which the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are $\perp$ to each other.

Solution :- $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$\Rightarrow (3\hat{i} + \hat{j} - 2\hat{k}) \cdot (\hat{i} + \lambda\hat{j} - 3\hat{k}) = 0$

$\Rightarrow 3 \times 1 + 1 \times \lambda + (-2)(-3) = 0$

$\Rightarrow 3 + \lambda + 6 = 0$

$\Rightarrow \lambda + 9 = 0$

$\Rightarrow \boxed{\lambda = -9}$

15. **Scalar project and vector projection**

(i) scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ (scalar quanity)

(ii) vector projection of $\vec{a}$ on $\vec{b} \cdot \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$.

N.B = [(S.P) of $\vec{a}$ on $\vec{b}$] $\times$ $\underset{\text{into}}{\hat{b}}$ = V.P of $\vec{a}$ on $\vec{b}$.

Example 1.1

find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution :- scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + 3\hat{j} + 2\hat{k})(\hat{i} + 2\hat{j} + \hat{k})$$
$$= 2 \times 1 + 3 \times 2 + 2 \times 1$$
$$= 2 + 6 + 2$$
$$= 10$$

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$s \cdot p = \frac{10}{\sqrt{6}}$$

vector projection of $\vec{a}$ on $\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right) \hat{b}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right) \vec{b}$$
$$= \frac{10}{(\sqrt{6})^2} (\hat{i} + 2\hat{j} + \hat{k})$$
$$= \frac{\cancel{10}}{\cancel{6}} (\hat{i} + 2\hat{j} + \hat{k})$$
$$= \frac{5}{3} (\hat{i} + 2\hat{j} + \hat{k})$$

vector projection of $\vec{a}$ on $\vec{b} = \frac{5\hat{i}}{3} + \frac{10}{3}\hat{j} + \frac{5\hat{k}}{3}$.

Assignment - 2.

Q1. Find the value of λ for which $\vec{a}$ & $\vec{b}$ are perpendicular, where $\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$.

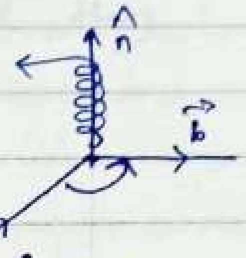
Q2. Find the angle between the vector $\vec{a}$ and $\vec{b}$ when, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Q3. Let $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$, find the projection of (i) $\vec{a}$ on $\vec{b}$ and (ii) $\vec{b}$ on $\vec{a}$.

16. **Vector (cross) product**

> Let $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors, and let θ be the angle between them such that $0 < \theta < \pi$.

Then the vector product is defined as



- $\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}}$.

>> $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \hat{n}$ form a right-handed system.

17. >> **sine Angle between two vectors**

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \quad [\because |\hat{n}| = 1 \text{ as a unit vector}]$$

$$\Rightarrow \boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}}$$

18. **Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$**

$$\boxed{\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}}$$

19. **properties**

① $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$ (not commutative)

② $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

③ $\vec{a} \parallel \vec{b} \Rightarrow \boxed{\vec{a} \times \vec{b} = 0}$

- N.B. $\hat{i} \times \hat{j} = \hat{j} \times \hat{k} = \hat{k} \times \hat{i} = 1$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

20. **Application of cross product**

(i) Area of parallelogram with two adjacent sides $\vec{a}$ & $\vec{b}$ is given by $|\vec{a} \times \vec{b}|$ sq units

(ii) Area of a parallelogram with diagonal $\vec{u}$ & $\vec{v}$ $= \frac{1}{2}|\vec{u} \times \vec{v}|$ sq. units

(iii) Area of ΔABC; where $\vec{AB} = \vec{c}$, $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ or, $\frac{1}{2}|\vec{b} \times \vec{c}|$ or, $\frac{1}{2}|\vec{c} \times \vec{a}|$ sq units.

21. **Working Rule of vector product (in terms of components)**

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Solved Examples

Example-1 If $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$. find $\vec{a}\times\vec{b}$ and $|\vec{a}\times\vec{b}|$.

Solution :-

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$$= (-2+20)\hat{i} - (-6+24)\hat{j} + (15-6)\hat{k}$$

$$= 18\hat{i} - 18\hat{j} + 9\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{18^2 + (-18)^2 + 9^2} = \sqrt{729} = 27$$

Example-2 :- find a Unit vector $\perp^{er}$ to each one of the vectors $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$.

Solution :-

We know that $\vec{a}\times\vec{b}$ is a vector $\perp^{er}$ to each one of $\vec{a}$ and $\vec{b}$.

so Unit vector $\perp^{er}$ to both $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix} = (1-6)\hat{i} - (-4-6)\hat{j} + (8+2)\hat{k}$

$= -5\hat{i} + 10\hat{j} + 10\hat{k}$

$$|\vec{a}\times\vec{b}| = \sqrt{(-5)^2 + (10)^2 + (10)^2} = \sqrt{225} = 15$$

Hence the required unit vector $= \dfrac{-5\hat{i}+10\hat{j}+10\hat{k}}{15}$

$$= \frac{5}{15}(-\hat{i}+2\hat{j}+2\hat{k})$$
$$= \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$$

__Example-3__

Find the sine angle between the vectors $\vec{a} = 2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b} = \hat{i}+3\hat{j}+2\hat{k}$.

__Solution:-__ $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (-2-9)\hat{i} - (4-3)\hat{j} + (6+1)\hat{k}$$
$$= -11\hat{i} - \hat{j} + 7\hat{k}.$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-11)^2 + (-1)^2 + 7^2} = \sqrt{171} = 3\sqrt{19}$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (3)^2} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\therefore \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{3\sqrt{19}}{\sqrt{14}\sqrt{14}} = \frac{3}{14}\sqrt{19} \text{ (Ans).}$$

__Example-4 :-__

Find the area of the parallelogram whose adjacent sides are represented by the vector $(3\hat{i}+\hat{j}-2\hat{k})$ and $\hat{i}-3\hat{j}+4\hat{k}$.

__Solution:-__

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k}$$
$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

Required area $= |\vec{a}\times\vec{b}|$

$= \sqrt{(-2)^2+(-14)^2+(-10)^2}$ sq units

$= \sqrt{300}$ sq units $= 10\sqrt{3}$ sq units.

Example-5 :-

Find the area of the parallelogram whose diagonals are represented by the vectors $\vec{d_1} = (2\hat{i}-\hat{j}+\hat{k})$ and $d_2 = (3\hat{i}+4\hat{j}-\hat{k})$

Solution :- Given that $\vec{d_1} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{d_2} = 3\hat{i}+4\hat{j}-\hat{k}$.

vector area of parallelogram is $\frac{1}{2}(d_1\times d_2)$

$$\text{Now} = (d_1\times d_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$= (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$

$= -3\hat{i}+5\hat{j}+11\hat{k}$

Required area $= \frac{1}{2}|d_1\times d_2|$

$= \frac{1}{2}\sqrt{(-3)^2+(5)^2+(11)^2}$ sq units

$= \frac{1}{2}\sqrt{155}$ sq units

Example-6 :-

Using vector method, find the area of the triangle whose vertices are A(1,1,1), B(1,2,3) and C(2,3,1).

Solution :-

position vector of $A = \hat{i}+\hat{j}+\hat{k}$

position vector of $B = \hat{i}+2\hat{j}+3\hat{k}$

position vector of $C = 2\hat{i}+3\hat{j}+\hat{k}$

$\therefore \vec{AB} = $ (p.v of B) $-$ (p.v of A)

$= (\hat{i}+2\hat{j}+3\hat{k}) - (\hat{i}+\hat{j}+\hat{k}) = \hat{j}+2\hat{k}$

And $\vec{AC} = $ (p. v of C) − (p. v of A)

$= (2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + \hat{j} + \hat{k})$

$= \hat{i} + 2\hat{j}$.

Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

Now $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$= (0-4)\hat{i} + (2-0)\hat{j} + (0-1)\hat{k} = -4\hat{i} + 2\hat{j} - \hat{k}$

$\Rightarrow \frac{1}{2}|(\vec{AB} \times \vec{AC})| = \frac{1}{2}|-4\hat{i} + 2\hat{j} - \hat{k}|$

$= \frac{1}{2}\sqrt{(-4)^2 + (2)^2 + (-1)^2}$

$= \frac{1}{2}\sqrt{16 + 4 + 1}$

$= \frac{1}{2}\sqrt{21} = \frac{\sqrt{21}}{2}$ sq units (Ans)

Assignment - 3

Q1. Find unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$, when.

(i) $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$

(ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$.

Q2. Find the area of the parallelogram whose adjacent sides are represented by the vectors.

(i) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$.

(ii) $\vec{a} = 2\hat{i} + \hat{j}$ and $\vec{b} \cdot \hat{j} - 3\hat{k}$.

Q3. Find the area of the parallelogram whose diagonals are represented by the vectors.

(i) $\vec{d_1} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{d_2} = \hat{i} - 3\hat{j} + 4\hat{k}$.

(ii) $\vec{d_1} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{d_2} = 3\hat{i} + 4\hat{j} - \hat{k}$.

Q4. Using vector method find the area of ΔABC, whose vertices are $A(3,-1,2)$, $B(1,-1,-3)$ and $C(4,-3,1)$.

Unit-2 (Limit & Continuity)

1. **Limit**

Let $y = f(x)$ be a function of x.
If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which is very near to 'a'.
If these values tends to a definite unique number as x tends to 'a', then the unique number, so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\boxed{\lim_{x \to a} f(x)}$

2. N.B.:- (Indeterminate forms)

If the value of $f(x)$ at $x = a$ comes out to one of the following form:-
$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^{0}$ and ∞^{0} ;

then, it is called indeterminate form.

3. **Evaluation of limits**:- (by direct substitution)

To find $\lim_{x \to a} f(x)$; we substitute $x = a$ in the function, if the value comes out to be a definite value, then it is the limit.
i.e. $\boxed{\lim_{x \to a} f(x) = f(a)}$ provided it exists.

Example :-

(i) $\lim_{x \to 2} 2x + 3 = 2(2) + 3 = 4 + 3 = 7$

(ii) $\lim_{x \to 2} \frac{x^2 + 3}{x - 1} = \frac{2^2 + 3}{2 - 1} = \frac{7}{1} = 7$

(iii) $\lim_{x \to 3} \frac{x^3 + 3}{x} = \frac{3^3 + 3}{3} = \frac{27 + 3}{3} = \frac{30}{3} = 10$

Rules of limit / (Algebra of limits)

(1) $\lim_{x \to a} K = K$.

(2) $\lim_{x \to a} K f(x) = K \lim_{x \to a} f(x)$

(3) $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(4) $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(6) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

Example.

(1) $\lim_{x \to 2} 5 = 5$

(2) $\lim_{x \to 3} (6x + 1) = \lim_{x \to 3} 6x + \lim_{x \to 3} 1 = 6 \lim_{x \to 3} x + 1$

$= (6 \times 3) + 1 = 18 + 1 = 19$

(3) $\lim_{x \to 5} 3x + x^2 - 5 = 3(5) + 5^2 - 5 = 15 - 25 - 5$
$= 15 - 30$
$= -15$ (Ans)

(4) $\lim_{x \to 2} e^x \sin x = e^2 \sin 2 = (2.17)^2 \times 0.035 = 0.075$

(5) $\lim_{x \to 3} \frac{x^2}{\log x} = \frac{3^2}{\log 3} = \frac{9}{0.4771} = 18.86$

Evaluation of limit (Indeterminate forms)

· These limit can be determined by some special rule for some standard function.

limit by factorisation :- $\left(\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1} \text{ form}\right)$

If the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ attains Indeterminate form.

... $x-a$ must be a factor of numerator and denominator which can be cancelled out.

e.g. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{\cancel{(x-a)}(x^{n-1}+x^{n-2}a+\cdots+a^{n-1})}{\cancel{(x-a)}}$

$= \lim_{x\to a}\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}\right)$

$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + a^{n-1}$

$= a^{n-1} + a^{n-1} + a^{n-1} + \cdots$ n terms

$\Rightarrow \boxed{\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}}$

Example - 1.

$\lim_{x\to 2}\frac{x^2-4}{x-2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 2}\frac{x^2-2^2}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2}(x+2) = 2+2$

$= 4$ (Ans)

Example-2

$\lim_{x\to 3}\frac{x^3-27}{x-3}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 3}\frac{x^3-3^3}{x-3} = \lim_{x\to 3}\frac{(x-3)(x^2+3\cdot x+3^2)}{x-3}$

$= 3^2+3\cdot 3+3^2 = 3(3^2) = 27$ (Ans).

(ii) Example - 3

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{x^4-2^4}{x-2} = 4(2)^{4-1} = 4(2)^3$$

$$= 4\times 8 = 32$$

(Applying formulae $\left[\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$)

Example - 4

$$\lim_{x\to 1} \frac{x^n-1}{x-1} = n(1)^{n-1} = 1 \text{ (Ans)}$$

(ii) simplification :- (limit of polynomial and rational function)

(i) for $\lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{g(a)}{h(a)}$, for $\frac{g(a)}{h(a)}$ is a definite value.

(ii) If $\frac{g(a)}{h(a)}$ is not definite, then we have to factorise it.

example $\lim_{x\to 2} \frac{x^2-4}{x^3-4x^2+4x} \left(\frac{0}{0}\right) = \lim_{x\to 2} \frac{(x+2)(x-2)}{x(x^2-4x+4)}$

$$\lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{x(x-2)^2} = \lim_{x\to 2} \frac{x+2}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0} = \infty$$

(iii) limit at infinite :- (Dividing highest powered x^n or x^m)

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_n x^m + b_{n-1}x^{m-1} + \cdots + b_1 x + b_0}$$

(i) if $m > n$, devide x^m with each term, then put x as '∞' to find the value.

(ii) if $m < n$, devide x^n

(iii) If $m = n$, devide any x^m or x^n as both are equal.

Example -1

$$\lim_{x \to \infty} \frac{4x^3+3x^2+1}{5x^2+2x+1}$$

Sol^n

$$\lim_{x \to \infty} \frac{\frac{4x^3}{x^3}+\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{5x^2}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}} = \lim_{x \to \infty} \frac{4+\frac{2}{x}+\frac{1}{x^3}}{\frac{5}{x}+\frac{2}{x^2}+\frac{1}{x^3}}$$

$$= \frac{4+\frac{2}{\infty}+\frac{1}{\infty}}{\frac{5}{\infty}+\frac{2}{\infty}+\frac{1}{\infty}} = \frac{4+0+0}{0+0+0} = \frac{4}{0} = \infty \text{ (Ans)}$$

Example- 2

$$\lim_{x \to \infty} \frac{x^2+2x+1}{x^3+4x^2+3x+2}$$

Sol^n

$$\lim_{x \to \infty} \frac{\frac{x^2}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}+\frac{4x^2}{x^3}+\frac{2x}{x^3}+\frac{2}{x^3}} = \frac{\frac{1}{x^2}+\frac{2}{x^2}+\frac{1}{x^3}}{1+\frac{4}{x}+\frac{2}{x^2}+\frac{2}{x^3}}$$

$$= \frac{\frac{1}{\infty}+\frac{2}{\infty}+\frac{1}{\infty}}{1+\frac{4}{\infty}+\frac{2}{\infty}+\frac{2}{\infty}} = \frac{0}{1} = 0 \text{ (Ans)}.$$

Example-3

$$\lim_{x \to \infty} \frac{4x^3+x^2+2x+1}{2x^3+3x+2}$$

Sol^n

$$\lim_{x \to \infty} \frac{\frac{4x^3}{x^3}+\frac{x^2}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}{\frac{2x^3}{x^3}+\frac{3x}{x^3}+\frac{2}{x^2}} = \lim_{x \to \infty} \frac{4+\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}}{2+\frac{3}{x^2}+\frac{2}{x^3}}$$

$$= \frac{4+\frac{1}{\infty}+\frac{2}{\infty}+\frac{1}{\infty}}{2+\frac{3}{\infty}+\frac{2}{\infty}} = \frac{4}{2} = 2 \text{ (Ans)}.$$

Assignments - 1(A)

(1) $\lim_{x\to 3} x+3$.

(2) $\lim_{x\to 4} \frac{4x+3}{x-2}$

(3) $\lim_{x\to -1} \frac{x^{10}+x^5+1}{x-1}$

(4) $\lim_{x\to 0} \frac{(x+1)^5-1}{x}$

(5) $\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4}$

(6) $\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3}$

(7) $\lim_{x\to 1} \frac{ax^2+bx+c}{cx^2+bx+a}$

(8) $\lim_{x\to 1} \frac{x^{15}-1}{x^{10}-1}$

(9) $\lim_{x\to 2} \frac{x^3-4x^2-4x}{x^2-4}$

(10) $\lim_{x\to 2} \frac{x^3-2x^2}{x^2-5x+6}$

(11) $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-6x+5}$

(12) $\lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{x-5}$

Assignment - 1(B)

(1) $\lim_{x\to 4} \frac{x^2-16}{x-4}$

(2) $\lim_{x\to b} \frac{x^8-b^8}{x^3-b^3}$

(3) $\lim_{x\to 1} \frac{x^{1/m}-1}{x^{1/n}-1}$

(4) $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$.

(5) .

Assignment - 1(C)

(1) $\lim_{x\to\infty} \frac{3x^2+4x-1}{2x^2+x+2}$

(2) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$

(3) $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$

(4) $\lim_{n\to\infty} \frac{1^3+2^3+\cdots+n^3}{n^4}$

(5) $\lim_{n\to\infty} \frac{n!}{(n+1)!-n!}$

(6) .

Limit of Some (standard function) / (standard forms)

1) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

proof $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{\cancel{(x-a)}\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}\right)}{\cancel{x-a}}$

$= \lim_{x \to a} \left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}\right)$

$= a^{n-1} + a^{n-2}.a + a^{n-3}.a^2 + \cdots + a^{n-1}$

$= a^{n-1} + a^{n-1} + a^{n-1} + \cdots$ n terms

$= n a^{n-1}$ (proved)

2)

Example-1

$\lim_{x \to 3} \frac{x^3 - 27}{x - 3} = \lim_{x \to 3} \frac{x^3 - 3^3}{x - 3} = 3(3)^{3-1} = 3 \times 3^2$

$= 27$ (Ans).

Example-2

$\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x \to 3} \frac{x^3 - 3^3}{(x-3)(x+3)} = \left(\lim_{x \to 3} \frac{x^3 - 3^3}{x - 3}\right) \times \left(\lim_{x \to 3} \frac{1}{x+3}\right)$.

$= 3 \times (3)^{3-1} \times \frac{1}{3+3} = 3 \times 3^2 \times \frac{1}{6_2} = \frac{9}{2}$ (Ans)

Example-3

$\lim_{x \to 25} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{5}}}{x - 25} \left(\frac{0}{0}\right)$ form

$\Rightarrow \lim_{x \to 25} \frac{\frac{\sqrt{5} - \sqrt{x}}{\sqrt{5x}}}{x - 25} = \lim_{x \to 25} \frac{\sqrt{5} - \sqrt{x}}{x - 25} \times \frac{1}{\sqrt{5x}}$

$= \lim_{x \to 25} \left(\frac{\sqrt{5} - \sqrt{x}}{\sqrt{5+x}} \times \frac{\sqrt{5} + \sqrt{x}}{x - 25} \times \frac{1}{\sqrt{5x}}\right)$

$= \lim_{x \to 25} \left(\frac{5 - x}{\sqrt{5+x}} \times \frac{1}{x - 25} \times \frac{1}{(\sqrt{5}+\sqrt{x})\sqrt{5x}}\right)$

$$\lim_{x \to 25} \frac{5-25}{\sqrt{5}+25} \times \frac{1}{25-25} \Big/ (\sqrt{5}+25)$$

$$= \frac{-20}{0} = \infty \quad (\text{Ans})$$

Example - 3

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{x+h-x}.$$

$$= \lim_{x+h \to x} \frac{(x+h)^3 - x^3}{x+h-x} \qquad \text{Let } x+h = y$$

$$= \lim_{y \to x} \frac{y^3 - x^3}{y-x}. \qquad \left[\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= 3(x)^{3-1}$$

$$= 3 \cdot x^2 \quad (\text{Ans}).$$

11) (a) $\boxed{\lim_{x\to 0} \frac{a^x-1}{x} = \log_e a}$ (b) $\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = \log_e e = 1}$

<u>proof</u>

Let $a^x - 1 = y$

$\Rightarrow a^x = y+1$

$\Rightarrow \log_e a^x = \log_e(1+y)$

$\Rightarrow x\log_e a = \log_e(1+y)$

$\Rightarrow x = \frac{\log_e(1+y)}{\log_e a}$

As $x \to 0 \Rightarrow y = a^0 - 1 = 1-1 = 0$

As $x \to 0 \Rightarrow y \to 0$

Now

$$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{y\to 0} \frac{y}{\frac{\log_e(1+y)}{\log_e a}}$$

$$= \lim_{y\to 0} \log_e a \frac{y}{\log(1+y)}$$

$$= \log_e a \left[\lim_{y\to 0} \frac{y}{\log(1+y)}\right]$$ [$\log_e a$ is a constant]

$$= \log_e a \left[\frac{1}{\lim_{y\to 0} \frac{\log(1+y)}{y}}\right]$$

$$= \log_e a \left[\frac{1}{\lim_{y\to 0} \frac{1}{y} \log(1+y)}\right]$$

$$= \log_e a \frac{1}{\lim_{y\to 0} \log_e(1+y)^{1/y}} = \log_e a \frac{1}{\log_e \lim_{y\to 0}\left[(1+y)^{1/y}\right]}$$

$$= \log_e a \frac{1}{\log_e e} \quad \left[\lim_{y \to 0} (1+y)^{y} = e \right]$$

$$= \log_e a \text{ (Ans)} \quad \left[\log_e e = 1 \right]$$

problems : [Solved Examples]

(1) $\lim_{x \to 0} \frac{a^x - b^x}{x}$

solution :- $\lim_{x \to 0} \frac{a^x - 1 - b^x + 1}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$

$$= \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b$$

$$= \log\left(\frac{a}{b}\right)$$

(2) $\lim_{x \to 0} \frac{3^x - 2^x}{x}$

solution :- $\lim_{x \to 0} \frac{3^x - 2^x - 1 + 1}{x} = \lim_{x \to 0} \left[\frac{3^x - 1}{x} - \frac{2^x - 1}{x}\right]$

$$= \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right) - \lim_{x \to 0} \left(\frac{2^x - 1}{x}\right)$$

$$= \log 3 - \log 2$$

$$= \log(3/2) \text{ (Ans)}$$

(iii) $\boxed{\lim_{x \to 0} \frac{\log(1+x)}{x} = 1}$

proof :- $\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}{x}$

$$= \lim_{x \to 0} 1 - \frac{x}{2} + \frac{x^2}{3} \cdots\cdots = 1.$$

OR $\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \log(1+x)^{1/x} = \log \lim_{x \to 0} (1+x)^{1/x}$

$$= \log e = 1$$

Solved Examples

(1). $\lim_{x\to 1} \frac{x-1}{\log_e x} = \frac{1}{\lim_{x\to 1} \frac{\log x}{x-1}} = \frac{1}{\lim_{x-1\to 0} \log \frac{x}{x-1}}$

$x-1 = y \Rightarrow x = 1+y$

$x \to 1 \Rightarrow y = 0$

$$\frac{1}{\lim_{y\to 0} \frac{\log(1+y)}{y}} = \frac{1}{1} = 1 \text{ (Ans)}.$$

Assignment - 2

(1) $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

(2) $\lim_{x\to 0} \frac{a^{mx} - b^{nx}}{x}$

(3) $\lim_{x\to 0} \frac{8^x - 2^x}{x}$

(4) $\lim_{x\to 0} \frac{e^{3x} - e^{2x}}{x}$

(5) $\lim_{x\to 2} \frac{x-2}{\log_a(x-1)}$

(6) $\lim_{x\to 0} \frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

(7) $\lim_{x\to a} \frac{\log x - \log a}{x-a}$.

(8) $\lim_{x\to 0} \frac{e^{bx} - e^{ax}}{x}$

(iv) (a) $\boxed{\lim_{x\to 0} (1+x)^{1/x} = e}$

(b) $\boxed{\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e}$

Solved Example

(1) $\lim_{x\to 0} (1+2x)^{1/x}$

$= \lim_{x\to 0} (1+2x)^{\frac{1}{2x}\times 2}$

$= \left[\lim_{x\to 0} (1+2x)^{1/2x}\right]^2$

$= e^2$ (Ans)

(2) $\lim_{x\to \infty} \left(1+\frac{1}{3x}\right)^x$

$= \lim_{x\to \infty} \left(1+\frac{1}{3x}\right)^{\frac{3x}{3}}$

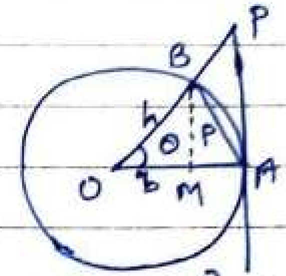
$= \left[\lim_{x\to \infty} \left(1+\frac{1}{3x}\right)^{3x}\right]^{1/3}$

$= e^{1/3}$ (Ans).

(V) (a) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ (b) $\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$

proof

(a) $\lim_{x\to 0} \frac{\sin x}{x}$.



Consider a circle of radius 'r',
'O' be the centre of the circle.
$\angle AOB = \theta$, measured in radians
Tangent produced at P meets at 'A'.
from figure, we can observe that

$\sin\theta = \frac{p}{h}$
$\Rightarrow \boxed{p = h\sin\theta}$

Area of ΔOAB < Area of sector OAB < Area of ΔOAP

$\frac{1}{2}(\text{base } OA \times \text{height } BM) < \frac{1}{2}(\text{radius } OA)^2 \times \theta < \frac{1}{2}(\text{base } OA \times \text{height } AP)$

$= \frac{1}{2}(\overline{OA} \times \overline{OB}\sin\theta) < \frac{1}{2} r^2\theta < \frac{1}{2}\, OA \cdot OA\tan\theta$

$= \frac{1}{2} r \times r\sin\theta < \frac{1}{2} r^2\theta < \frac{1}{2} r^2 \tan\theta$

In ΔOAP: $\tan\theta = \frac{p}{b}$
$\Rightarrow \tan\theta = \frac{AP}{OA}$
$\Rightarrow AP = OA\tan\theta$
$(OA = OB = r)$

$\Rightarrow \frac{1}{2} r^2\sin\theta < \frac{1}{2} r^2\theta < \frac{1}{2} r^2\tan\theta$

$\Rightarrow \sin\theta < \theta < \tan\theta$

$\Rightarrow \frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$

$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta \cdot \sin\theta}$

$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$.

$\Rightarrow \lim_{\theta\to 0} 1 > \lim_{\theta\to 0} \frac{\sin\theta}{\theta} > \lim_{\theta\to 0} \cos\theta$

$\Rightarrow 1 < \lim_{\theta\to 0} \frac{\sin\theta}{\theta} < 1 \Rightarrow \boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$

(b) ~~similarly~~

we have, $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{\theta \to 0} \frac{1}{\cos\theta} = 1 \times \frac{1}{1} = 1$

Solved Example

Q1. $\lim_{x \to 0} \frac{\sin 3x}{x}$.

$= \lim_{x \to 0} \left(3 \times \frac{\sin 3x}{3x} \right) = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3 \times (1) = 3$

Q2. $\lim_{x \to 0} \frac{\sin 5x}{2x}$

$= \lim_{x \to 0} \left(\frac{\sin 5x}{5x} \cdot 5x \right) \times \frac{1}{2x}$

$= \left(\lim_{x \to 0} \frac{\sin 5x}{5x} \right) \times \frac{5x}{2x}$ $\boxed{\because \lim_{x \to 0} \frac{\sin 5x}{5x} = 1}$

$= \frac{5}{2} \times (1) = 5/2$ (Ans).

Q3. $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$.

$= \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \cdot ax}{\frac{\sin bx}{bx} \cdot bx} = \frac{a}{b} \frac{\lim_{x \to 0} \left(\frac{\sin ax}{ax} \right)}{\lim_{x \to 0} \left(\frac{\sin bx}{bx} \right)} = \frac{a \cdot (1)}{b(1)} = \frac{a}{b}$

Q4. $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2 \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2$

$= 2 \times (1)^2 = 2$ (Ans)

· Assignment - 3

Q1. $\lim_{x \to 0} \frac{\sin 3x}{5x}$

Q2. $\lim_{x \to 0} \frac{\sin x^\circ}{x}$

[Hints $180^\circ = \pi^c$
$1^\circ = \pi/180^\circ$
$x^\circ = \frac{\pi x}{180}$]

Q3. $\lim_{x \to 0} \frac{\tan mx}{\tan nx}$

Q4. $\lim_{x \to 0} \frac{\tan 8x}{\sin 2x}$

Q5. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

Q6. $\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$

Q7. $\lim_{x \to \pi/2} \frac{\cos x}{\pi/2 - x}$

Q8 $\lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x}$.

Q9. $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

Q10. $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$.

Continuity

A real function $f(x)$ is said to be **continuous** at a point a if $\lim_{x\to a} f(x)$ exist and equals $f(a)$.

If $f(x)$ is not continuous at a point, it is said to be discontinuous at that point.

working Rule

$f(x)$ is continuous at $x = $ 'a' if

(i) $f(a)$ has definite value at $x = a$

(ii) $\lim_{x\to a} f(a)$ exist.

(iii) $\lim_{x\to a} f(x) = f(a)$

N.B $\boxed{\text{Existance of limit}}$

$\lim_{x\to a} f(x)$ exists iff $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$

both exist and are equal.

$\boxed{\text{Evaluation of Left hand limit } \left(\lim_{x\to a^-} f x\right)}$

"$x \to a^-$" means that x is tending to a from the left hand side, i.e. x is a number less than a but very very close to 'a'.

Steps

> write $\lim_{x\to a^-} f(x)$

> put $x = a-h$ and replace $x\to a^-$ by $h\to 0$. to obtain $\lim_{h\to 0} f(a-h)$

> Simplify $\lim_{h\to 0} f(a-h)$ by using the formula for the given function.

E.g.

Evaluate left hand limit of the function.

$$f(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases} \quad \text{at } x = 4.$$

Solution:-

(L.H.L. of $f(x)$ at $x = 4$)

$$\lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4-h)$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{4-h-4}$$ [Replacing x by $4-h$ in $f(x) = \frac{|x-4|}{x-4}$]

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \left(\frac{h}{-h}\right)$$ [$\because h > 0$]

$$= \lim_{h \to 0} (-1) = -1$$ (Ans)

To Evaluate Right hand limit (RHL) $\left(\lim_{x \to a^+} f(x)\right)$

$x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$, where $h > 0$, always.

steps

> write $\lim_{x \to a^+} f(x)$

> put $x = a+h$ and replace $x \to a^+$ by $h \to 0$ to obtain $\lim_{h \to 0} f(a+h)$.

> simplify $\lim_{h \to 0} f(a+h)$ by using the formula.

e.g.

Evaluate the right hand limit of the function.

$$f(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases} \quad \text{at } x = 4.$$

solution:

(RHL of $f(x)$ at $x=4$)

$\lim_{x \to 4^+} f(x)$

$= \lim_{h \to 0} f(4+h)$

$= \lim_{h \to 0} \frac{|4+h-4|}{4+h-4}$

$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (1) = 1$. (Ans)

Solved Examples

Test the continuity of the function $f(x)$ at the origin.

$$f(x) = \begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 1; & x = 0 \end{cases}$$

solution: we have.

(LHL at $x=0$).

$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$.

and
(RHL at $x=0$)

$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$.

$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$.

Thus, we have $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

$\Rightarrow$ limit doesn't exist.
Hence, $f(x)$ is not continuous at the origin.

Q2. Discuss the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} 2-x, & x<2 \\ 2+x, & x \geq 2. \end{cases}$$

<u>Solution</u> we have,
(LHL at $x=2$)

$= \lim_{x \to 2^-} f(x) = \lim_{x \to 2} 2-x$ $\quad (\because f(x) = 2-x \text{ for } x<2)$

$= 2-2 = 0$
(RHL at $x=2$)

$= \lim_{x \to 2^+} f(x) = \lim_{x \to 2} 2+x$

$= 2+2 = 4$.

$\because \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

Hence $f(x)$ is not continuous at $x=2$, as limit doesn't exist.

<u>N.B.</u>: Here $f(2) = 2+2 = 4$ (as $f(x) = 2+x$ at $x \geq 2$)
(Definite)

Assignments - 3

A. Examine the continuity of the function on $f(x)$ at the given point.

Q1. $$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases} \quad \text{at } x = 2$$

Q2. $$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{for } x \neq 1 \\ 2 & \text{for } x = 1. \end{cases} \quad \text{at } x = 1.$$

Q3. $$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}, \quad \text{at } x = 0,$$

Q4. $$f(x) = \begin{cases} 2x-1 & x < 0 \\ 2x+1 & x \geq 0 \end{cases} \quad \text{at } x = 0,$$

(B) Q1. Determine the value of 'a' if the function $f(x)$ defined by $f(x) = \begin{cases} 2x-1 & ; x < 2 \\ a & , x = 2 \\ x+1 & x > 2 \end{cases}$ is continuous at $x = 2$

Q2. For what value of k is the function

$$f(x) \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ k, & x = 0, \end{cases}$$

continuous at $x = 0$.

Unit-III | Differentiation

Introduction

Let $y = f(x)$ be a continuous function. Then the value 'y' depends upon the value of x and it changes with a change in the value of 'x';

so 'x' is called the independent variable. and 'y' is called the dependent variable.

Let Δx be a small change (+ve or negative) in 'x'.

Let Δy be the corresponding change in $y = f(x)$. Then, the value of x changes from x to $x + \Delta x$ and the value of $f(x)$ changes from $f(x)$ to $f(x + \Delta x)$.

So the change in the value of f is $f(x+\Delta x) - f(x)$

or, $\boxed{\Delta y = f(x + \Delta x) - f(x)}$

Now, the average rate of change of y with respect to 'x' is equal to $\frac{\Delta y}{\Delta x}$

as $\Delta x \to 0$, we find that Δy also tends to zero, therefore,

Instataneous rate of change of y with respect to $x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$\Rightarrow \boxed{\frac{dy}{dy} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}.$$

measures the rate of change of $y = f(x)$. with respect to 'x' which is called derivative of $f(x)$ w.r.t 'x'.

Geometrical meaning of Derivative at a point

Consider the curve $y = f(x)$,
Let $f(x)$ be differentiable at $x = c$.
Let $P(c, f(c))$ be a point on the curve.
and $Q(x, f(x))$ be a neighbouring point on it. then,

Slope of the chord $PQ = \dfrac{f(x) - f(c)}{x - c}$

Taking limit as $Q \to P$, ie. $x \to c$, we have.

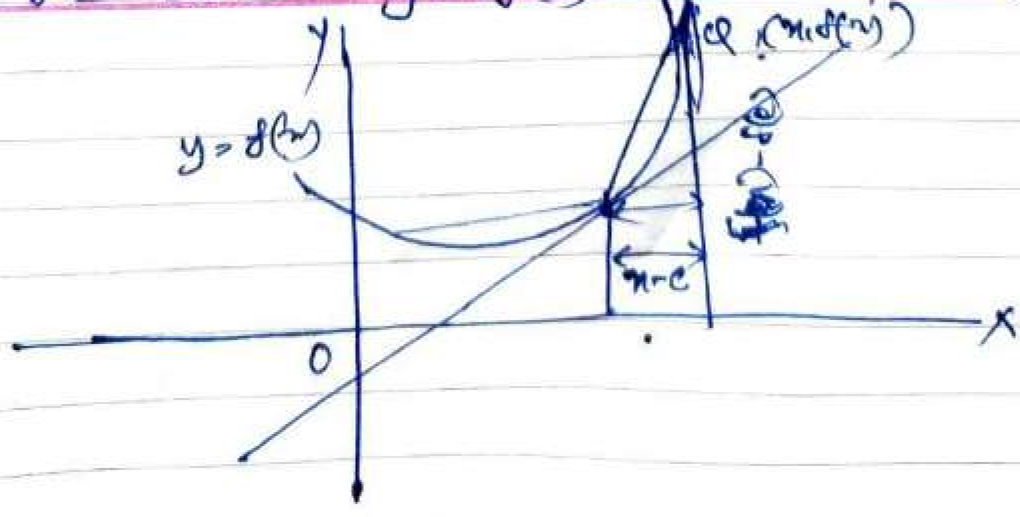
$$\lim_{Q \to P} (\text{slope of the chord } PQ) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

As $Q \to P$, chord PQ becomes tangent at P. therefore, from (i), we have,

$$\text{Slope of the tangent at } P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \left[\frac{d\, f(x)}{dx}\right]_{x = c} \text{ or } f'(c).$$

> Thus the derivative of a function at a point $x = c$, is the slope of the tangent to the curve $y = f(x)$. at the point $(c, f(c))$



Differentiation from first principle

The derivative of a function is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The process of finding the derivative of a function by using the above defination is called the differentiation from first principle or by ab-initio method or by delta method.

N.B = $f'(x)$ can be also written as $\frac{dy}{dx} = Dy = y_1 = y'$

We will find the derivative of some standard function by first principle

Some Trigonometric formula to Recall.

(i) $\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$

(ii) $\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$

(iii) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

(iv) $\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$

(v) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$

(vi) $\sin C - \sin D = 2 \sin \frac{C-D}{2} \cdot \cos \frac{C+D}{2}$

(vii) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$

(viii) $\cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$.

Date : ....

(ix) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(x) $2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$.

(xi) $2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$

(xii) $2\sin A\cdot\sin B = \cos(A-B) - \cos(A+B)$

~~(xiii) sin~~

Some limit formula recall

(i) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$.

(ii) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$

(iii) $\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \frac{1}{\log_e a}$.

(iv) $\lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1$

(v) $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

(vi) $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = \lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$.

(vii) $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

We will find out the following standard derivatives

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$

(2) $\frac{d}{dx}(a^x) = a^x\log_e a$

(3) $\frac{d}{dx}(e^x) = e^x$

(4) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(5) $\frac{d}{dx}(\cos x) = \sin x$

(6) $\frac{d}{dx}(\sin x) = -\cos x$

(7) $\frac{d}{dx}(\tan x) = \sec^2 x$

(8) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(9) $\frac{d}{dx}(\sec x) = \sec x\tan x$

(10) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cdot\cot x$

(1) $\boxed{\dfrac{d(x^n)}{dx} = nx^{n-1}}$

proof: Let $f(x) = x^n$, then $f(x+h) = (x+h)^n$

$$\therefore \frac{d(f(x))}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}$$ where $z = x+h$ and $z \to x$ as $h \to$

$$= nx^{n-1}$$

Example:-

(i) $\dfrac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$

(ii) $\dfrac{d}{dx}\left(\dfrac{1}{x^3}\right) = \dfrac{d}{dx}\left(x^{-3}\right) = -3x^{-3-1} = \dfrac{-3}{x^4}$

(iii) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}\left(x^{1/2}\right) = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2\sqrt{x}}$

(iv) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}\left(x^{-1/2}\right) = -\dfrac{1}{2}x^{-1/2-1} = -\dfrac{1}{2}x^{-3/2}$

(v) $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

(vi) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}\left(x^{-1}\right) = -1x^{-1-1} = \dfrac{-1}{x^2}$

(7) $\boxed{\frac{d}{dx}(e^x)\ e^x}$

proof : Let $f(x) = e^x$ Then $f(x+h) = e^{x+h}$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right) = e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right).$$

$$= e^x \cdot 1 = e^x \qquad \left[\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = 1\right]$$

Hence $\frac{d}{dx}(e^x) = e^x$

(8) $\boxed{\frac{d}{dx}(a^x) = a^x \log_e a}$

proof :- Let $f(x) = a^x$ then $f(x+h) = a^{x+h}$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h}\right) = a^x \log_e a \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a\right]$$

$$\frac{d}{dx}(a^x) = a^x \log_e a.$$

Example :-

(i) $\frac{d}{dx}(5^x) = 5^x \log_e 5$

(ii) $\frac{d}{dx}(10^x) = 10^x \log_e 10$

(iii) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}\left[(e^2)^x\right] = (e^2)^x \log e^2 = e^{2x} \cdot 2 \log e$

$= 2e^{2x}$

NB | ('δx' is 'h'
δx or $h \approx 0$)

$$\boxed{\frac{d}{dx}(\sin x)}$$

$f(x) = \sin x$

$f(x+\delta x) = \sin(x+\delta x)$

$$\frac{d}{dx}\sin x = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x} \quad \text{(by first principle)}$$

$$= \lim_{\delta x \to 0} \frac{2\sin\left(\frac{x+\delta x - x}{2}\right)\cdot\cos\left(\frac{x+\delta x + x}{2}\right)}{\delta x} \quad \left[\sin C - \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}\right]$$

$$= \lim_{\delta x \to 0} \frac{2\sin\left(\frac{\delta x}{2}\right)\cdot\cos\left(\frac{\delta x}{2}+x\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cancel{2}\sin\frac{\delta x}{2}}{\cancel{2}\times\frac{\delta x}{2}}\cdot\cos\left(\frac{\delta x}{2}+x\right)$$

$$= \lim_{\delta x \to 0}\left(\frac{\sin \delta x/2}{\delta x/2}\right)\times\cos\left(\frac{\delta x}{2}+x\right)$$

$$= 1 \times \lim_{\delta x \to 0}\cos\left(\frac{\delta x}{2}+x\right) = \cos\left(\frac{0}{2}+x\right)$$

$$\Rightarrow \boxed{\frac{d}{dx}\sin x = \cos x}$$

Assignment: I (prove by first principle / by defination)

① $\frac{d}{dx}(\cos x) = -\sin x$

② $\frac{d}{dx}(\tan x) = \sec^2 x$

③ $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

Rules to find derivative.

(1) $\frac{d}{dx} c\{f(x)\} = c \frac{d}{dx} f(x)$

(2) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

(3) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

(4) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d f(x)}{dx} - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$ gx

Example: (find derivate of following function)

Q1 $y = 2x^3$

Ans $\frac{d}{dx}(2x^3) = 2 \frac{d}{dx}(x^3) = 2(3x^2) = 6x^2$

Q2 $y = x^3 - x^2 + 6$

Ans $\frac{d}{dx}(x^3 - x^2 + 6) = \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) + \frac{d}{dx}(6)$

$= 3x^2 - 2x + 0 = 3x^2 - 2x$ Ans

Q3 $y = x^2 \sin x$

Ans. $\frac{dy}{dx} = \frac{d(x^2 \sin x)}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$

$= x^2 \cos x + \sin x (2x)$

$= x(x \cos x + 2 \sin x)$ (Ans)

Q4. $y = \frac{\tan x}{e^x(1+x^2)}$

Ans. $\frac{dy}{dx} = \frac{d(\tan x / e^x(1+x^2))}{dx}$

$= \frac{e^x(1+x^2) \frac{d(\tan x)}{dx} - \tan x \frac{d}{dx} e^x(1+x^2)}{[e^x(1+x^2)]^2}$

$= \frac{e^x(1+x^2) \sec^2 x - \tan x \left[e^x \frac{d}{dx}(1+x^2) + (1+x^2) \frac{d(e^x)}{dx}\right]}{[e^x(1+x^2)]^2}$

$$= \frac{e^x \sec^2 x\,(1+x^2) - \tan x\,[e^x(2x) + (1+x^2)e^x]}{[e^x(1+x^2)]^2}$$

$$= \frac{e^x\,[(1+x^2)\sec^2 x - 2x\tan x - (1+x^2)\tan x]}{[e^x(1+x^2)]^2}$$ Ans

Assignment 12 Find derivative of following function w.r.t. x.

(i) $\frac{a^x - b^x}{x}$ (ii) $\frac{\tan x}{\log x}$ (iii) $\frac{e^x}{1+x^2} - x\sin x$

Derivative of composite function by chain Rule

Composite function: A function formed by composition of more than one function is called composite function.

Example: 1) $\sin x^2$ (composite of $\sin x$ & x^2)

2) $a^{\ln x}$ (composite of a^x & $\ln x$)

3) $\sqrt{\sin\sqrt{x}}$ (composite of $\sqrt{x}$, $\sin x$ & $\sqrt{x}$)

(4) $\log(\sin x^2)$ (composite of $\log x$, ~~again~~ $\sin x$, x^2)

Chain Rule:-

If $y = f(t)$, t is a function x defined by $t = g(x)$,

then $\frac{dy}{dx} = \frac{d f(t)}{dx} = \frac{df(t)}{dt} \times \frac{dt}{dx} = f'(t) \times g'(x)$

In general: If $y = f(u)$

$u = g(v)$

$v = h(t)$

.... and so on,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdots\cdots \frac{dt}{dx}$$ [t is a funct^n x a independent variable at the end.

Example:- find $\frac{dy}{dx}$ if $y = 5^{\sin x^2}$

Sol^n $\frac{d}{dx}\left(5^{\sin x^2}\right) = \frac{d\left(5^{\sin x^2}\right)}{d(\sin x^2)} \times \frac{d(\sin x^2)}{dx^2} \times \frac{dx^2}{dx}$

$= \left(5^{\sin x^2} \times \log a\right) \times \left(\cos x^2\right) \times (2x)$

$\frac{d}{dx} a^x = a^x \log a$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx} x^2 = 2x$

PARTIAL DERIVATIVE

(Imagine a partial derivative of function comprising of two independent variables, namely, "science" and "Humanity")

Partial Derivative:-

A partial derivative of a function of several variables is its derivative with respect to one of those variables, with others held constant.

N.B. The symbol used to denote partial derivatives is '∂' called 'del'.

N.B. **Several variable function:-**

Suppose that 'A' denotes the area of a rectangle of side x and y units of length. Then $A = xy$

It is a function of two independent variables x & y. This fact is expressed by $A = f(x, y) = xy$.

Examples: pressure (P) : depends upon volume (V) & temp (T).
Output of a factory : depends upon independent (variable) factors · investment (capital), labour (input), work hours (time) etc.

Notation: $f(x, y, z, t, \ldots\ldots)$

N.B. **Function of two variable:**

Let X, Y, and Z be three non-empty sets. f is a function of two variables defined from $X \times Y$ to Z. Usually the notation is $\boxed{z = f(x, y)}$.

$z \rightarrow$ dependent variable

$x, y \rightarrow$ independent variable.

Geometrical Interpretation:-

A function $z = f(x, y)$ of two variables represents a surface in three dimension.

When we keep one variable constant say $y = b$, then the points on the surface intercepted by the plane $y = b$ determine a curve $z = f(x, b)$.

The derivative of z w.r.t x at $x = a$ now gives us the slope

Mathematics may not teach us how to add love or minus hate. But it gives us every reason to hope that "Every problem has a solution"

of this curve in the plane $y=b$ at the point $z=f(a,b)$

Hence $\frac{\partial z}{\partial x}$ means slope of certain curve when y is kept constant. and so as we can define $\frac{\partial z}{\partial y}$.

N.B. 1st order partial derivative $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ also written as f_x & f_y.

Example: $z = 2x^2y + xy^2 + 5x + 5y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

solution: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[2x^2y + xy^2 + 5x + 5y]$

$$= \left[2y\frac{\partial}{\partial x}(x^2) + y^2\frac{\partial x}{\partial x} + 5\frac{\partial x}{\partial x} + \frac{\partial}{\partial x}(5y)\right]$$

Taking y as a constant.

$$= 2y(2x) + y^2 + 5 + 0$$

$$= 4xy + y^2 + 5$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[2x^2y + xy^2 + 5x + 5y]$$

$$= 2x^2\frac{\partial y}{\partial y} + x\frac{\partial y^2}{\partial y} + \frac{\partial(5x)}{\partial y} + 5\frac{\partial y}{\partial y}$$

Taking x as constant.

$$= 2x^2 + x(2y) + 0 + 5$$

$$= 2x^2 + 2xy + 5$$

solved problems:-

Q1. Given $z = \sin\left(\frac{x}{y}\right)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: $\frac{\partial z}{\partial x} = \frac{\partial\left[\sin\left(\frac{x}{y}\right)\right]}{\partial x} = \frac{\partial\left[\sin\left(\frac{x}{y}\right)\right]}{\partial\left(\frac{x}{y}\right)} \times \frac{\partial\left(\frac{x}{y}\right)}{\partial x}$ [chain Rule]

$$= \cos\left(\frac{x}{y}\right) \times \frac{1}{y}\frac{\partial x}{\partial x} \quad \text{[treating } y \text{ as constant]}$$

$$= \frac{1}{y}\cos\left(\frac{x}{y}\right) \quad \text{(Ans)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\sin\left(\frac{x}{y}\right)\right] = \frac{\partial}{\partial\left(\frac{x}{y}\right)}[\sin(x/y)] \times \frac{\partial\left(\frac{x}{y}\right)}{\partial y}$$

$$= \cos\left(\frac{x}{y}\right) \times x\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \left[\cos\left(\frac{x}{y}\right)\right]\left(\frac{-x}{y^2}\right) \quad \text{(Ans)}$$

Date : .... "pure mathematicians just love to try unsolved problems" – They love a "challenge" Page No. ....

Q2. Find f_x and f_y of $f(x,y) = x^y + y^x$.

solution) $f_x = \frac{\partial f}{\partial x} = \frac{\partial (x^y + y^x)}{\partial x} = \frac{\partial (x^y)}{\partial x} + \frac{\partial (y^x)}{\partial x}$

$= y x^{y-1} + y^x \ln y$. (Ans)

treating y as constant we can apply the formula as →

$\frac{d(x^n)}{dx} = n x^{n-1}$

$\frac{d(a^x)}{dx} = a^x \ln a$

$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^y + y^x]$.

$= \frac{\partial (x^y)}{\partial y} + \frac{\partial (y^x)}{\partial y}$ ← (treating 'x' as constant)

$= x^y \ln x + x y^{x-1}$ (Ans).

Q3. Find f_x, f_y for $f(x,y) = \log(x^2 + y^2 - 2xy)$.

solution) $\log(x^2 + y^2 - 2xy) = \log(x-y)^2 = 2\log(x-y)$.

Now, $f_x = \frac{\partial}{\partial x}[2\log(x-y)] = 2\frac{\partial}{\partial x}\log(x-y)$.

$= 2 \frac{\partial \log(x-y)}{\partial (x-y)} \times \frac{\partial (x-y)}{\partial x} = 2\frac{1}{x-y} \times \left[\frac{\partial x}{\partial x} - \frac{\partial y}{\partial x}\right]$

$= \frac{2}{x-y} \times (1-0) = \frac{2}{x-y}$. (Ans) [taking y as constant]

$f_y = \frac{\partial}{\partial y}[2\log(x-y)] = 2\frac{\partial}{\partial y}\log(x-y) = 2\frac{\partial \log(x-y)}{\partial (x-y)} \times \frac{\partial (x-y)}{\partial y}$

$= 2\frac{1}{x-y} \times \frac{\partial (x)}{\partial y} - \frac{\partial y}{\partial y} = \frac{2}{x-y} \cdot (0-1) = \frac{-2}{x-y}$ (Ans)

Q4. If $z = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

solution) $\frac{\partial z}{\partial x} = \frac{\partial f\left(\frac{y}{x}\right)}{\partial x} = \frac{\partial f(y/x)}{\partial (y/x)} \times \frac{\partial (y/x)}{\partial x} = f'\left(\frac{y}{x}\right) \times y \frac{\partial}{\partial x}\left(\frac{1}{x}\right)$

$= f'(y/x) \cdot y\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2} f'(y/x)$

$\frac{\partial z}{\partial y} = \frac{\partial f(y/x)}{\partial y} = \frac{\partial f(y/x)}{\partial (y/x)} \times \frac{\partial (y/x)}{\partial y} = f'\left(\frac{y}{x}\right) \times \frac{1}{x}\frac{\partial y}{\partial y}$

$= \frac{1}{x} f'\left(\frac{y}{x}\right)$.

Now, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(-\frac{y}{x^2}\right) f'\left(\frac{y}{x}\right) + y\frac{1}{x} f'\left(\frac{y}{x}\right)$.

$= \left[-\frac{y}{x} + \frac{y}{x}\right] f'\left(\frac{y}{x}\right) = 0$ Ans.

Date : .... "History" make men wise "poets" " witty. the mathematics subtle; natural philosophy deep; moral grave; logic & rhetoric able to contend. Page No. ....

Assignment - 1 (PD)

1. Find f_x, f_y, where $f(x,y)$ is given as follows.

(i) $\dfrac{x^2y+y^2x}{x+y}$ (ii) $\sqrt{x^2+y^2}$ (iii) $\sin^{-1}\left(\dfrac{x}{y}\right)$ (iv) e^{2x+3y}

(v) a^{xy} (vi) $e^{2x}\cos 3y$ (vii) $\tan(xy)$:

2. If $f(x,y,z) = e^{xyz}$ then find $xf_x+yf_y+zf_z$

2nd order, refer to next section

3. If $U = (x^2+y^2+z^2)^{-1/2}$ show that,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

[Hints: take $x^2+y^2+z^2 = r$.

$$\Rightarrow u = r^{-1/2} \Rightarrow u = \frac{1}{\sqrt{r}} \Rightarrow u^2 = \frac{1}{r}$$

$$\Rightarrow \frac{\partial}{\partial x}(u^2) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) \Rightarrow 2u\frac{\partial u}{\partial x} = -\frac{1}{r^2}\times\frac{\partial r}{\partial x} \Rightarrow (2u)\frac{\partial u}{\partial x} = -\frac{1}{r^2}(2x)$$

$$\Rightarrow u\frac{\partial u}{\partial x} = -\frac{x}{r^2} \Rightarrow \frac{1}{\sqrt{r}}\frac{\partial u}{\partial x} = -\frac{x}{r^2} \Rightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{x}{r\sqrt{r}}}$$

Again differentiate $\frac{\partial u}{\partial x}$ w.r.t. x to find $\frac{\partial^2 u}{\partial x^2}$ & similarly $\frac{\partial^2 u}{\partial y^2}$ & $\frac{\partial^2 u}{\partial z^2}$ in this manner]

4. If $u = \sqrt{x}+\sqrt{y}+\sqrt{z}$, find $xU_x+yU_y+zU_z$.

5. If $z = xy\,f\left(\dfrac{y}{x}\right)$, Show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}+z\dfrac{\partial z}{\partial z} = 3z$.

6. Given $f(u,v) = \dfrac{2u-3v}{u^2+v^2}$, find $f_u(2,1)$ and $f_v(2,1)$.

(Hints: find $f_u = \frac{\partial f}{\partial u}$ & $f_v = \frac{\partial f}{\partial v}$ by taking v & u as constant respectively, then put $u=2$ & $v=1$.)

Mathematics is not about number, equation, computation or algorithms, it is about understanding)

2nd order partial derivative :

for $z = f(x,y)$, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves function of two variables x and y.

Hence they also posses higher order partial derivative as following notations.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = f_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = f_{xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{yx}$$

Note: $f_{yx} = f_{xy}$ when partial derivatives are continuous.

Solved problems:

Example:

Q. find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ for $z = e^{2x}\cos 3y$.

solution $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(f_x)$ & $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(f_y)$

$$f_x = \frac{\partial}{\partial x}(e^{2x}\cos 3y) = \cos 3y \frac{\partial (e^{2x})}{\partial x} = \cos 3y (2e^{2x})$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\cos 3y)(2e^{2x}) = 2\cos 3y \frac{\partial (e^{2x})}{\partial x} = 2\cos 3y(2e^{2x})$$

$$f_y = \frac{\partial}{\partial y}(e^{2x}\cos 3y) = e^{2x}\frac{\partial(\cos 3y)}{\partial y} = e^{2x}(-\sin 3y)\,3$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-3e^{2x}\sin 3y) = -3e^{2x}\frac{\partial}{\partial y}(\sin 3y) = -3e^{2x}(\cos 3y)\,3$$

Assignment 1) Solve Q. No - '3' of Assignment - 1 (PD).

2) Find $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ of Q No. 1 (Assignment - 1 (PD))

Mathematics has beauty in its practise.

Homogeneous function:

Defination:- A function f is said to be homogeneous in x and y of degree 'n' if $f(tx, ty) = t^n f(x,y)$. where 't' is any constant.

Notes: To test where a function is homogeneous or not we have to replace x and y as tx & ty then simplify the expression and taking common t^n (n is the least common power) we can conclude that form $t^n \{f(x,y)\}$.

Example-1: Test $f(x,y) = 2xy^2 - 3x^2y$ is homogeneous or not.

Solution: Here

$$f(tx, ty) = 2(tx)(ty)^2 - 3(tx)^2(ty)$$
$$= 2(tx)t^2y^2 - 3t^2x^2(ty)$$
$$= 2t^3xy^2 - 3t^3x^2y$$
$$= t^3(2xy^2 - 3x^2y).$$
$$= t^3 f(x,y).$$

Hence f is a homogeneous of degree '3'.

Example-2: Test $\sin^{-1}\left(\frac{x}{y}\right)$ is homogeneous function and hence find its degree.

Solution:

$$f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$$
$$f(tx,ty) = \sin^{-1}\left(\frac{tx}{ty}\right) = \sin^{-1}\left(\frac{x}{y}\right) = f(x,y)$$

$$\Rightarrow f(tx,ty) = t^0 f(x,y) \qquad [\because t^0 = 1]$$

Hence $\sin^{-1}\left(\frac{x}{y}\right)$ is a homogeneous function of degree '0'.

Note: (1) A function is homogeneous if each term in the expression is of the same degree.

(2) If z is a homogeneous function of x and y of degree n, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also homogeneous of degree '$n-1$'.

(3) A homogeneous function of degree 'n' can put in the form.

$$\boxed{z = x^n \phi\left(\frac{y}{x}\right)}$$

"The only way to learn mathematics is to do mathematics!"

Euler's Theorem : (1st order.)

If z is a homogeneous function of degree n, then,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

proof: Given z is a homogeneous function of degree 'n'.
we can put in the form as: $z = x^n \phi\left(\frac{y}{x}\right)$.

Now, $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^n \phi\left(\frac{y}{x}\right)\right]$.

[x^n & $\phi\left(\frac{y}{x}\right)$ both have variable 'x' so we can apply $\left[\frac{d}{dx}[f(x)\, g(x)] = f'g + g'f\right]$]

$$= x^n \frac{\partial}{\partial x}\left[\phi\left(\frac{y}{x}\right)\right] + \phi\left(\frac{y}{x}\right)\frac{\partial (x^n)}{\partial x}$$

$$= x^n \phi'\left(\frac{y}{x}\right) \times \frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right) n x^{n-1}$$

$$= x^n \phi'\left(\frac{y}{x}\right) y\left(\frac{\partial \cancel{x}}{\partial x}\right) + n x^{n-1}\phi\left(\frac{y}{x}\right)$$

$$= x^n \phi'\left(\frac{y}{x}\right) \times \frac{-y}{x^2} + n x^{n-1}\phi\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} = \frac{-y x \cdot x^n}{x^2}\phi'\left(\frac{y}{x}\right) + n x \cdot x^{n-1}\phi\left(\frac{y}{x}\right)$$

$$= -y x^{n-1}\phi'\left(\frac{y}{x}\right) + n x^n \phi\left(\frac{y}{x}\right). \text{ ——— (i)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[x^n \phi\left(\frac{y}{x}\right)\right]$$

$$= x^n \frac{\partial}{\partial y}\left[\phi\left(\frac{y}{x}\right)\right] \quad \text{[Taking } x \text{ as constant]}$$

$$= x^n \phi'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= x^n \phi\left(\frac{y}{x}\right)\frac{1}{x}\frac{\partial y}{\partial y} = x^{n-1}\phi'\left(\frac{y}{x}\right).$$

$$y\frac{\partial z}{\partial x} = y x^{n-1}\phi'\left(\frac{y}{x}\right) \text{ ——— (ii)}$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -y x^{n-1}\phi'\left(\frac{y}{x}\right) + n x^n \phi\left(\frac{y}{x}\right) + y x^{n-1}\phi'\left(\frac{y}{x}\right) = n x^n \phi\left(\frac{y}{x}\right)$$

$$= \boxed{nz}$$

Mathematics is the music of reason"

Postulates of Euler's theorem : (2nd order)

If z is a homogeneous function of x and y of degree n then show that .

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z} .$$

proof: Given that $z=f(x,y)$ is homogeneous of degree 'n'.
by Euler's theorem , we have

$$\boxed{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz} \quad \text{---} (1)$$

Differentiating eqⁿ (1) with respect to 'x',

$$\frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=\frac{\partial}{\partial x}(nz)$$

$$\Rightarrow \left[\frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial x}\right]\right]+\left[y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right]=n\frac{\partial z}{\partial x} \quad \text{(taking 'y' as constant)}$$

$$\Rightarrow \left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\frac{\partial x}{\partial x}\right]+\left[y\frac{\partial^2 z}{\partial x\partial y}\right]=n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$

multiplying 'x' on both side

$$x^2\frac{\partial^2 z}{\partial x^2}+xy\frac{\partial^2 z}{\partial x\partial y}=(n-1)x\frac{\partial z}{\partial x} \quad \text{---} (2)$$

Differentiating eqⁿ (1) with respect to 'y',

$$\frac{\partial}{\partial y}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=\frac{\partial}{\partial y}(nz)$$

$$\Rightarrow \left[x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]+\left[\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial y}\right)\right]=n\frac{\partial z}{\partial y}.$$

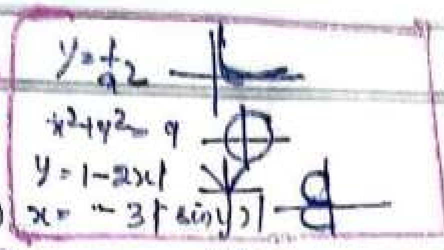
$$\Rightarrow x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} .$$

multiplying 'y' on both side .

$$xy\frac{\partial^2 z}{\partial y\partial x}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)y\frac{\partial z}{\partial y} . \quad \text{---} (3)$$

All you need



Adding (2) and (3)

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y} + xy\frac{\partial^2 z}{\partial y \partial x} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)nz \quad \text{[proved]}$$

Examples: (Application)

Q1. Verify Euler's theorem for $z = \frac{y}{x}$.

Ans:
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = y\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = y\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}\left(\frac{\partial y}{\partial y}\right) = \frac{1}{x}(1) = \frac{1}{x}$$

Here $z = f(x,y) = \frac{y}{x}$

$f(tx, ty) = \frac{ty}{tx} = \frac{y}{x} = f(x,y)$.

$\Rightarrow f(tx, ty) = t^0 f(x,y)$,

$f(x,y) = \frac{y}{x}$ is homogeneous of degree '0'.

by statement of Euler's theorem. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ $(n=0)$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(-\frac{y}{x^2}\right) + y\left(\frac{1}{x}\right) = -\frac{y}{x} + \frac{y}{x} = 0$$
(verified)

Q2. If $z = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$

Solution: Let $\frac{x^2+y^2}{x+y} = u$ → which is clearly a homogeneous function of degree '1'

Now $z = \sin^{-1} u$

$\Rightarrow \sin z = u$

$\Rightarrow \boxed{u = \sin z}$ is a homogeneous function of degree '1'

by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ $(n = 1)$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \cdot u$

$\Rightarrow x\frac{\partial(\sin z)}{\partial x} + y\frac{\partial(\sin z)}{\partial y} = 1(\sin z) \Rightarrow x\cos z\frac{\partial z}{\partial x} + y\cos z\frac{\partial z}{\partial y} = \sin z$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\sin z}{\cos z} = \tan z$ (proved).

Date : ....

Deare math, I Don't want to solve your problems. I have my own to solve 😂

Page No. ....

Assignment:

Q1. If $z = \frac{x-y}{x+y}$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

2. Verify Euler's theorem for $u = x^2 \log\left(\frac{y}{x}\right)$.

3. If $z = xy f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

Q4. If $z = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2z$.

Q5. If $z = \cos^{-1}\frac{(x^2+y^2)}{x+y}$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\cot z$

Q6. If $z = \ln\left(\frac{x^2+y^2}{x+y}\right)$ then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

Q7. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Q8. If $u = x^3 - 3xy^2$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Q9. If $u = x\sin^{-1}\frac{y}{x}$ then show that $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0$.

Q10. If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2$, then show that
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Q11. If $u = f\left(\frac{x^2+y^2}{xy}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Q12. Recall the geometrical meaning of partial derivative of 1st order two variable function. $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

"Thank you:"

Definite Integral :-

The definite integral is exactly defined as the limit and summation.

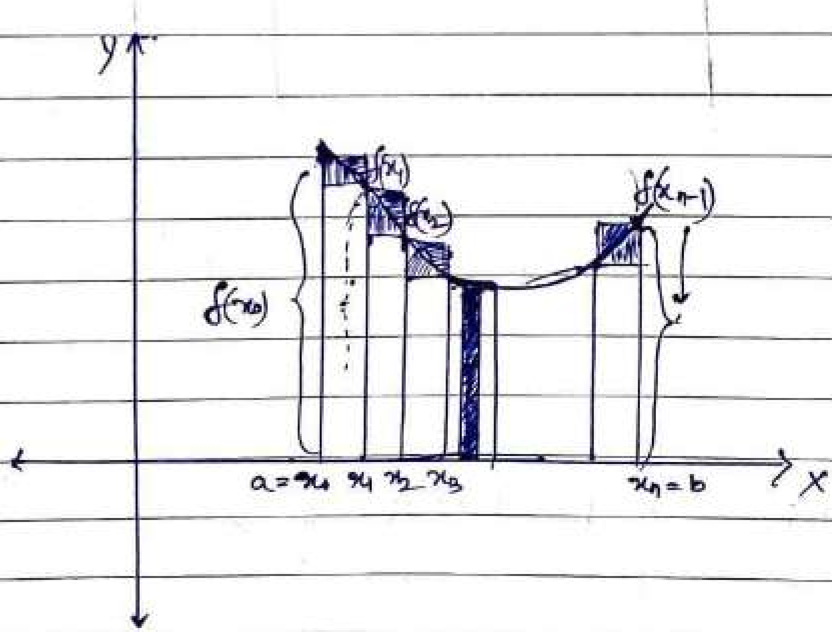
Derivation : (Definite integral as a limit of sum) :

(Approximating the area under a curve) :-

Geometric Concept: By dividing a region into many small shapes (rectangle) that have known area 'formula'; we can sum these area and obtain a reasonable estimate of the true area.

Consider Let $f(x)$ be a continuous and non negative function defined on the closed interval $[a,b]$.

The region is bounded by curve $f(x)$ above, the x-axis below, the line $x=a$ on the left and $x=b$ on the right.



Let the interval be divided into sub-intervals of length $\frac{b-a}{n}$ of equal width by the points $x_0, x_1, x_2, \cdots x_n$ such that $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$ and

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = \frac{b-a}{n} = \Delta x$$

Let construct a rectangle with width Δx and height equal to $f(x_{i-1})$.
the area of the rectangle is 'length x breadth' $= f(x_{i-1}) \times \Delta x$.
Hence the total area enclosed by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis is by taking sum of area of all rectangle as $\Rightarrow f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Let the Sum be $\boxed{S = \sum_{i=0}^{n} f(x_i)\Delta x}$ = 'Area under the curve'

(Note: 'Σ' summation notation for 'n' numbers of term to be added.)

· But the actual area 'A' under the curve $y = f(x)$ will be nearly accurate. as $n \to \infty$, $\Delta x \to 0$,

Then $A = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x = \boxed{\int_a^b f(x)\,dx}$

Thus the definite integral $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$ above the x-axis and between the ordinates $x = a$ and $x = b$.

Fundamental Theorem of Integral Calculus

(The relation between definite integral and indefinite integral)

Statement:

· If $f(x)$ is continuous in the interval $[a, b]$ and $\phi(x)$ is an anti-derivative of $f(x)$ $\left[\text{i.e. } \frac{d\phi(x)}{dx} = f(x)\right]$

$$\boxed{\int_a^b f(x)\,dx = \phi(b) - \phi(a)}$$

proof:- $\int f(x)dx = \phi(x) + k$ (where k is an arbitrary constant) $\Rightarrow \int_a^b f(x)dx = \left[\phi(x) + k\right]_a^b = \phi(b) + k - \phi(a) - k$
$= \phi(b) - \phi(a)$.

Example:-

$\int_1^2 x^3 dx = \left|\frac{x^4}{4}\right|_1^2 = \frac{1}{4}\left|x^4\right|_1^2 = \frac{1}{4}\left|2^4 - 1^4\right| = \frac{1}{4}\left|16 - 1\right| = \frac{15}{4}$ (Ans) properties of definite integral. (Elementary Rules)

(1) $\int_a^b [g(x) + h(x)]\, dx = \int_a^b g(x)\, dx + \int_a^b h(x)\, dx$

ample: $\int_0^{\pi/2} (3x^2 + 2x + \cos x)\, dx = \int_0^{\pi/2} 3x^2 dx + \int_0^{\pi/2} 2x\, dx + \int_0^{\pi/2} \cos x\, dx = |x^3|_0^{\pi/2} + |x^2|_0^{\pi/2} + |\sin x|_0^{\pi/2}$

$= \frac{\pi^3}{8} + \frac{\pi^2}{4} + 1$

(2) $\int_a^b \lambda\, g(x)\, dx = \lambda \int_a^b g(x)\, dx$.

(3) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(4) $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy = \int_a^b f(z)\, dz$.

(5) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad a < c < b$

some useful properties enlarging the scope of definite integrals.

(6) $\int_0^a f(x)\, dx = \int_0^a (a-x)\, dx$

(7) $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$ (f is even)

or 0 (if f(x) is odd)

(8) $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(2a-x) = f(x)$

and 0 if $f(2a-x) = -f(x)$.

Date : ....   Page No. ....

Total = 40 marks

Assignments - I (D.I) (2 marks) × 8 = (16 marks)

Q1. $\int_0^{\pi/4} \tan^2 x \, dx$

5. $\int_1^2 e^{4x+1} \, dx$

2. $\int_0^{\pi/4} \sin 2x \cos x \, dx$

6. $\int_0^2 3^{x+2} \, dx$

3. $\int_{\pi/6}^{-\pi/6} \frac{1+\cos 2x}{1-\cos 2x} \, dx$

7. $\int_0^2 x^2 e^{x^3} \, dx$

4. $\int_{\pi/4}^{\pi/2} \cot x \, dx$.

8. $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

(3 marks) × 4 = (12 marks)

9. $\int_0^1 \frac{dx}{\sqrt{e^{2x}-1}}$

(11) $\int_0^{3/2} [2x] \, dx$

10. $\int_0^4 \sqrt{x^2+9} \, dx$

(12) $\int_0^2 [x^2] \, dx$

4 marks × 3 = 12 (marks)

13. $\int_1^2 x \tan^{-1} x \, dx$

(15) $\int_0^{\pi/2} \frac{x}{(2x+1)(x+1)} \, dx$

(14) $\int_0^{\pi/2} \frac{\cos x \, dx}{(\sin x+1)(\sin x+2)}$

(16)

Page No. ....

Assignment - 2 (Long Questions) 5×5 = 25 (marks)

(1) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$

(2) $\int_0^{\pi/4} \log(1+\tan\alpha)$

(3) $\int_0^{\pi} \frac{dx}{1+\cot x}$

(4) $\int_0^{\pi/2} \log(\sin x)\,dx$

(5) $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

"You are never too old to set another goal or to dream a new dream" - C.S. Lewis

Differential Equations.

An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable is called a differential equation.

Example :- Each of the following equation is a differential equation :-

(i) $\frac{dy}{dx} + 5y = e^x$ (ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \sin x$.

(iii) $\frac{dy}{dx} = \frac{x^3 - y^3}{xy^2 - x^2y}$ (iv) $x^2dx + y^2dy = 0$.

Order of Differential Equation :-

The order of the highest-order derivative occurring in a differential equation is called the order of the differential equation.

Degree of a Differential Equation:-

The power of the highest-order derivative occurring in a differential equation, after it is made free from radicals and fractions, is called the degree of the differential equation.

Example - (1) Consider the equation $\left(\frac{dy}{dx}\right)^2 + 5y = \sin x$

The power of highest order and only one derivative term is '2'.

so its order is = 1

degree is = 2

(ii) $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^2 = 0$, order = 2

degree = 3

"Do what you can with all you have, wherever you are"
— Theodore Roosevelt

Some more example

(iii) $\dfrac{d^2y}{dx^2} = \sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$

we have to make it free from radicals by squaring both the sides.

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2.$$

Clearly Order = 2
degree = 2,

Assignment -1

Determine the order and degree of each of the following equations.

1. $x^2\left(\dfrac{d^2y}{dx}\right) + 2xy - 6x^3 = 0$

2. $\left(\dfrac{d^2y}{dx^2}\right)^3 + 2\left(\dfrac{dy}{dx}\right)^4 + 9 = \sin x$

3. $a\dfrac{d^2y}{dx^2} = \left\{1+\left(\dfrac{dy}{dx}\right)^3\right\}^{3/2}$

4. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

5. $x\left(\dfrac{dy}{dx}\right) + \dfrac{2}{\left(\dfrac{dy}{dx}\right)} = y^2$

~~$\sqrt{1-y^2}\,dx + y$~~

6. $\left(\dfrac{dy}{dx}\right)^4 + 3y\dfrac{d^2y}{dx^2} = 0$

7. $\tan^{-1}\sqrt{\dfrac{dy}{dx}} = x$

8. $\ln\left(\dfrac{d^2y}{dx^2}\right) = y$

(9) $\dfrac{\frac{dy}{dt}}{y+\frac{dy}{dt}} = \dfrac{y t}{\frac{dy}{dt}}$

(10) $\dfrac{d^2y}{du^2} = \dfrac{3y + \frac{dy}{du}}{\sqrt{\frac{d^2y}{du^2}}}$

(11) $e^{\frac{dy}{dx}} = x^2$

(12) $\dfrac{d^ny}{dx^n} + \dfrac{dy}{dx} = e^{3x}$.

"Fake it untill you make it! Act as if you had all the confidence you required untill it becomes your reality"
— Brian Tracy

Solution of a differential Equation

A function of the form $\boxed{y = f(x) + C}$ which satisfies a given differential equation is called its solution.

Solving Differential Equation with variable separation.

(1st order and 1st degree D.E.)

(i) **Differential equation of the types** $\boxed{\frac{dy}{dx} = f(x)}$.

Method

$\Rightarrow$ write $\frac{dy}{dx} = fx$

$\Rightarrow dy = f(x)\,dx.$

$\Rightarrow$ Integrating both side,

$$\int dy = \int f(x)\,dx + c$$

$$\Rightarrow \boxed{y = \int f(x)\,dx + C}.$$

Example :- solve $\frac{dy}{dx} = \frac{1}{x^2+1}$

$\Rightarrow dy = \frac{1}{x^2+1}\,dx$

$\Rightarrow \int dy = \int \frac{1}{x^2+1}\,dx.$

$\Rightarrow \boxed{y = \tan^{-1}x + C}$

(ii) **Differential equation of the types** $\boxed{\frac{dy}{dx} = g(y)}$

Method :-

write $\frac{dy}{dx} = g(y)$ form

$\Rightarrow \frac{dy}{g(y)} = dx$

$\Rightarrow$ Integrate $\int \frac{dy}{g(y)} = \int dx \Rightarrow \boxed{\int \frac{dy}{g(y)} = x + C}$

"Dare to be free; dare to go as far as your thought leads and dare to carry that out in your life - "Swami Vivekananda"

Example

$$\frac{dy}{dx} + y = 1$$

solution $\frac{dy}{dx} = 1 - y$

$\Rightarrow \frac{dy}{1-y} = dx$

$\Rightarrow \int \frac{dy}{1-y} = \int dx$

$\Rightarrow \log|1-y| = x + C$

$\Rightarrow \log|1-y| = \log(e^{x+C})$

$\Rightarrow 1-y = e^{x+C}$

$\Rightarrow \boxed{y = 1 - e^{x+C}}$ (solution)

N.B Always try to ~~exprees~~ express the solution as $y = f(x) + c$ form if possible.

(ii) Differential equation of the form $\frac{dy}{dx} = (f(x))(g(y))$

$\Rightarrow \boxed{f(x)\, dx = g(y)\, dy}$ can be put in the form

• The solution is given by $\int f(x)\, dx = \int g(y)\, dy + C$

Example $(x+1)\frac{dy}{dx} = 2xy$

$\Rightarrow (x+1)dy = 2xy\, dx$

$\Rightarrow \cancel{(x+1)} \frac{dy}{y} = \frac{2x}{x+1} dx.$

Limitations live only in our minds. But if we use our imaginations, our possibilities become limitless. — Jamie Paolinetti

$$\Rightarrow \int \frac{1}{y}\,dy = 2\int \frac{x}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = 2\int \frac{x+1-1}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = 2\int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \log y = 2[x - \log(x+1)] + C.$$

Assignment - 2

(1) **Solve.**

(i) $\frac{dy}{dx} = x^2 + x - \frac{1}{x}$ (iv) $\frac{dy}{dx} = \log x$

(ii) $\frac{dy}{dx} + 2x = e^{3x}$ (v) $\frac{dy}{dx} = \tan^{-1}x$

(iii) $(x^2+1)\frac{dy}{dx} = 1$

(2) **Solve.**

(i) $\frac{dy}{dx} = \sec y$

(ii) $\frac{dy}{dx} = \sin^2 y$

(iii) $\frac{dy}{dx} + y = 1$

(3) **Solve**

(i) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

(ii) $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$

(iii) $\frac{dy}{dx} = e^{x+y}$.

If you are offered a seat on a rocket ship, don't ask what seat! Just get on - Sheryl Sandberg

(iv) $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1+x^2}} = 0$

(v) $\frac{dy}{dx} = (e^x+1)y$

(vi) $x\sqrt{1-y^2}\,dy = (y+1)e^x\,dx$

(vii) $x\frac{dy}{dx} + \cot y = 0$

(viii) $\tan y\,dx + \sec^2 y \tan x\,dy = 0$

(ix) $\frac{dy}{dx} = y\sin 2x \quad y(0) = 1$

(x) $\tan^{-1} y = x + \frac{x^3}{3} + C$.

Linear Differential Equation

(i) The most general form of differential equation is $\boxed{\frac{dy}{dx} + Py = Q}$, where P is a constant,
Q is a constant or a function of 'x'.

(ii) The other common form of linear differential equation is $\boxed{\frac{dx}{dy} + Px = Q}$.

where P is a constant and Q is a constant or a function of 'y'.

Example

$$\boxed{x\frac{dy}{dx} - y = x^2}$$

which can be written as $\frac{dy}{dx} - \frac{1}{x}y = x$
is a linear differential equation.
where $P = -\frac{1}{x}$ and $Q = x$

we can't help everyone, but everyone can help someone.
- Ronald Reagan

$$\boxed{\text{solution of } \frac{dy}{dx} + Py = Q}$$

steps

1) first write the differential equation in linear form and find out P & Q.

2) Next to find out $\boxed{e^{\int P dx}}$ which is known as the integrating factor, ie I.F.

3) multiply $e^{\int P dx}$ in the equation as

$$e^{\int P dx} \times \frac{dy}{dx} + Py\, e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\boxed{\left(e^{\int P dx} \times y\right) = Q\, e^{\int P dx}}$$

4) Integrate w.r.t 'x'

$$\Rightarrow \int \frac{d}{dx}\left(e^{\int P dx} \times y\right) dx = \int Q e^{\int P dx} dx$$

5) $$\Rightarrow \boxed{y \cdot e^{\int P dx} = \int Q e^{\int P dx} dx + C}$$

which is the required solution.

* N.B you can also omit the steps 3, 4. and directly find out the solution as

$$\boxed{y \times I.F = \int (Q \cdot I.F) dx + C}$$ which is general sol for all linear differential equation.

we can easily forgive a child who is afraid of the dark. the real tragedy of life when men are afraid of the light.
- plato -

Solved Examples

Q1. solve $x\frac{dy}{dx} - y = x^2$

Solution the given differential equation may be written as . $\frac{dy}{dx} - \frac{1}{x} y = x$

This is of the form $\frac{dy}{dx} + Py = Q$.

where $P = -\frac{1}{x}$ and $Q = x$

Thus, the given equation is linear,

$$I.f = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

so the required solution is

$$y \times I.f = \int (Q \times I.f)\, dx + C.$$

i.e. $y \times \frac{1}{x} = \int \left(x \times \frac{1}{x}\right) dx + C$

$$= \int dx + C = x + C.$$

$$\therefore \frac{y}{x} = x + C \Rightarrow y = x^2 + Cx$$

Hence $\boxed{y = x^2 + Cx}$ is the required solution.

Q2. solve $(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$

Solution : the given differential equation may be written as

$$\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1} x}{1+x^2}$$

this is of the form $\frac{dy}{dx} + Py = Q$.

where $P = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1} x}{1+x^2}$.

Thus, the given equation is linear

$$I.f = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

Be who you are and say what you feel, because those who mind don't matter and those who matter don't mind - Dr. Seuss

$\therefore$ the required solution is

$$y \times I.f = \int (Q \times I.f)\, dx + C.$$

i.e. $y \times e^{\tan^{-1}x} = \int \left\{ \frac{\tan^{-1}x}{1+x^2} \times e^{\tan^{-1}x} \right\} dx + C.$

$$= \int t e^t dt + C \quad \text{where } \tan^{-1}x = t.$$

$$= te^t - \int 1 \cdot e^t dt + C$$

$$= te^t - e^t + C = e^t(t-1) + C$$

$$= e^{\tan^{-1}x}(\tan^{-1}x - 1) + C.$$

Hence $y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$ is required solution.

Example-3

Solve $\frac{dy}{dx} + (\sec x)y = \tan x$.

Sol^n the given equation is of the form,

$\frac{dy}{dx} + Py = Q$ where $P = \sec x$ and $Q = \tan x$

$\therefore I.f = e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log|\sec x + \tan x|}$

$= \sec x + \tan x$

So the required solution is

$$y \times I.f = \int (Q \times I.f)\, dx + C$$

i.e $y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$

$$= \int \sec x \cdot \tan x\, dx + \int \tan^2 x\, dx + C$$

$$= \sec x + \int (\sec^2 x - 1)dx + C$$

$$= \sec x + \tan x - x + C$$

So $y(\sec x + \tan x) = \sec x + \tan x - x + C$ is the required solution.