C.V. RAMAN POLYTECHNIC, BHUBANESWAR

LECTURE NOTE

STRUCTURAL MECHANICS,
(Th.1)

SEM-3rd

BRANCH- CIVIL ENGINEERING

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STRUCTURAL MECHANICS

LEARNING OBJECTIVE

CHAPTER-

1. **Properties of Materials** Classification of materials, elastic materials, plastic materials, ductile materials, brittle materials. Tensile test, compressive test, impact test, fatigue test, torsion test.
2. **Simple Stresses and Strains** Concept of stress, normal and shear stresses due to torsion Concept of strain, strain and deformation, longitudinal and lateral strain, poison's ratio, Volumetric strain Hooke's law, modulli of elasticity and rigidity, Bulk modulus of elasticity, relationship between the elastic constants. Stresses and strains in bars subjected to tension and compression. Extension of uniform bar under its own weight, stress produces in compound bars (two or three) due to axial load. Stressstrain diagram for mild steel, mechanical properties, factor of safety
   Temperature stresses and strains
3. **Bending Moment and Shear Force** Concept of a beam and supports (Hinges, Roller and Fixed), types of beams: simply supported, cantilever, fixed and continuous beams
   Types of loads (point, uniformly distributed and varying loads) Concept of bending moment and shear force, sign conventions Bending Moment and shear force diagrams for cantilever, simply supported and over hanging beams subjected to concentrated, uniformly distributed and uniformly varying loads (B.M. and S.F. diagrams should preferably be drawn on graph paper. Relationship between load, shear force and bending moment, point of maximum bending moment and contraflexure.
4. **Second Moment of Area** Concept of second moment of area, radius of gyration Theorems of parallel and perpendicular axes Second moment of area for sections of Rectangle, Triangle, Circle, Trapezium, Angle, Tee, I, Channel and Compound sections. (No derivation)
5. **Bending and Shear Stresses** Theory of simple bending Application of the equation M / I= sigma / Y= E/R (No derivation is required) Moment of resistance, sectional modulus and permissible bending stresses in circular, rectangular, I, T and L sections; Comparision of strengths of the above sections.
6. **Slope and Deflection** Necessity for determination of reflection Moment area theorems (no derivation) Computation of slopes and deflections using moment area theorems for: (a) Simple supported beam with UDL over entire span and concentrated load at any point
   (c) Cantilever with UDL over entire span and concentrated load at free end
7. **Columns Theory of columns**, Euler, Rankine's and I.S. formulae. Combined Direct and Bending Stresses Concentric and eccentric loads, eccentricity Effect of eccentric load on the section, stresses due to eccentric loads, examples in the case of short columns. Effect of wind pressure on walls and chimneys; water pressure on dams and earth pressure on retaining walls their causes of failures and their stability.
8. **Analysis of Trusses** Concept of a frame, redundant and deficient frame, End supports, ideal and practical trusses. Analysis of trusses by: (i) Methods of joints (ii) Method of sections and (iii) Graphical method

CHAPTER-1 PROPERTIES OF MATERIAL

- Elasticity: Ability of a body to resist a distorting influence or stress and to return to its original size and shape when the stress is removed.
- Plasticity: Ability of a material to undergo irreversible or permanent deformations without breaking or rupturing; opposite of brittleness.
- Malleability: Ability of the material to be flattened into thin sheets under applications of heavy compressive forces without cracking by hot or cold working means.
- Ductility: Ability of a material to deform under tensile load (% elongation).
- Flexibility: Ability of an object to bend or deform in response to an applied force; pliability; complementary to stiffness.
- Toughness: Ability of a material to absorb energy (or withstand shock) and plastically deform without fracturing (or rupturing); a material's resistance to fracture when stressed; combination of strength and plasticity
- Brittleness: Ability of a material to break or shatter without significant deformation when under stress; opposite of plasticity,examples:glass,concrete,cast iron,ceramics etc.

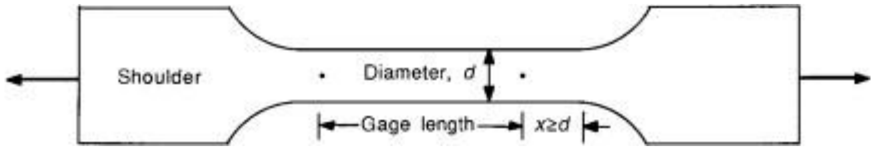
TEST OF MATERIALS

1.TENSILE TEST-

TENSILE TESTS are performed for several reasons. The results of tensile tests are used in selecting materials for engineering applications. Tensile properties frequently are included in material specifications to ensure quality. Tensile properties often are measured during development of new materials and processes, so that different materials and processes can be compared. Finally, tensile properties often are used to predict the behavior of a material under forms of loading other than uniaxial tension. The strength of a material often is the primary concern. The strength of interest may be measured in terms of either the stress necessary to cause appreciable plastic deformation or the maximum stress that the material can withstand. These measures of strength are used, with appropriate caution (in the form of safety factors), in engineering design. Also of interest is the material's ductility, which is a measure of how much it can be deformed before it fractures. Rarely is ductility incorporated directly in design; rather, it is included in material specifications to ensure quality and toughness. Low ductility in a tensile test often is accompanied by low resistance to fracture under other forms of loading. Elastic properties also may be of interest, but special techniques must be used to measure these properties during tensile testing, and more accurate measurements can be made by ultrasonic techniques.

Tensile Specimens and Testing Machines

Tensile Specimens- Consider the typical tensile specimen . It has enlarged ends or shoulders for gripping. The important part of the specimen is the gage section. The cross-sectional area of the gage section is reduced relative to that of the remainder of the specimen so that deformation and failure will be localized in this region. The gage length is the region over which measurements are made and is centered within the reduced section. The distances between the ends of the gage section and the shoulders should be great enough so that the larger ends do not constrain deformation within the gage section, and the gage length should be great relative to its diameter. Otherwise, the stress state will be more complex than simple tension. Detailed descriptions of standard specimen shapes are given and in subsequent chapters on tensile testing of specific materials. There are various ways of gripping the specimen, some of which are illustrated .The end may be screwed into a threaded grip, or it may be pinned; butt ends may be used, or the grip section may be held between wedges. There are still other methods .The most important concern in the selection of a gripping method is to ensure that the specimen can be held at the maximum load without slippage or failure in the grip section. Bending should be minimized.



2-COMPRESSIVE TEST

Compressive strength or **compression strength** is the capacity of a material or structure to withstand loads tending to reduce size, as opposed to tensile strength, which withstands loads tending to elongate. In other words, compressive strength resists compression (being pushed together), whereas tensile strength resists tension (being pulled apart). In the study of strength of materials, tensile strength, compressive strength, and shear strength can be analyzed independently.

Some materials fracture at their compressive strength limit; others deform irreversibly, so a given amount of deformation may be considered as the limit for compressive load. Compressive strength is a key value for design of structures.

3-TORSION TEST

A torsion test measures the strength of any material against maximum twisting forces. Itis an extremely common test used in material mechanics to measure how much of a twist acertain material can withstand before cracking or breaking. This applied pressure is referred to astorque. Materials typically used in the manufacturing industry, such as metal fasteners andbeams, are often subject to torsion testing to determine their strength under duress.There are three broad categories under which a torsion test can take place: failure testing, prooftesting and operational testing. Failure testing involves twisting the material until it breaks. Prooftesting observes whether a material can bear a certain amount of torque load over a given periodof time. Operational testing tests specific products to confirm their elastic limit before going onthe market.It is critical for the results of each torsion test to be recorded. Recording is done through creatinga stress-strain diagram with the angle of twist values on the X-axis and the torque values on theY-axis. Using a torsion testing apparatus, twisting is performed at quarter-degree increments withthe torque that it can withstand recorded. The strain corresponds to the twist angle, and the stresscorresponds to the torque measured.

4-IMPACT TEST

The **Charpy impact test**, also known as the **Charpy V-notch test**, is a standardized high strain-rate test which determines the amount of energy absorbed by a material during fracture. This absorbed energy is a measure of a given material's notch toughness and acts as a tool to study temperature-dependent ductile-brittle transition. It is widely applied in industry, since it is easy to prepare and conduct and results can be obtained quickly and cheaply. A disadvantage is that some results are only comparative.

The Test was developed around 1900 by S.B. Russell (1898, American) and Georges Charpy (1901, French).The test became known as the Charpy test In the early 1900s due to the technical contributions and standardization efforts by Charpy. The test was pivotal in understanding the fracture problems of ships during World War II.

Today it is utilized in many industries for testing materials, for example the construction of pressure vessels and bridges to determine how storms will affect the materials used.

he apparatus consists of a pendulum of known mass and length that is dropped from a known height to impact a notched specimen of material. The energy transferred to the material can be inferred by comparing the difference in the height of the hammer before and after the fracture (energy absorbed by the fracture event).

The notch in the sample affects the results of the impact test, thus it is necessary for the notch to be of regular dimensions and geometry. The size of the sample can also affect results, since the dimensions determine whether or not the material is in plane strain. This difference can greatly affect the conclusions made.

CHAPTER-2 SIMPLE STRESS AND STRAIN BEHAVIOUR OF MATERIALS

1. Introduction

When a force is applied on a body it suffers a change in shape, that is, it deforms. A force to resist the deformation is also set up simultaneously within the body and it increases as the deformation continues. The process of deformation stops when the internal resisting force equals the externally applied force. If the body is unable to put up full resistance to external action, the process of deformation continues until failure takes place. The deformation of a body under external action and accompanying resistance to deform are referred to by the terms strain and stress respectively.

2. Stresses

Stress is defined as the internal resistance set up by a body when it is deformed. It is measured in N/m^2 and this unit is specifically called Pascal (Pa). A bigger unit of stress is the mega Pascal (MPa).

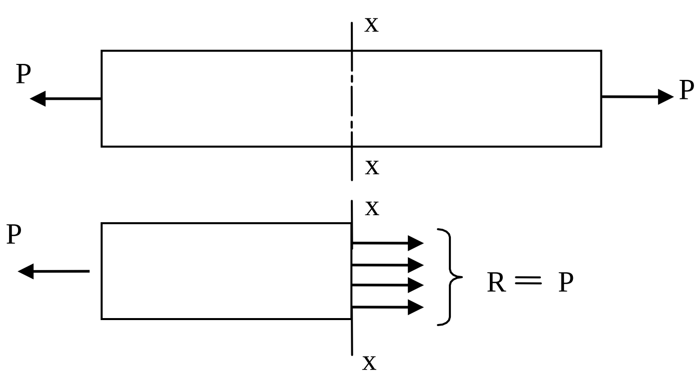
$1\ Pa = 1N/m^2$,

$1MPa = 10^6\ N/m^2 = 1N/mm^2$.

Three Basic Types of Stresses

Basically three different types of stresses can be identified. These are related to the nature of the deforming force applied on the body. That is, whether they are tensile, compressive or shearing.

Tensile Stress



Consider a uniform bar of cross sectional area A subjected to an axial tensile force P. The stress at any section x-x normal to the line of action of the tensile force P is specifically called tensile stress p_t . Since internal resistance R at x-x is equal to the applied force P, we have,

p_t = (internal resistance at x-x)/(resisting area at x-x)

$= R/A$

$= P/A$.

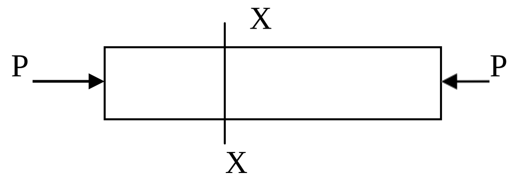
Under tensile stress the bar suffers stretching or elongation.

Compressive Stress

If the bar is subjected to axial compression instead of axial tension, the stress developed at x-x is specifically called compressive stress p_c.

$p_c = R/A$

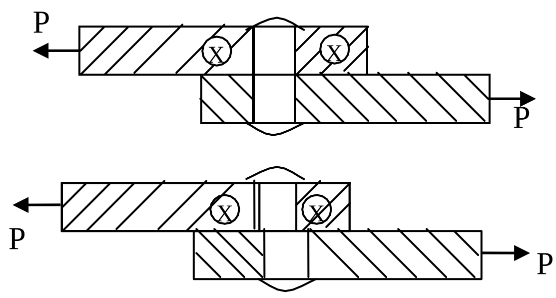
$= P/A$.

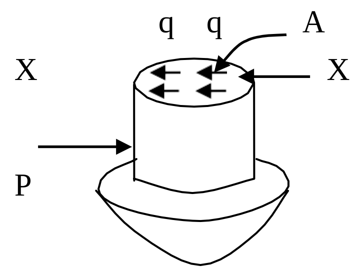


Under compressive stress the bar suffers shortening.

Shear Stress

Consider the section x-x of the rivet forming joint between two plates subjected to a tensile force P as shown in figure.





The stresses set up at the section x-x acts along the surface of the section, that is, along a direction tangential to the section. It is specifically called shear or tangential stress at the section and is denoted by q.

$q = R/A$

$= P/A.$

Normal or Direct Stresses

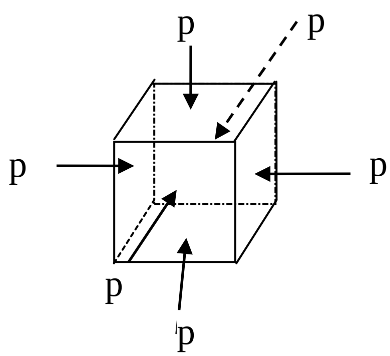
When the stress acts at a section or normal to the plane of the section, it is called a normal stress or a direct stress. It is a term used to mean both the tensile stress and the compressive stress.

Simple and Pure Stresses

The three basic types of stresses are tensile, compressive and shear stresses. The stress developed in a body is said to be simple tension, simple compression and simple shear when the stress induced in the body is (a) single and (b) uniform. If the condition (a) alone is satisfied, the stress is called pure tension or pure compression or pure shear, as the case may be.

Volumetric Stress

Three mutually perpendicular like direct stresses of same intensity produced in a body constitute a volumetric stress. For example consider a body in the shape of a cube subjected equal normal pushes on all its six faces. It is now subjected to equal compressive stresses p in all the three mutually perpendicular directions. The body is now said to be subjected to a volumetric compressive stress p.



Volumetric stress produces a change in volume of the body without producing any distortion to the shape of the body.

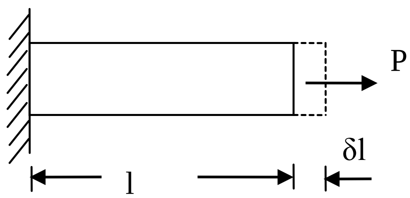
3. Strains

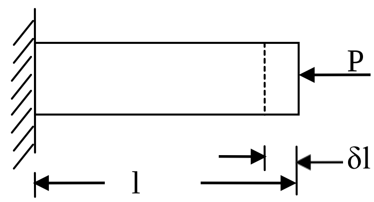
Strain is defined a the ratio of change in dimension to original dimension of a body when it is deformed. It is a dimensionless quantity as it is a ratio between two quantities of same dimension.

Linear Strain

Linear strain of a deformed body is defined as the ratio of the change in length of the body due to the deformation to its original length in the direction of the force. If l is the original length and δl the change in length occurred due to the deformation, the linear strain e induced is given by

$$\mathbf{e = \delta l / l}.$$





Linear strain may be a tensile strain , $\mathbf{e_t}$ or a compressive strain $\mathbf{e_c}$ according as δl refers to an increase in length or a decrease in length of the body. If we consider one of these as +ve then the other should be considered as –ve, as these are opposite in nature.

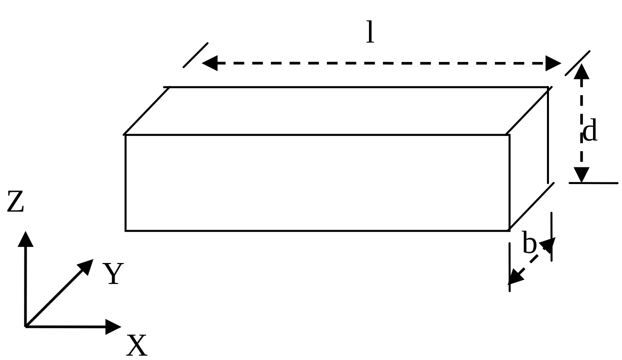
Lateral Strain

Lateral strain of a deformed body is defined as the ratio of the change in length (breadth of a rectangular bar or diameter of a circular bar) of the body due to the deformation to its original length (breadth of a rectangular bar or diameter of a circular bar) in the direction perpendicular to the force.

Volumetric Strain

Volumetric strain of a deformed body is defined as the ratio of the change in volume of the body to the deformation to its original volume. If V is the original volume and δV the change in volume occurred due to the deformation, the volumetric strain e_v induced is given by $\mathbf{e_v = \delta V/V}$

Consider a uniform rectangular bar of length l, breadth b and depth d as shown in figure. Its volume V is given by,



$V = lbd$

$\delta V = \delta l\, bd + \delta b\, ld + \delta d\, lb$

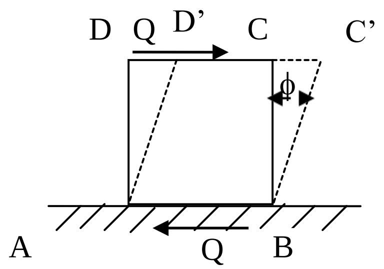
$\delta V / V = (\delta l / l) + (\delta b / b) + (\delta d / d)$

$e_v = e_x + e_y + e_z$

This means that volumetric strain of a deformed body is the sum of the linear strains in three mutually perpendicular directions.

Shear Strain

Shear strain is defined as the strain accompanying a shearing action. It is the angle in radian measure through which the body gets distorted when subjected to an external shearing action. It is denoted by Φ.



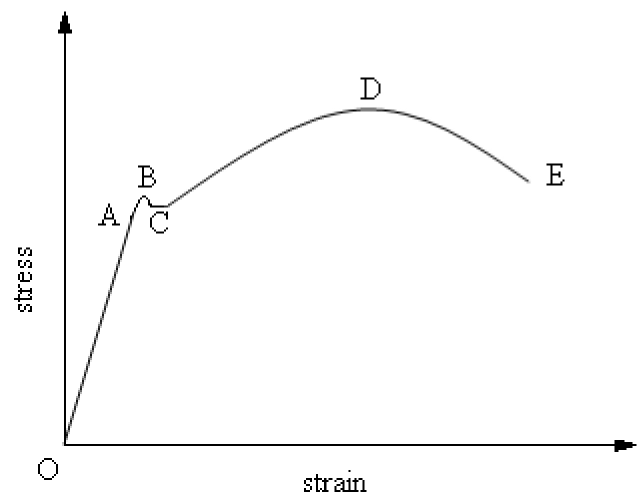
Consider a cube ABCD subjected to equal and opposite forces Q across the top and bottom forces AB and CD. If the bottom face is taken fixed, the cube gets distorted through angle ϕ to the shape ABC'D'. Now strain or deformation per unit length is

Shear strain of cube = CC' / CD = CC' / BC = ϕ radian

4. Relationship between Stress and Strain

Relationship between Stress and Strain are derived on the basis of the elastic behaviour of material bodies.

A standard mild steel specimen is subjected to a gradually increasing pull by Universal Testing Machine. The stress-strain curve obtained is as shown below.



A -Elastic Limit
B - Upper Yield Stress
C - Lower Yield Stress
D -Ultimate Stress
E -Breaking Stress

Elasticity and Elastic Limit

Elasticity of a body is the property of the body by virtue of which the body regains its original size and shape when the deformation force is removed. Most materials are elastic in nature to a lesser or greater extend, even though perfectly elastic materials are very rare.

The maximum stress upto which a material can exhibit the property of elasticity is called the elastic limit. If the deformation forces applied causes the stress in the material to exceed the elastic limit, there will be a permanent set in it. That is the body will not regain its original shape and size even after the removal of the deforming force completely. There will be some residual strain left in it.

Yield stress

When a specimen is loaded beyond the elastic limit the stress increases and reach a point at which the material starts yielding this stress is called yield stress.

Ultimate stress

Ultimate load is defined as maximum load which can be placed prior to the breaking of the specimen. Stress corresponding to the ultimate load is known as ultimate stress.

Working stress

Working stress= Yield stress/Factor of safety.

Hooke's Law

Hooke's law states that stress is proportional to strain upto elastic limit. If I is the stress induced in a material and e the corresponding strain, then according to Hooke's law,

p / e = E, a constant.

This constant E is called the modulus of elasticity or Young's Modulus, (named after the English scientist Thomas Young).

It has later been established that Hooke's law is valid only upto a stress called the limit of proportionality which is slightly less than the elastic limit.

Elastic Constants

Elastic constants are used to express the relationship between stresses and strains. Hooke's law , is stress/strain = a constant, within a certain limit. This means that any stress/corresponding strain = a constant, within certain limit. It follows that there can be three different types of such constants. (which we may call the elastic constants or elastic modulae) corresponding to three distinct types of stresses and strains. These are given below.

(i) Modulus of Elasticity or Young's Modulus (E)

Modulus of Elasticity is the ratio of direct stress to corresponding linear strain within elastic limit. If p is any direct stress below the elastic limit and e the corresponding linear strain, then **E = p / e**.

(ii) Modulus of Rigidity or Shear Modulus (G)

Modulus of Rigidity is the ratio of shear stress to shear strain within elastic limit. It is denoted by N,C or G. if q is the shear stress within elastic limit and ϕ the corresponding shear strain, then $\mathbf{G = q / \phi}$.

(iii) Bulk Modulus (K)

Bulk Modulus is the ratio of volumetric stress to volumetric strain within the elastic limit. If p_v is the volumetric stress within elastic limit and e_v the corresponding volumetric strain, we have $K = p_v / e_v$.

5. Poisson's Ratio

Any direct stress is accompanied by a strain in its own direction and called linear strain and an opposite kind strain in every direction at right angles to it, lateral strain. This lateral strain bears a constant ratio with the linear strain. This ratio is called the Poisson's ratio and is denoted by (1/m) or μ.

Poisson's Ratio = Lateral Strain / Linear Strain.

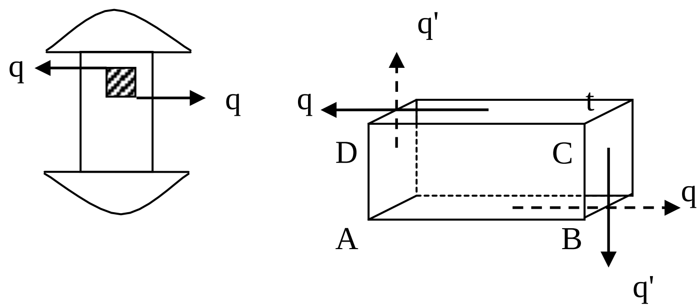
Value of the Poisson's ratio for most materials lies between 0.25 and 0.33.

6. Complementary Strain

Consider a rectangular element ABCD of a body subjected to simple shear of intensity q as shown. Let t be the thickness of the element.

Total force on face AB is , F_{AB} = stress X area = q X AB X t.

Total force on face CD is, F_{CD} = q X CD Xt = q XAB Xt.



FAB and FCD being equal and opposite, constitute a couple whose moment is given by,

M =F_{AB} X BC = q XAB X BC X t

Since the element is in equilibrium within the body, there must be a balancing couple which can be formed only by another shear stress of some intensity q' on the faces BC and DA. This shear stress is called the complementary stress.

F_{BC} = q' X BC X t

F_{DA} = q' X DA X t = q' X BC Xt

The couple formed by these two forces is M' = F_{BC} X AB = q' X BC X t

For equilibrium, M' = M.

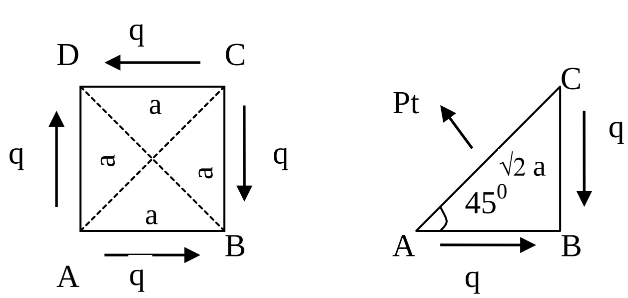
Therefore q' = q

This enables us to make the following statement.

In a state of simple shear a shear stress of any intensity along a plane is always accompanied by a complementary shear stress of same intensity along a plane at right angles to the plane.

<u>7. Direct Stresses Developed Due to Simple Shear.</u>

Consider a square element of side a and thickness t in a state of simple shear as shown in figure. It is clear that the shear stress on the forces of element causes it to elongate in the direction of the diagonal BD. Therefore a tensile stress of same intensity pt is induced in the elements along BD. ie, across the plane of the diagonal AC. The triangular portion ABC of the element is in equilibrium under the action of the following.



(1)F_{AC} = Normal force on face AC = pt X AC X t = pt X √2 aXt

(2)F_{AB} = Tangential force on face AB = q X BC X t = q aXt

(3)FBC = Tangential force on face BC = q X BC X t = q aXt

For equilibrium in the direction normal to AC,

FAC – F_{AB} cos45 – F_{BC} cos45 = 0

Pt X √2 at – q at X 1/√2 - q at X 1/√2 = 0

√2 pt – 2 q /√2 = 0

pt = q

It can also be seen that the shear stress on the faces of the element causes it to foreshorten in the direction of the diagonal BD. Therefore a compressive stress pc is induced in the element in the direction AC, ie across the plane of the diagonal BD. It can also be shown that pc = q.

It can thus be concluded that simple shear of any intensity gives rise to direct stresses of same intensity along the two planes inclined at 45º to the shearing plane. The stress along one of these planes being tensile and that along the other being compressive.

8. Relationship among the elastic constants

Relationship between modulus of elasticity and modulus of rigidity

Consider a square element ABCD of side 'a' subjected to simple shear of intensity q as shown in figure.

It is deformed to the shape AB'C'D under the shear stress. Drop perpendicular BE to the diagonal DB'. Let Φ be the shear strain induced and let N be the modulus of rigidity.

The diagonal DB gets elongated to DB'. Hence there is tensile strain et in the diagonal.

et = (DB' – DB) / DB = EB' / DB

since this deformation is very small we can take L BB'E = 45º

EB' = BB' / √2 = AB tan Φ / √2 = a tan Φ / √2

DB = √2 a

et = (a tan Φ / √2)/ √2 a = tan Φ /2 = = Φ / 2 since Φ is small

ie et = ½ X q/N---------------------------(1)

We know that stress along the diagonal DB is a pure tensile stress pt = q and that along the diagonal AC is a pure compressive stress pc also equal to q. hence the strain along the diagonal DB is et = q/E + 1/m X q/E

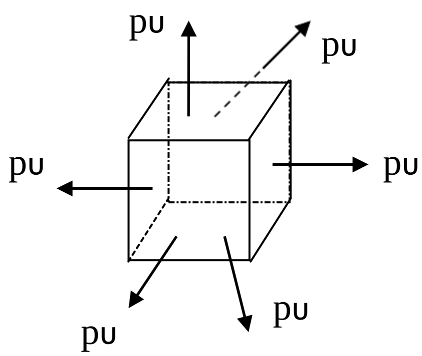
Ie et = q/E (1+1/m) ----------------------(2)

From (1) and (2) we have,

E = 2N(1+1/m)

This is the required relationship between E and N.

Relationship between Modulus of Elasticity E and Bulk Modulus K



Consider a cube element subjected to volumetric tensile stress pυ in X,Y and Z directions. Stress in each direction is equal to pυ. ie px = py = pz =pυ

Consider strains induced in X-direction by these stresses. px induces tensile strain , while py and pz induces compressive strains. Therefore,

ex = px/E – 1/m[py/E + pz/E] = pυ/E[1-2/m]

due to the perfect symmetry in geometry and stresses

ey = pυ/E[1-2/m]

ez = pυ/E[1-2/m]

K = p υ / e υ =p υ/(ex+ey+ez) = pυ/[3pυ/E(1-2/m)]

ie E = 3K(1-2/m) is the required relationship.

Relationship among the constants

From above,

E = 2N[1+(1/m)] and E = 3K[1-(2/m)]

$E = 3K[1-2(E/2N -1)] = 3K[1-E/N +2]$
$9K = E[1+(3K/N)] = E[(N+3K)/N]$
$E = 9NK/(N+3K)$

9. Bars of uniform section

Consider a bar of length l and Cross sectional area A. Let P be the axial pull on the bar,p the stresss induced ,e the strain in the bar and δl is the elongation.

Then $p= P/A$

$e= p/E =P/(AE)$---------------------(1)

$e= \delta l/l$-----------------------(2)

equating (1) and (2)

$$\delta l = Pl / (AE)$$

CHAPTER-3 BENDING MOMENT AND SHEAR FORCE

TYPES OF LOADS

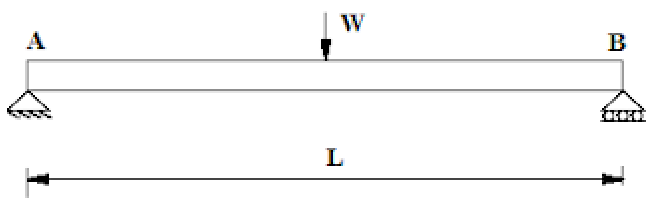
A beam is usually horizontal member and load which will be acting over the beam will be usually vertical loads. There are following types of loads as mentioned here and we will discuss each type of load in detail.

- Point load or concentrated load
- Uniformly distributed load
- Uniformly varying load

- **Point load or concentrated load**

  Point load or concentrated load, as name suggest, acts at a point on the beam. If we will see practically, point load or concentrated load also distributed over a small area but we can consider such type of loading as point loading and hence such type of load could be considered as point load or concentrated load.

  Following figure displayed here indicates the beam AB of length L which will be loaded with point load W at the midpoint of the beam. Load W will be considered here as the point load.

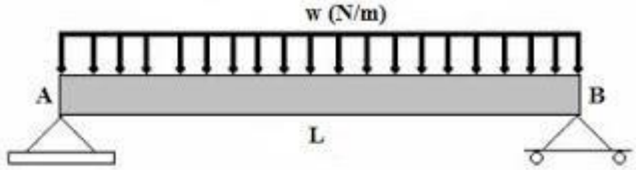


- **Uniformly distributed load**

  Uniformly distributed load is the load which will be distributed over the length of the beam in such a way that rate of loading will be uniform throughout the distribution length of the beam.

  Uniformly distributed load is also expressed as U.D.L and with value as w N/m. During determination of the total load, total uniformly distributed load will be converted in to point load by multiplying the rate of loading i.e. w (N/m) with the span of load distribution i.e. L and will be acting over the midpoint of the length of the uniformly load distribution.

  Let us consider the following figure, a beam AB of length L is loaded with uniformly distributed load and rate of loading is w (N/m).



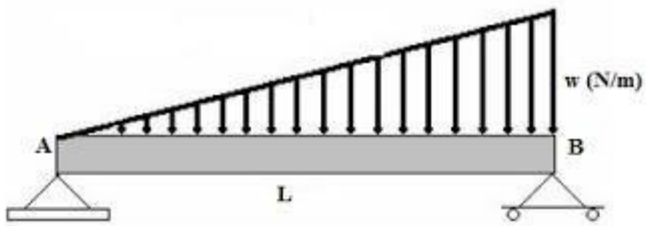
Total uniformly distributed load, P = w*L

- **Uniformly varying load**

  Uniformly varying load is the load which will be distributed over the length of the beam in such a way that rate of loading will not be uniform but also vary from point to point throughout the distribution length of the beam.

  Uniformly varying load is also termed as triangular load. Let us see the following figure, a beam AB of length L is loaded with uniformly varying load.

  We can see from figure that load is zero at one end and increases uniformly to the other end. During determination of the total load, we will determine the area of the triangle and the result i.e. area of the triangle will be total load and this total load will be assumed to act at the C.G of the triangle.



Total load, P = w*L/2

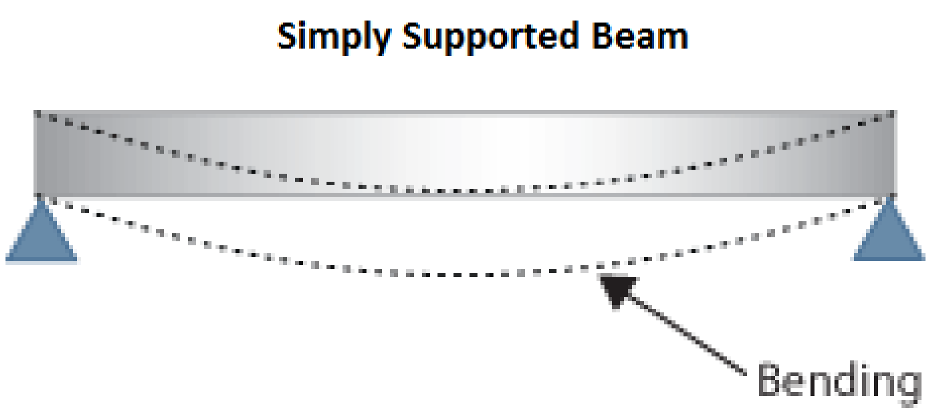
TYPES OF BEAMS

The four different types of beams are:

1. Simply Supported Beam
2. Fixed Beam
3. Cantilever Beam
4. Continuously Supported Beam

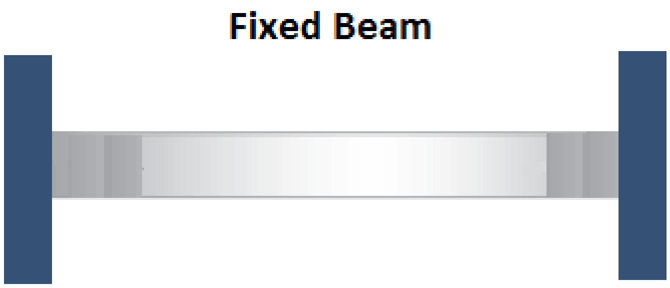
1. Simply Supported Beam

If the ends of a beam are made to rest freely on supports beam, it is called a simple (freely) supported beam.



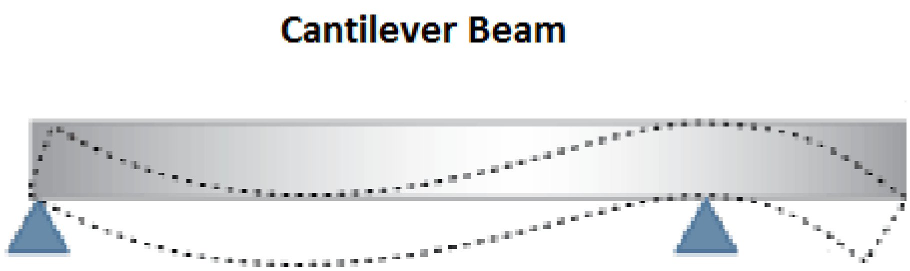
2. Fixed Beam

If a beam is fixed at both ends it is free called fixed beam. Its another name is a built-in beam.



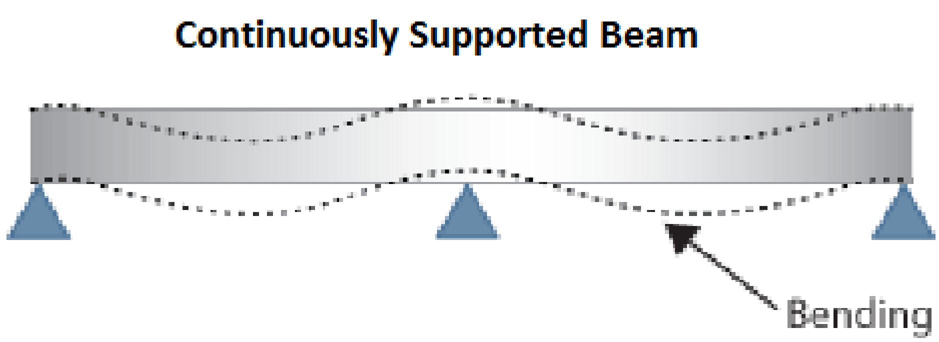
3. Cantilever Beam

If a beam is fixed at one end while the other end is free, it is called cantilever beam.



4. Continuously Supported Beam

If more than two supports are provided to the beam, it is called continuously supported beam.

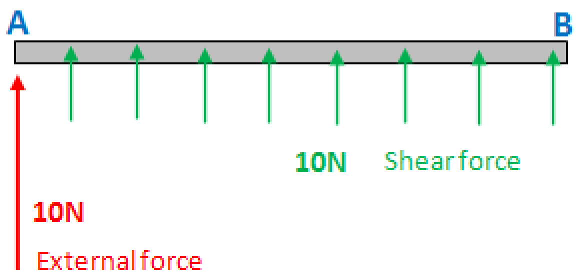


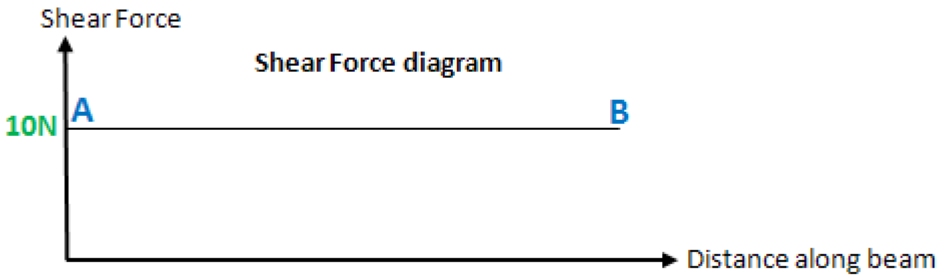
TYPES OF SUPPORT

Different types of external supports are as follows:

- Fixed support
- Pinned support or hinged support
- Roller support
- Link support
- Simple support

Below a force of 10N is exerted at point A on a beam. This is an external force. However because the beam is a rigid structure,the force will be internally transferred all along the beam. This internal force is known as shear force. The shear force between point A and B is usually plotted on a shear force diagram. As the shear force is 10N all along the beam, the plot is just a straight line, in this example.



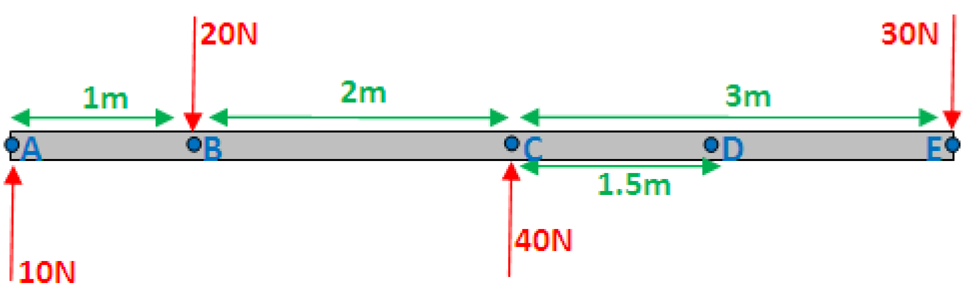


The idea of shear force might seem odd, maybe this example will help clarify. Imagine pushing an object along a kitchen table, with a 10N force. Even though you're applying the force only at one point on the object, it's not just that point of the object that moves forward. The whole object moves forward, which tells you that the force must have transferred all along the object, such that every atom of the object is experiencing this 10N force.

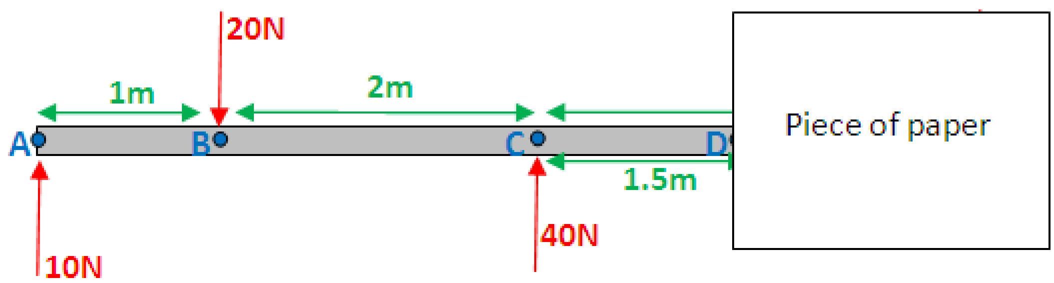
Please note that this is not the full explanation of what shear force actually is.

Basic shear diagram

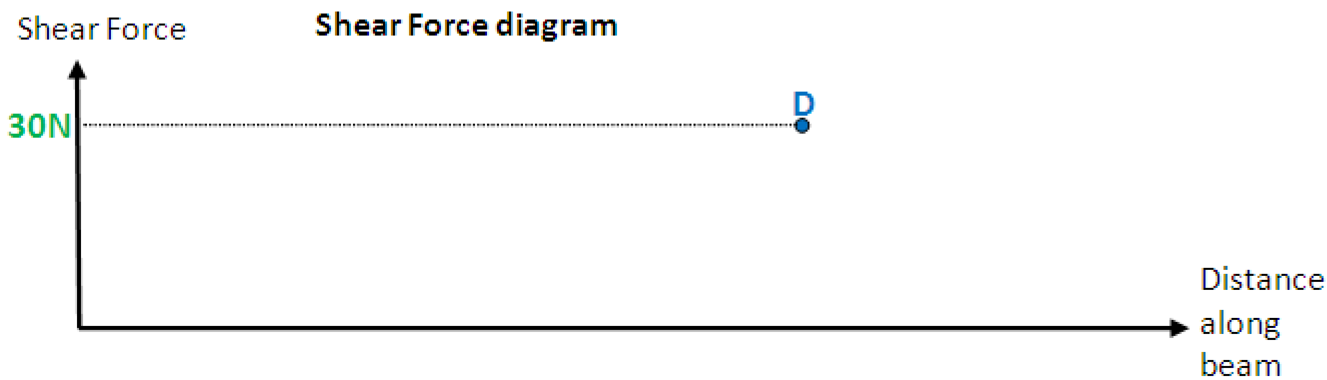
What if there is more than one force, as shown in the diagram below, what would the shear force diagram look like then?



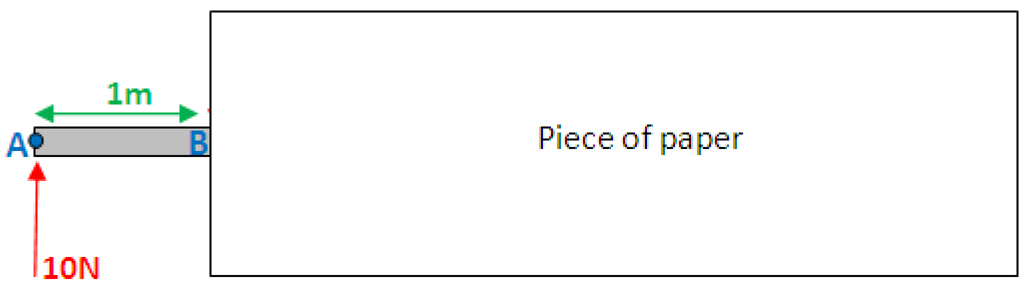
The way you go about this is by figuring out the shear force at points A,B,C,E (as there is an external force acting at these points). The way you work out the shear force at any point, is by covering (either with your hand or a piece of paper), everything to right of that point, and simply adding up the external forces. Then plot the point on the shear force diagram. For illustration purposes, this is done for point D:



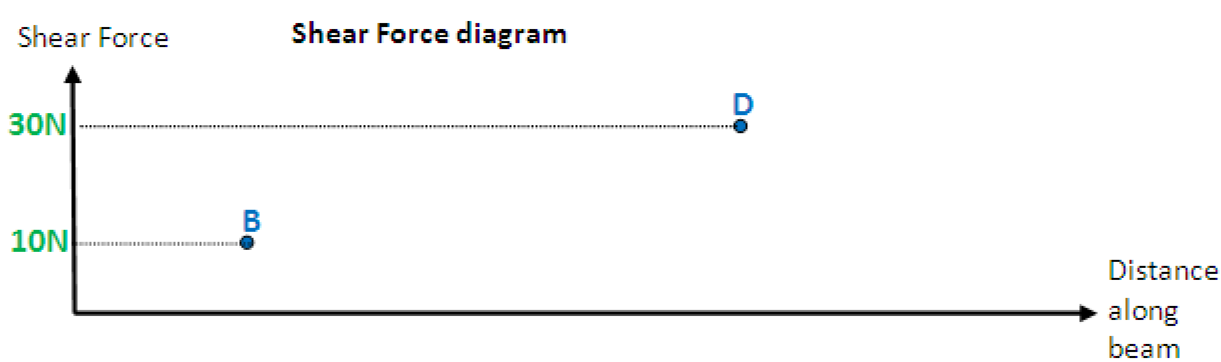
Shear force at D = 10N - 20N + 40N = 30Newtons



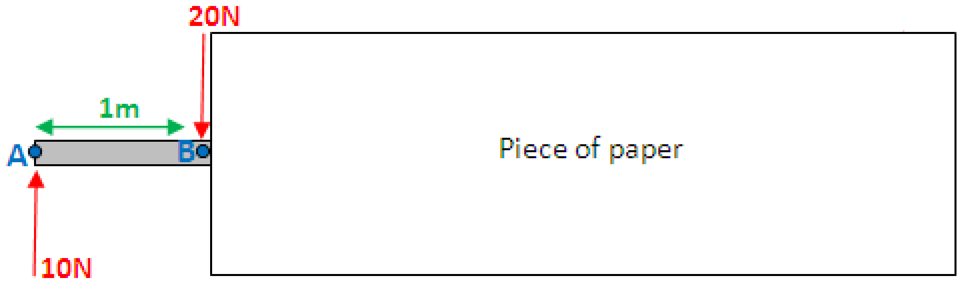
Now, let's do this for point B. BUT - slight complication - there's a force acting at point B, are you going to include it? The answer is both yes and no. You need to take 2 measurements. Firstly put your piece of paper, so it's JUST before point B:



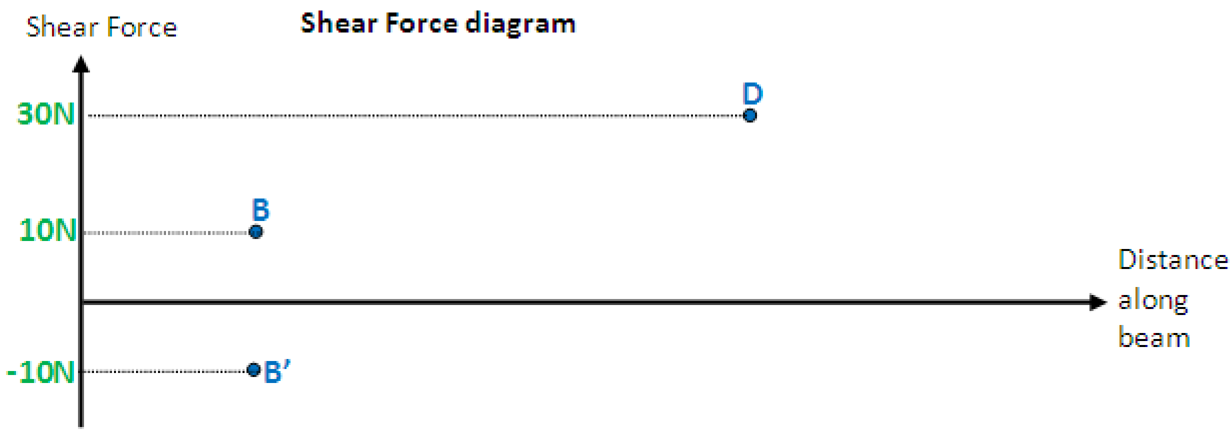
Shear force at B = 10N



Now place your paper JUST after point B:

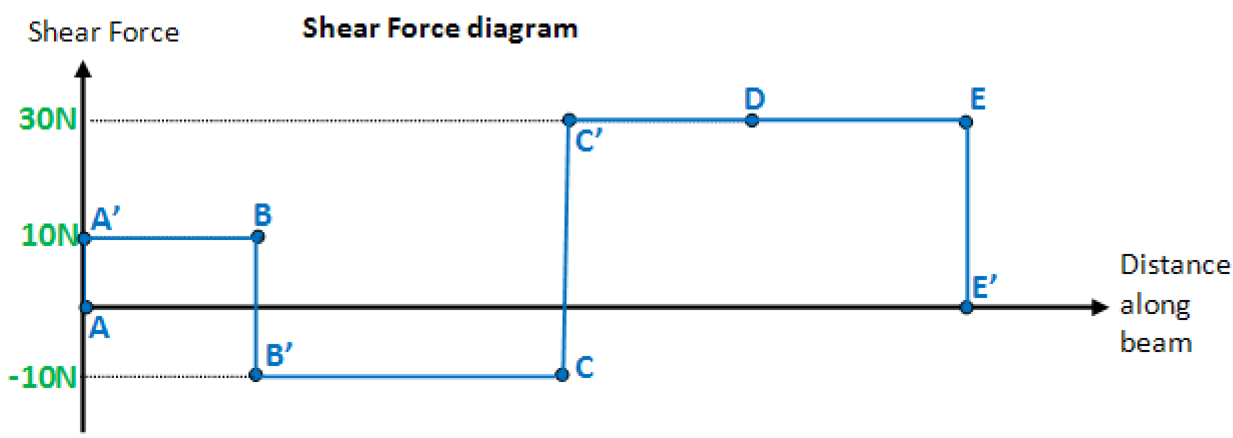


Shear force at B = 10N - 20N = -10N



(B' is vertically below B)

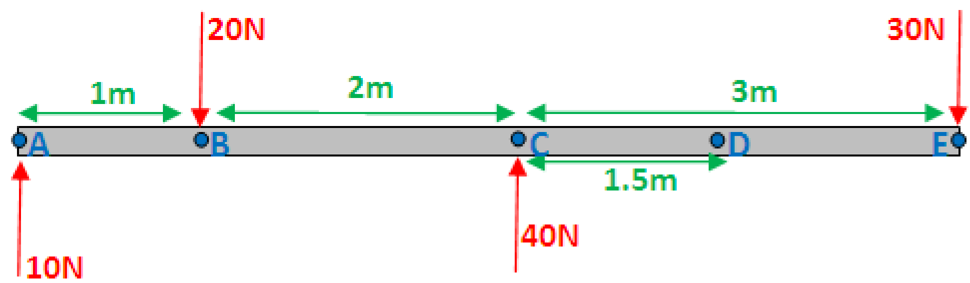
Now, do point A, D and E, and finally join the points. your diagram should look like the one below. If you don't understand why, leave a message on the discussion section of this page (its at the top), I will elaborate on the explanation:



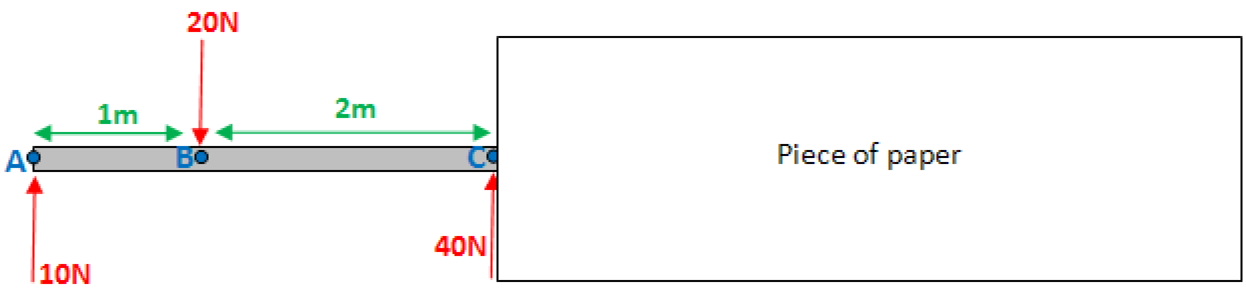
Notice how nothing exciting happens at point D, which is why you wouldn't normally analyse the shear force at that point. For clarity, when doing these diagrams it is recommended you move you paper from left to right, and hence analyse points A,B, C, and E, in that order. You can also do this procedure covering the left side instead of the right, your diagram will be "upside down" though. Both diagrams are correct.

Basic bending moment diagram

Bending moment refers to the internal moment that causes something to bend. When you bend a ruler, even though apply the forces/moments at the ends of the ruler, bending occurs all along the ruler, which indicates that there is a bending moment acting all along the ruler. Hence bending moment is shown on a bending moment diagram. The same case from before will be used here:

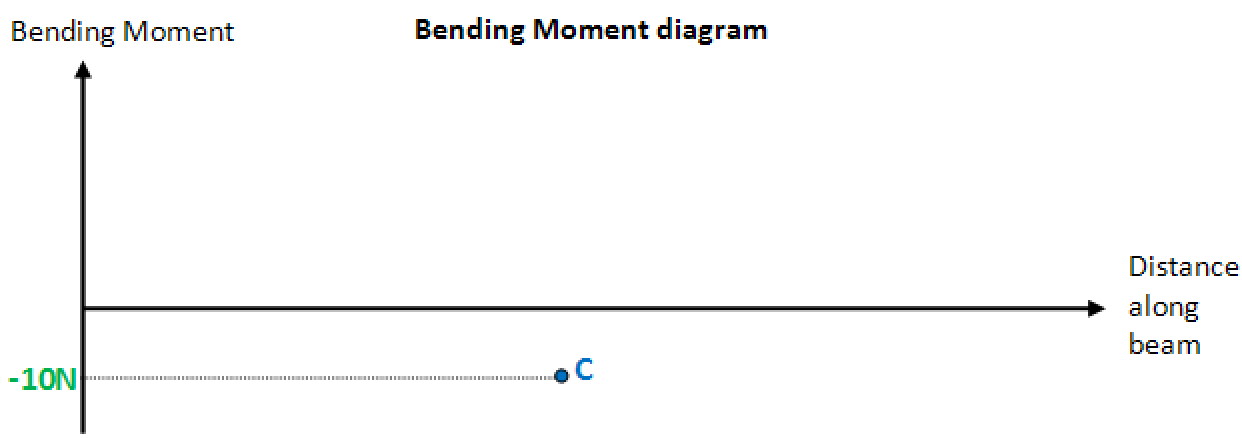


To work out the bending moment at any point, cover (with a piece of paper) everything to the right of that point, and take moments about that point. (I will take clockwise moments to be positive). To illustrate, I shall work out the bending moment at point C:



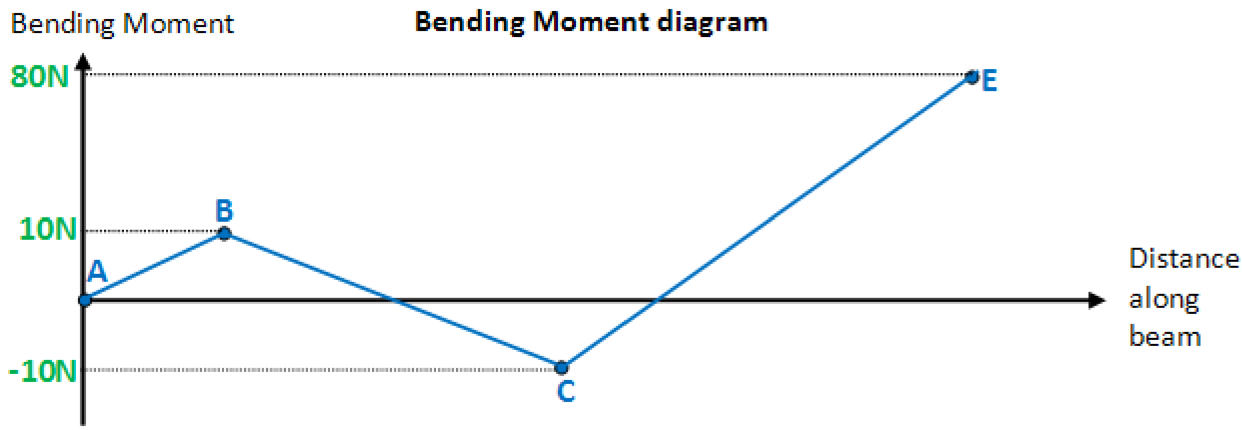
Bending moment at C = 10Nx3m - 20Nx2m = -10Nm

(Please note that the two diagrams below should show units in "Nm", not in "N" as it is currently showing)



Notice that there's no need to work out the bending moment "just before and just after" point C, (as in the case for the shear force diagram). This is because the 40N force at point C exerts no moment about point C, either way.

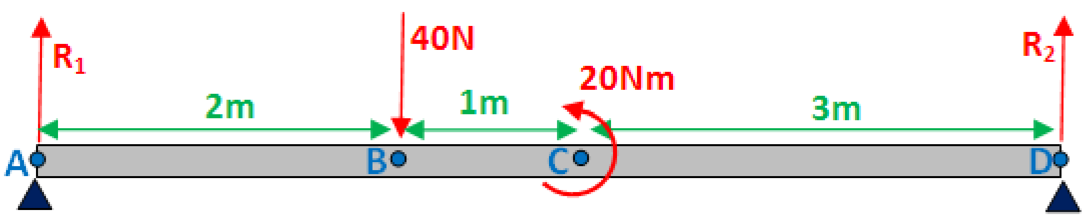
Repeating the procedure for points A,B and E, and joining all the points:



Normally you would expect the diagram to start and end at zero, in this case it doesn't. This is my fault, and it happened because I accidentally chose my forces such that there is a moment disequilibrium. i.e. take moments about any point (without covering the right of the point), and you'll notice that the moments aren't balanced, as they should be. It also means that if you're covering the left side as opposed to the right, you will get a completely different diagram. Sorry about this... Upon inspection, the forces are unbalanced, so it is immediately expected that the diagram will most likely not be balanced.

Point moments

Point moments are something that you may not have come across before. Below, a point moment of 20Nm is exerted at point C. Work out the reaction of A and D:



Force equilibrium: $R_1 + R_2 = 40$

Taking moments about A (clockwise is positive): $40\cdot2 - 20 - 6\cdot R_2 = 0$

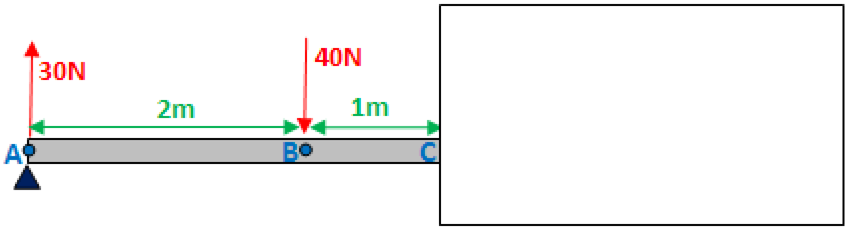
$R_1 = 30N$, $R_2 = 10N$

If instead you were to take moments about D you would get: $- 20 - 40\cdot4 + 6\cdot R_1 = 0$

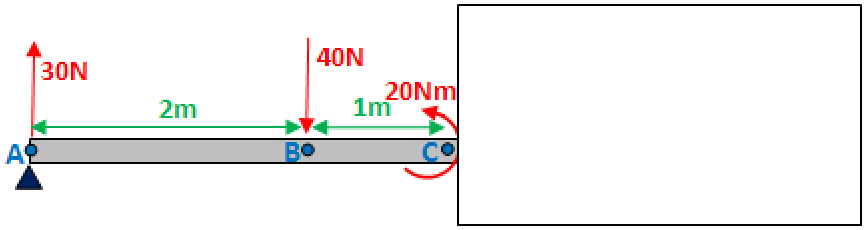
I think it's important for you to see that wherever you take moments about, the point moment is always taken as a negative (because it's a counter clockwise moment).

So how does a point moment affect the shear force and bending moment diagrams?
Well. It has absolutely no effect on the shear force diagram. You can just ignore point C when drawing the shear force diagram. When drawing the bending moment diagram you will need to work out the bending moment just before and just after point C:

Just before C:



Just after C:



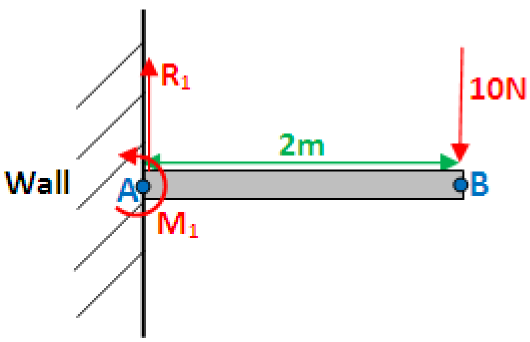
Just before: bending moment at C = 3·30 - 1·40 = 50Nm

Just after: bending moment at C = 3·30 - 1·40 - 20 = 30Nm

Then work out the bending moment at points A, B and D (no need to do before and after for these points). And plot.

Cantilever beam

Until now, you may have only dealt with "simply supported beams" (like in the diagram above), where a beam is supported by 2 pivots at either end. Below is a cantilever beam, which means - a beam that rigidly attached to a wall. Just like a pivot, the wall is capable of exerting an upwards reaction force R_1 on the beam. However it is also capable of exerting a point moment M_1 on the beam.

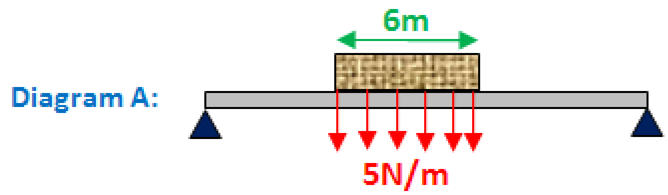


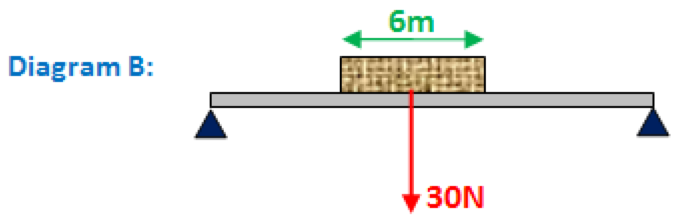
Force equilibrium: $R_1 = 10N$

Taking moments about A: $-M_1 + 10 \cdot 2 = 0 \rightarrow M_1 = 20Nm$

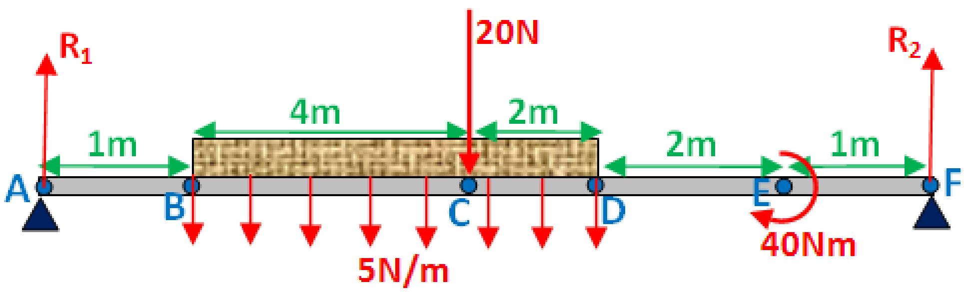
Uniformly Distributed Load (UDL)

Below is a brick lying on a beam. The weight of the brick is uniformly distributed on the beam (shown in diagram A). The brick has a weight of 5N per meter of brick (5N/m). Since the brick is 6 meters long the total weight of the brick is 30N. This is shown in diagram B. Diagram B is a simplification of diagram A. As you will see, you will need to be able to convert a type A diagram to a type B.





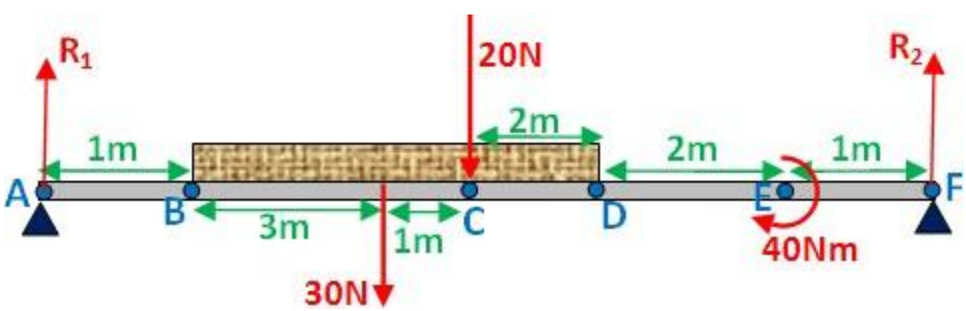
To make your life more difficult I have added an external force at point C, and a point moment to the diagram below. This is the most difficult type of question I can think of, and I will do the shear force and bending moment diagram for it, step by step.



Firstly identify the key points at which you will work out the shear force and bending moment at. These will be points: A,B,C,D,E and F.

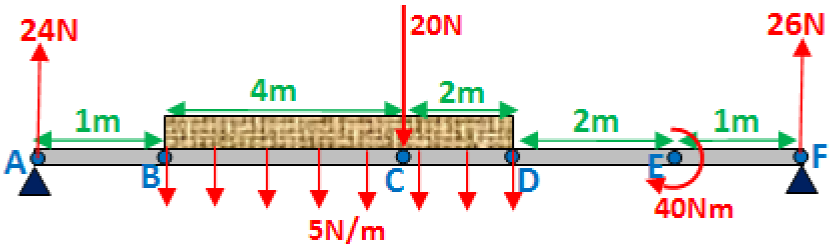
As you would have noticed when working out the bending moment and shear force at any given point, sometimes you just work it out at the point, and sometimes you work it out just before and after. Here is a summary: When drawing a shear force diagram, if you are dealing with a point force (points A,C and F in the above diagram), work out the shear force before and after the point. Otherwise (for points B and D), just work it out right at that point. When drawing a bending moment diagram, if you are dealing with a point moment (point E), work out the bending moment before and after the point. Otherwise (for points A,B,C,D, and F), work out the bending moment at the point.

After identifying the key points, you want to work out the values of R_1 and R_2. You now need to convert to a type B diagram, as shown below. Notice the 30N force acts right in the middle between points B and D.



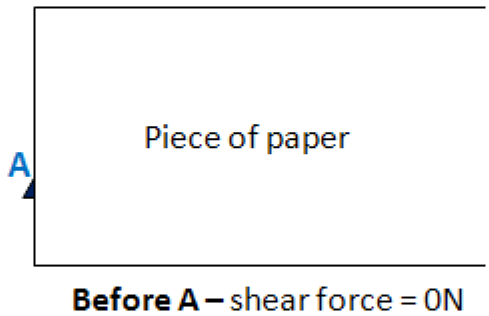
Force equilibrium: $R_1 + R_2 = 50$
Take moments about A: $4 \cdot 30 + 5 \cdot 20 + 40 - 10 \cdot R_2 = 0$
$R_1 = 24N$, $R_2 = 26N$

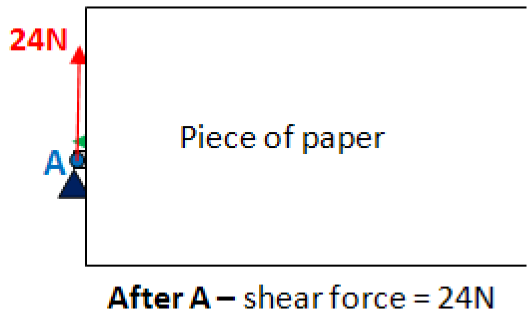
Update original diagram:



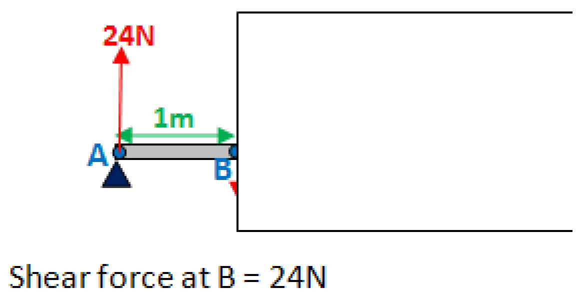
Shear force diagram

point A:





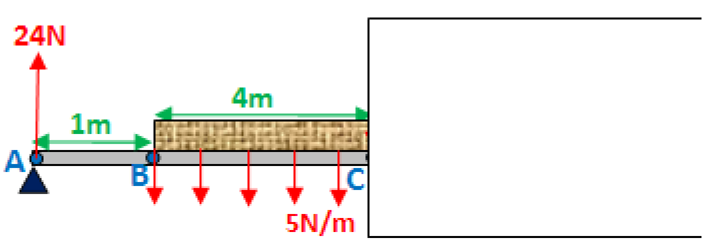
point B:



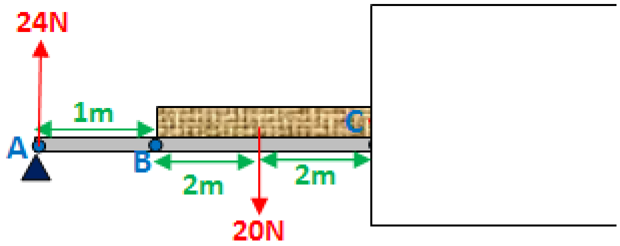
Notice that the uniformly distributed load has no effect on point B.

point C:

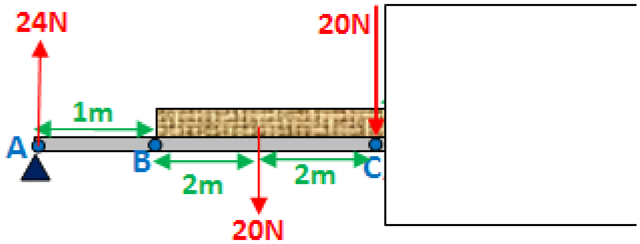
Just before C:



Now convert to a type B diagram. Total weight of brick from point B to C = 5x4 = 20N

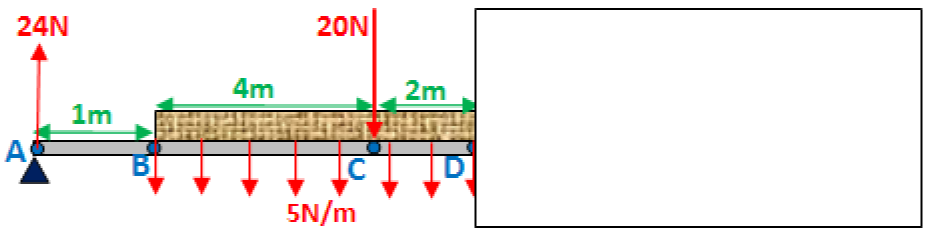


Shear force before C: 24 - 20 = 4N

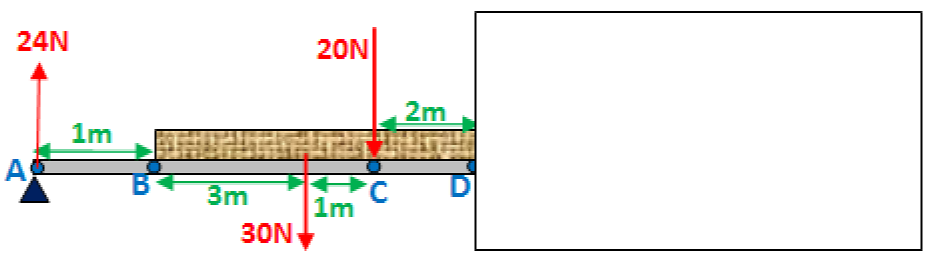


Shear force after C: 24 - 20 - 20 = -16N

point D:



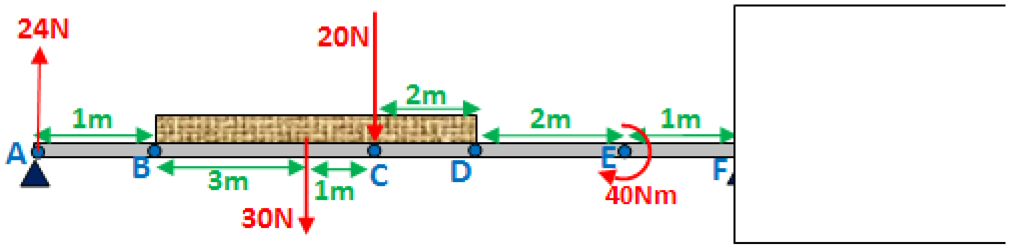
Convert to type B diagram:



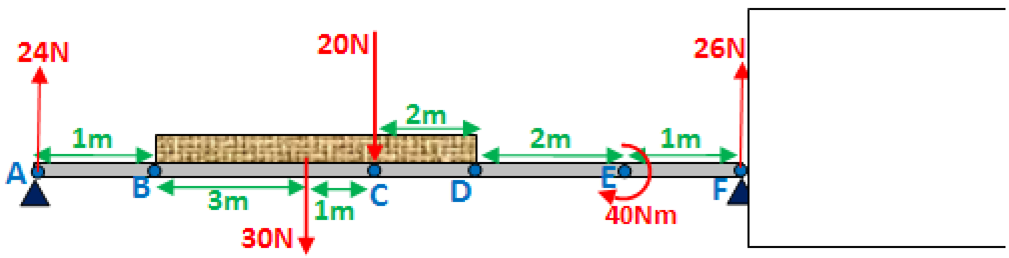
Shear force at D: 24 - 30 - 20 = -26N

point F:

(I have already converted to a type B diagram, below)

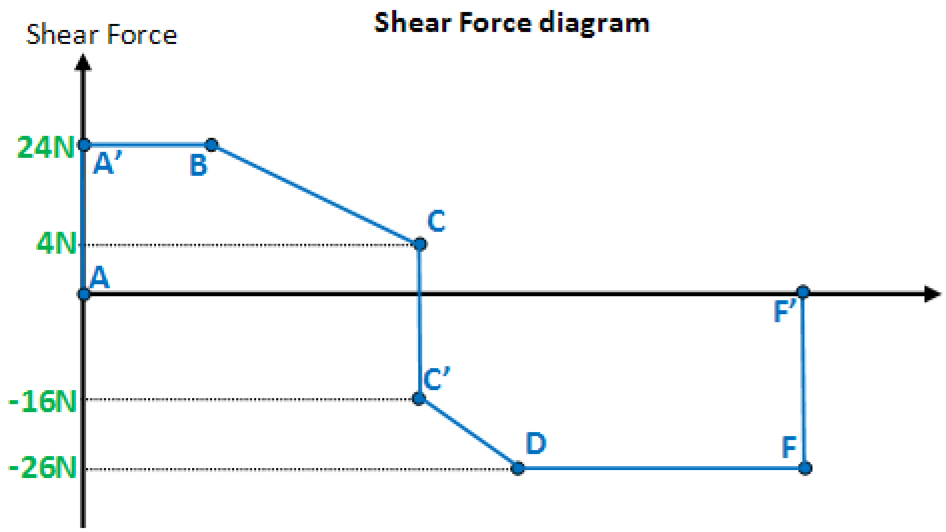


Shear force before F: 24 -30 -20 = -26N



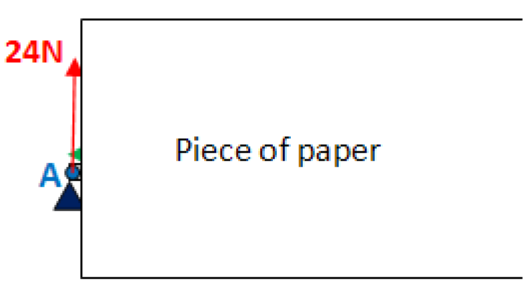
Shear force after F: 24 -30 -20 +26 = 0N

Finally plot all the points on the shear force diagram and join them up:



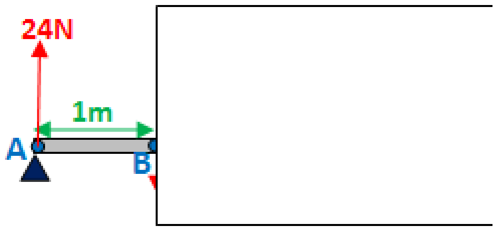
Bending moment diagram

Point A



Bending moment at A: 0Nm

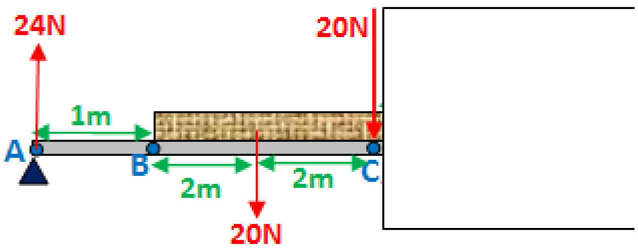
Point B



Bending moment at B: 24·1 = 24Nm

point C:

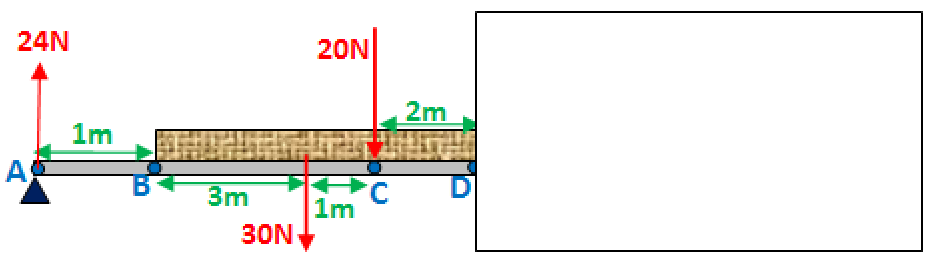
(I have already converted to a type B diagram, below)



Bending moment at C: 24·5 - 20·2 = 80Nm

point D:

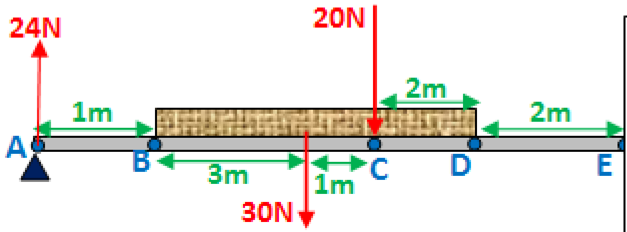
(I have already converted to a type B diagram, below)



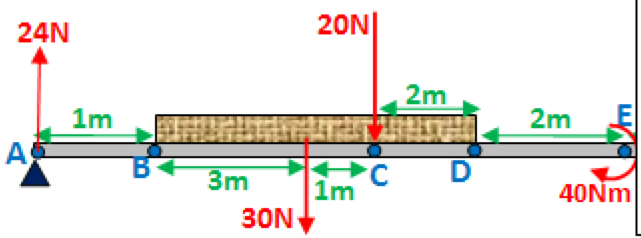
Bending moment at D: 24·7 - 30·3 - 20·2 = 38Nm

point E:

(I have already converted to a type B diagram, below)



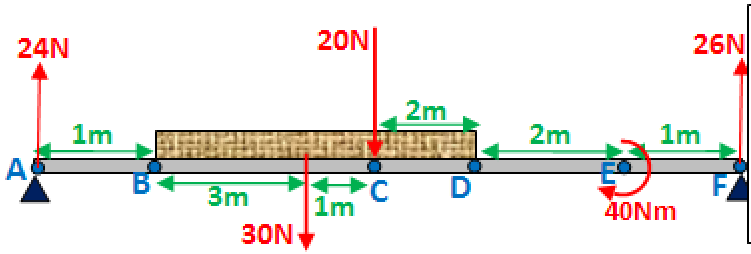
Bending moment just before E: 24·9 - 30·5 -20·4 = -14Nm



Bending moment just after E: 24·9 - 30·5 -20·4 + 40 = 26Nm

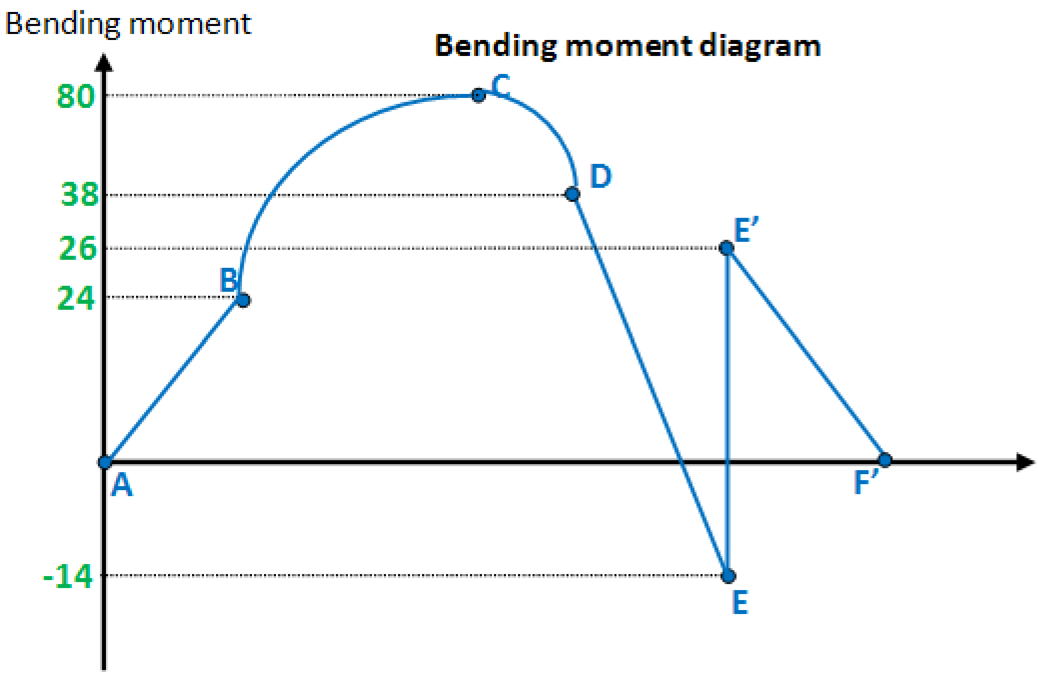
point F:

(I have already converted to a type B diagram, below)



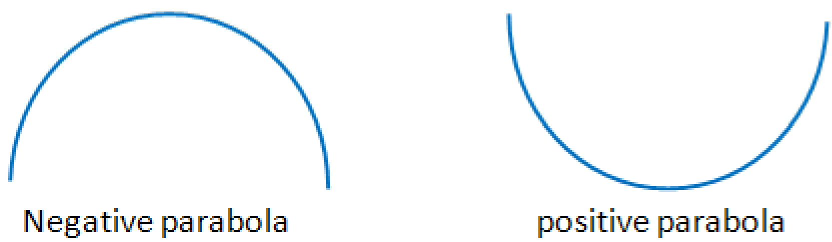
Bending moment at F: 24·10 - 30·6 - 20·5 + 40 = 0Nm

Finally, plot the points on the bending moment diagram. Join all the points up, EXCEPT those that are under the uniformly distributed load (UDL), which are points B,C and D. As seen below, you need to draw a curve between these points. Unless requested, I will not explain why this happens.



Note: The diagram is not at all drawn to scale.

I have drawn 2 curves. One from B to C, one from C to D. Notice that each of these curves resembles some part of a negative parabola.

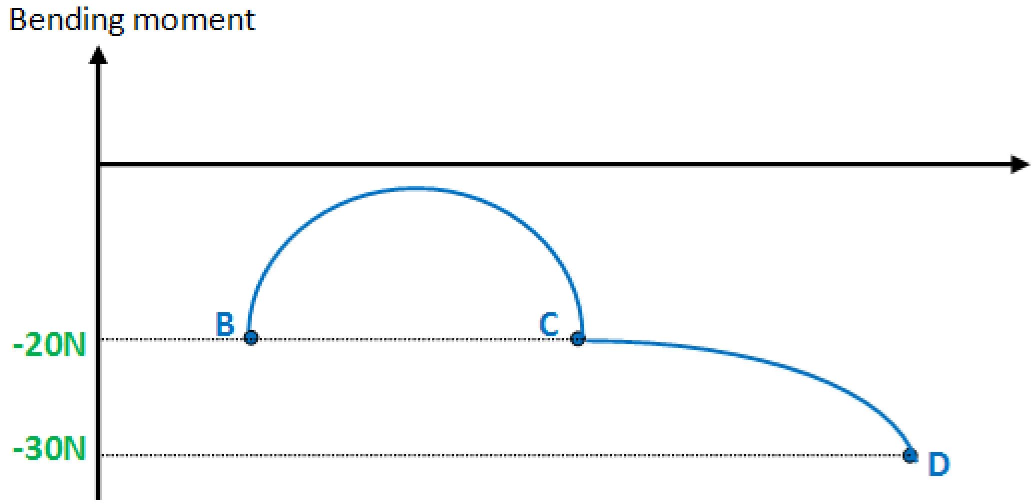


Rule: When drawing a bending moment diagram, under a UDL, you must connect the points with a curve. This curve must resemble some part of a negative parabola.

Note: The convention used throughout this page is "clockwise moments are taken as positive". If the convention was "counter-clockwise moments are taken as positive", you would need to draw a positive parabola.

Hypothetical scenario

For a hypothetical question, what if points B, C and D, were plotted as shown below. Notice how I have drawn the curves for this case.



If you wanted to find the peak of the curve, how would you do it? Simple. On the original diagram (used at the start of the question) add an additional point (point G), centrally between point B and C. Then work out the bending moment at point G.

That's it! If you have found this article useful, please comment in the discussion section (at the top of the page), as this will help me decide whether to write more articles like this. Also please comment if there are other topics you want covered, or you would like something in this article to be written more clearly.

IMPORTANT QUESTIONS

OBJECTIVE

1. 1. The bending moment for a certain portion of the beam is constant. For that section, shear force would be

- Zero
- Increasing
- Decreasing
- Constant

2. Hoop stress induced in a thin cylinder by winding it with wire under tension will be

- Compressive
- Tensile
- Shear
- Zero

3. What is the limiting value of Poisson's ratio?

- 0 and 0.5
- 1 and -0.5
- -1 and -0.5
- -1 and 0.5

4. Slenderness ratio has a dimension of

- m
- m^-1
- m^2
- Dimensionless quantity

5. Young's modulus of elasticity for a perfectly rigid body is

- Zero
- Unity
- Infinity
- None of these

6. Which material has the highest value of Poisson's ratio?

- Rubber
- Wood
- Copper
- Steel

7. Radial stress in a thin spherical pressure vessel is

- Equal to hoop stress
- Double the hoop stress
- Half the hoop stress
- Zero

8. The area under stress strain curve represents

- Breaking strength of material
- Toughness of material
- Hardness of material
- Energy required to cause failure

9. For a thin spherical shell subjected to internal pressure , the ratio of volumetric strain to diametrical circumferential strain is

- 1.25
- 1.5
- 2.0
- 3.0

10. Which of the following beam is likely to have the point of contraflexture?

- Cantilever beam

- Simply supported beam
- Beam with overhangs
- Beam fixed at both ends

11. The______________forces are used are used in the method of sections for the calculation of the internal forces.
a) Internal rotational
b) Couple rotational
c) Translational
d) External

12. Every point on the force vector which is the internal force is having the same magnitude and the same direction as the whole force vector have.
a) True
b) False

13. For getting the normal force on the supports, we do what?
a) Make the vertical sum of the forces equal to zero
b) Make the horizontal sum of the forces equal to zero
c) Make the moment sum of the forces equal to zero
d) Make the rotational sum of the forces equal to zero

14. For getting the horizontal component of the support reactions what do we do?
a) Make the vertical sum of the forces equal to zero
b) Make the horizontal sum of the forces equal to zero
c) Make the moment sum of the forces equal to zero
d) Make the rotational sum of the forces equal to zero

15. Twisting moment is also called as ___________
a) Moment of line
b) Moment of section
c) Moment of plane
d) Torsional moment

16. The loading generally act upon the_____________of the body.
a) Centroid
b) Symmetrical centre
c) Rotational centre
d) Chiral centre

17. The area of does make the difference in the internal forces, that is if the area is large the internal force acting is also large and vice versa.
a) True
b) False

18. The magnitude of each loading will be____________at various points along the axis of the member of the beam.
a) Same
b) Different
c) Slightly different
d) Slightly same

19. Torsional moment is applied at the____________part of the beam.
a) The centroid
b) The left end
c) The right end
d) The axis beyond the body of the beam

20. Normal force is equal to ________________
a) The net horizontal force
b) The net vertical force with a negative sign
c) The net horizontal force with a negative sign
d) The net vertical force

21. If the normal force creates a tension then the force is said to be _____________
a) Positive
b) Negative
c) Rotational
d) Collinear

22. If the shear force creates a clockwise rotation then the force is said to be _____________
a) Positive
b) Negative
c) Rotational
d) Collinear

SHORT QUESTIONS

1. Define Elastic Materials ?
2. Define Plastic Materials ?
3. Define Brittle Materials ?
4. Define ductile Materials ?
5. Give Any Two Example Of Ductile Materials ?
6. Give Any Two Example Of Brittle Materials ?
7. Describe tensile test?
8. Rigid body?
9. Describe compresion test?
10. Bulk modulus?
11. FOS
12. define load?
13. Define tensile load ?
14. Define compressive load ?
15. Define shear stress load ?
16. define stress?
17. define strain?
18. Define hooke's law?
19. define volumetric strain ?
20. unit of stress, strain. Modulus of rigidity?
21. define bending moment?
22. Define shear force?
23. define types of load?
24. define types of beam?
25. define types of supports ?
26. define types of reactions ?
27. define point of contraflexure?
28. define elastisity ?
29. relation between modulus canstant?
30. define bulk modulus ?

LONG QUESTIONS

1. Describe compresion test?
2. Describe tensile test?
3. discribe stress-strain digram/graph ?
4. define hooke's law?
5. derivate the formula of elongation of bars?
6. Describe E,C,K. Relationship also?
7. Define mechanical properties of materials ?
8. defferentiate between load and stress?
9. describe sfd &bmd and point of contraflexure.?

Shear Force and Bending Moment

TYPES OF FORCES: Basically, structural members experience two types of forces.

External Forces: Actions of other bodies on the structure under consideration are known as external forces.

Internal Forces: Forces and couples exerted on a member or portion of the structure by the rest of the structure. Internal forces always occur in equal but opposite pairs.

TYPES OF LOAD

The following are the important types of load which act on a beam.

1. Concentrated or point load,
2. Uniformly distributed load, and
3. Uniformly varying load

1. **Concentrated or Point Load:** Load acting at a point or over very limited area compared to the length of the beam is known as concentrated load or point load.

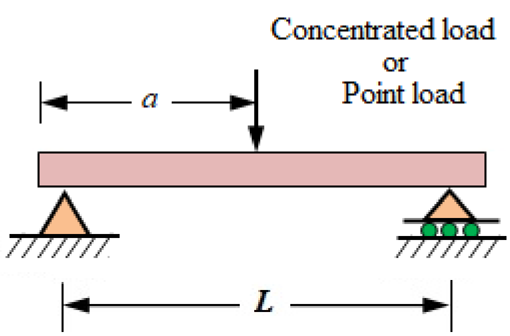


Fig. 1

2. **Uniformly Distributed Load:** Load that is spread over a beam with uniform rate of loading, (‘*w*’ per unit run) is known as uniformly distributed load or *UDL*. Uniformly distributed load is also known as rectangular load.

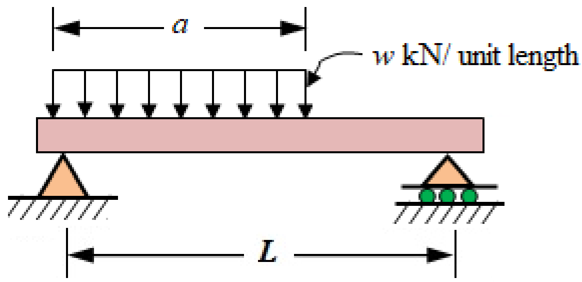


Fig. 2

3. **Uniformly Varying Load:** Load that is spread over a beam with the rate of loading uniformly from one point to the other along the beam is known as uniformly varying load. Uniformly varying distributed load is also known as triangular load.

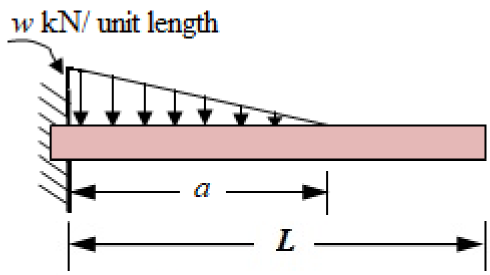


Fig. 3

4. **Parabolic Load:** If the variation of load distribution follows the equation of parabola, it is known as parabolic distributed load or simply parabolic load.

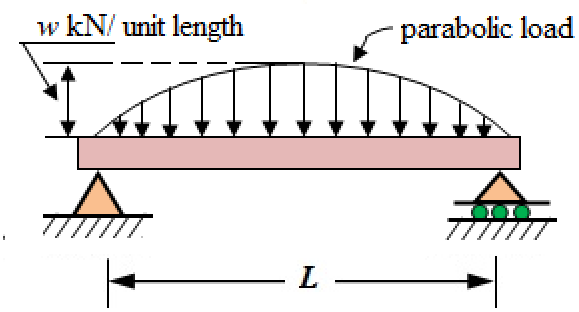


Fig. 4

TYPES OF SUPPORTS

1. Simple support
2. Roller Support

3. Pin (or) Hinge Support
4. Fixed support

Simple Supports

Simple support is just a support on which structural member rests. It is idealized to be a frictionless surface support. It only resists vertical movement of support. A simple support is free to rotate and translate along the surface upon which it rests. The resulting reaction force is always a single force perpendicular to the plane of support.

The horizontal or lateral movement allowed is up to a limited extent and after that the structure loses its support. For example, if a plank is laid across gap to provide a bridge, it is assumed that the plank will remain in its place. It will do so until a foot kicks it or moves it. At that moment the plank will move because the simple connection cannot develop any resistance to the lateral load.

This type of support is not commonly used in structural purposes. However, Simple supports are often found in zones of frequent seismic activity.

Roller Supports

Roller supports are free to rotate and translate along the surface upon which they rest. The surface can be horizontal, vertical, or sloped at any angle. They cannot resist parallel or horizontal forces and moment. They only resist perpendicular forces. Hence, the resulting reaction force is always a single force that is perpendicular to the plane of support.

This type of support is provided at one end of bridge spans. The reason for providing roller support at one end is to allow contraction or expansion of bridge deck with respect to temperature differences in atmosphere. If roller support is not provided then it will cause severe damage to the banks of bridge. But this horizontal force should be resisted by at least one support to provide stability so, roller support should be provided at one end only not at both ends.

Pinned Supports

A pinned support is same as hinged support. It can resist both vertical and horizontal forces but not a moment. It allows the structural member to rotate, but not to translate in any direction.

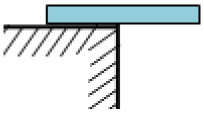
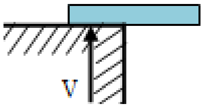
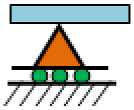
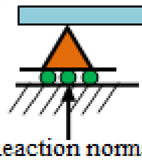
Many connections are assumed to be pinned connections even though they might resist a small amount of moment in reality. It is also true that a pinned connection could allow rotation in only one direction; providing resistance to rotation in any other direction. In human body knee is the best example of hinged support as it allows rotation in only one direction and resists lateral movements. Ideal pinned and fixed supports are rarely found in practice, but beams supported on walls or simply connected to other steel beams are regarded as pinned. The distribution of moments and shear forces is influenced by the support condition.

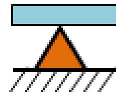
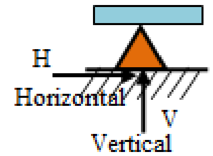
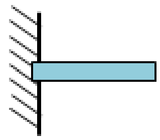
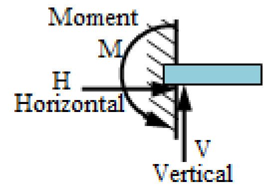
Best example for hinged support is door leaf which only rotates about its vertical axis without any horizontal or vertical movement.

Fixed Supports

Fixed support can resist vertical and horizontal forces as well as moment since they restrain both rotation and translation. They are also known as rigid support for the stability of a structure there should be one fixed support. A flagpole at concrete base is common example of fixed support In RCC structures the steel reinforcement of a beam is embedded in a column to produce a fixed support as shown in above image. Similarly all the riveted and welded joints in steel structure are the examples of fixed supports Riveted connection are not very much common now a days due to the introduction of bolted joints.

Table 1. Idealized Structural Supports

Types of supports	Real life Example	Symbol	Movement allowed and prevented	Unknown reactions
Frictionless or Simple support			Prevented: vertical translation Allowed: horizontal translation and rotation	
Roller support			Prevented: vertical translation Allowed: horizontal translation and rotation	

Pinned or hinged support			Prevented: horizontal translation and vertical translation Allowed: Rotation	
Fixed or Built-in support			Prevented: horizontal translation, vertical translation and rotation	

BEAM:

A **Beam** is defined as a structural member subjected to transverse shear loads (load normal to the axis of the beam) during its functionality. Due to the transverse shear loads, a beam is subjected to variable shear force and bending moment. Beam is a flexural member, designed primarily for bending. Analysis of beam pertains to the calculations of shear forces and bending moments along the length of the beam and drawing of shear force diagram and bending moment diagram.

TYPES OF BEAMS: Depending upon the degrees of freedom and support conditions beams are of various types.

Beams

- **Statically determinate beam**
  Number of reactions in the beam is equal to number of useful static equations of equilibrium
  - Cantilever beam
  - Simply supported beam
  - Overhanging beam
- **Statically indeterminate beam**
  Number of reactions in the beam is more than number of useful static equations of equilibrium
  - Fixed beam
  - Propped Cantilever beam
  - Continuous beam

Statically Determinate Beam

A beam is said to be statically determinate if all its reaction components can be calculated by applying three conditions of static equilibrium.

Statically Indeterminate Beam

When the number of unknown reaction components exceeds the static conditions of equilibrium, the beam is said to be statically indeterminate. To determine the unknown reactions additional equations of deformations are required.

The following are the important types of beam

1. Cantilever beam,
2. Simply supported beam,
3. Overhanging beam,
4. Fixed beams, and
5. Continuous beam.

1. Cantilever beam

A beam which is fixed or built into a rigid support at one end and free at the other end is known as cantilever beam. Such beam is shown in Fig. The built-in support prevents displacements as well as rotations of the end of the beam. Cantilever is statically determinate.

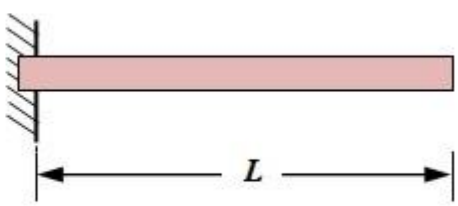


Fig.5 Cantilever beam

2. Simply Supported beam

A beam supported or resting freely on the supports at its both ends is known as simply supported beam. Such beam is shown in Fig. The end supports are free to rotate and have no moment of resistance. Simply supported beam is statically determinate beam.

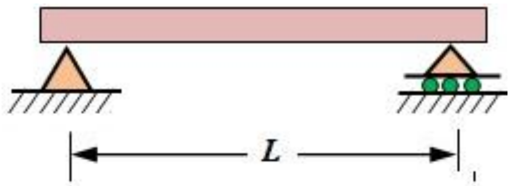


Fig. 6 Simply supported beam

3. Overhanging Beam

A beam supported over two supports and extended beyond one or both the supports is known as overhanging beam. An ***overhanging beam***, shown in Fig., is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports. It is a statically determinate beam.

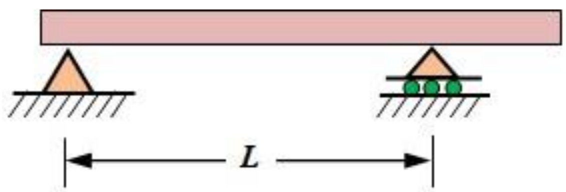


Fig. 7 Overhanging beam

4. Fixed Beam

A beam with both ends fixed or built into the supports or walls, is known as fixed beam. Such beam is shown in Fig. A fixed beam is also known as a built-in or encastred beam. It is a statically indeterminate beam.

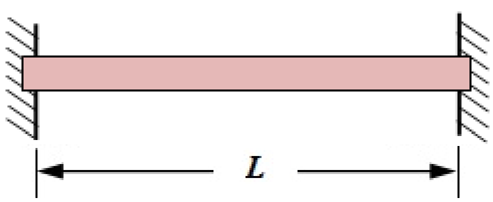


Fig. 8 Fixed beam

5. Propped cantilever beam

A beam with one end fixed and the other end simply supported over a roller is known as propped cantilever beam or simply propped cantilever. Propped cantilever is statically indeterminate.

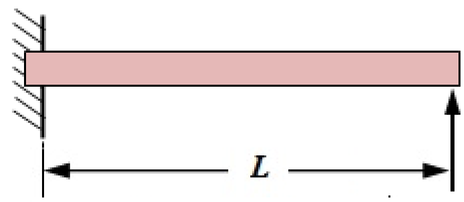


Fig.9 Propped cantilever

6. Continuous Beam

A beam which is supported over more than two supports is known as continuous beam. Continuous beam is also statically indeterminate.

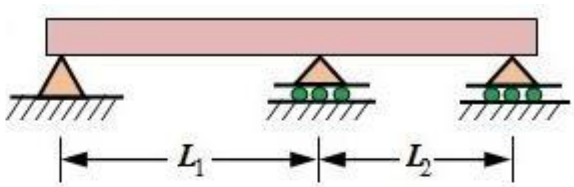


Fig.10 Continuous beam

SHEAR FORCE AND BENDING MOMENT:

The beams transfer the transverse (vertical) loads to the supports. In the process of load transfer, they experience shear force and bending moments.

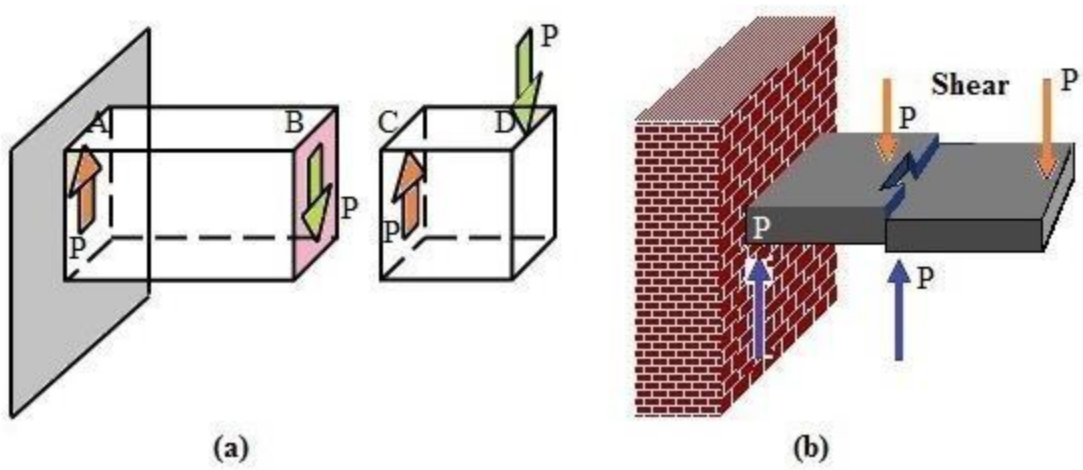


Fig.11 Shearing off beam

Shear force at any section of a beam is defined as the ***net or unbalanced vertical force on either side of the section***. It is the algebraic sum of vertical components of all the forces acting on the beam on either left side or right side of the section. The effect of shear force is to shear off or cut the member at a section. It is similar to the effect of scissor cutting the page of paper.

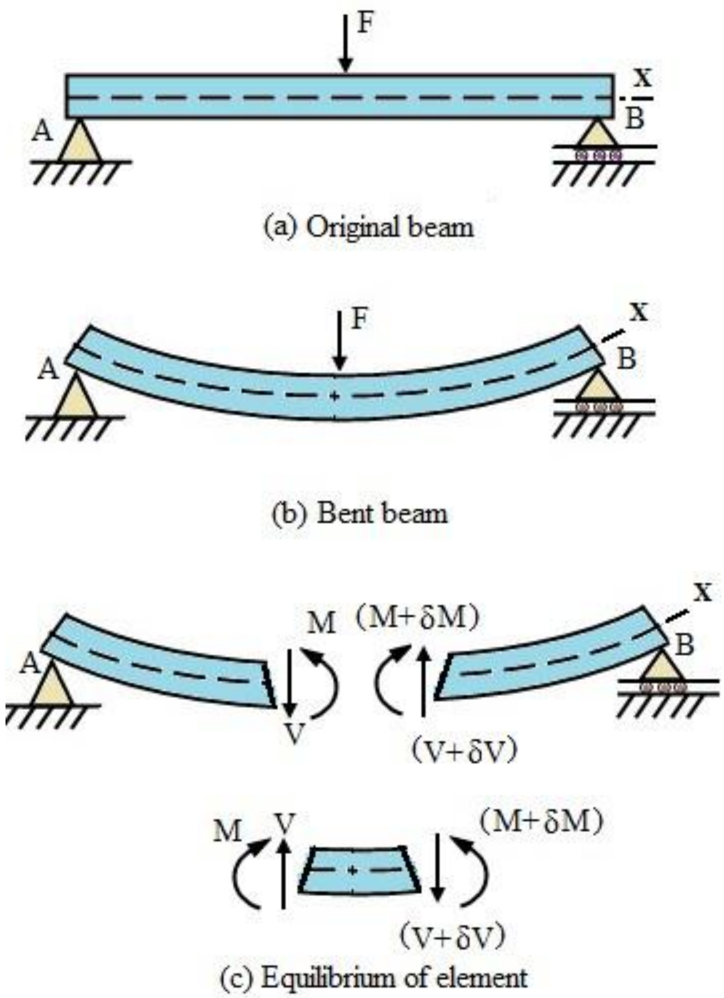


Fig.11 Bending of beam

The moment which tends to bend the beam in plane of load is known as bending moment. In other word bending moment at any section of a beam is the ***net or unbalanced moment due to all forces on either side of the section***. Bending moment at any section is the algebraic sum of the moments due to all forces acting on the beam on either right or left side of the section. The effect of bending moment is to bend the element.

Sign convention:

The shear force and bending moment are vector quantities and as a matter of convenience are assigned the following sign convention.

Shear force acting in the upward direction to the left hand side of the section and downward direction to the right hand side of the section is considered to be positive & vice-versa.

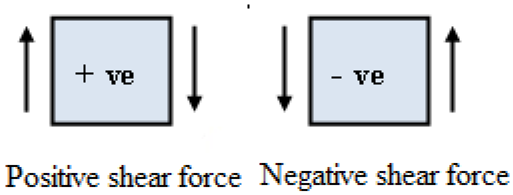


Fig.12

Bending moment is considered to be positive when it is acting in the clockwise direction on the left hand side of the section (L.H.S) (or) when it is acting in the counter-clockwise direction on the right hand side of the section (R.H.S) as the section & vice versa.

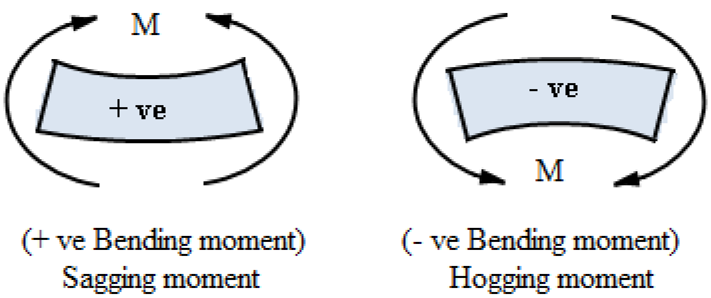


Fig. 13

SHEAR FORCE AND BENDING MOMENT DIAGRAMS:

Graphical representation of variation of shear force along the length of the beam for any given loading condition is known as ***shear force diagram*** (SFD). If *x* denotes the length of the beam, then shear force '*F*' is function of *x*, i.e. $F(x)$.

Similarly, graphical representation of variation of bending moment along the length of the beam for any given loading condition is known as ***bending moment diagram*** (BMD). If *x* denotes the length of the beam, then bending moment is function of *x*, and is denoted as $M(x)$.

Shear force diagram and bending moment diagram are helpful for further analysis and design of beam.

SFD and BMD of a beam reveal the following important information at salient points in the beam. These are maximum shear force, maximum bending moment, point of contralexture or point of inflexion, etc.

RELATIONS BETWEEN LOAD, SHEAR FORCE AND BENDING MOMENT

Consider a beam *AB* carrying generalized loading as shown in the figure. Take an element of infinitesimal length δx between section 1-1 and 2-2 at a distance of x from the left hand support *A*. The free body diagram of the element is drawn with positive sense of the shear forces and bending moments.

The intensity of loading over the length of the element may be taken as constant, i.e., w. Considering equilibrium of the element,

Resolving the forces vertically, $\sum V = 0$

$$F = w\delta x + F + \delta F$$

$$\delta F = -w\delta x$$

$$\frac{\delta F}{\delta x} = -w$$

In the limiting case, as $\delta x \to 0$, $\frac{dF}{dx} = -w$ (1)

So, the rate of change of shear force is equal to the intensity or rate of loading.

Taking moments of the forces and couples about the section 2-2, $\sum M_2 = 0$

$$M + \delta M + w\frac{(\delta x)^2}{2} = M + F\delta x$$

Neglecting small quantities of higher order, we have

$$\frac{\delta M}{\delta x} = F$$

In the limiting case as $\delta\, x \to 0,\ \dfrac{dM}{dx} = F$ (2)

The above equation shows that the rate of change of bending moment is equal to the shear force at the section. Also bending moment would be maximum at a section where shear force is zero.

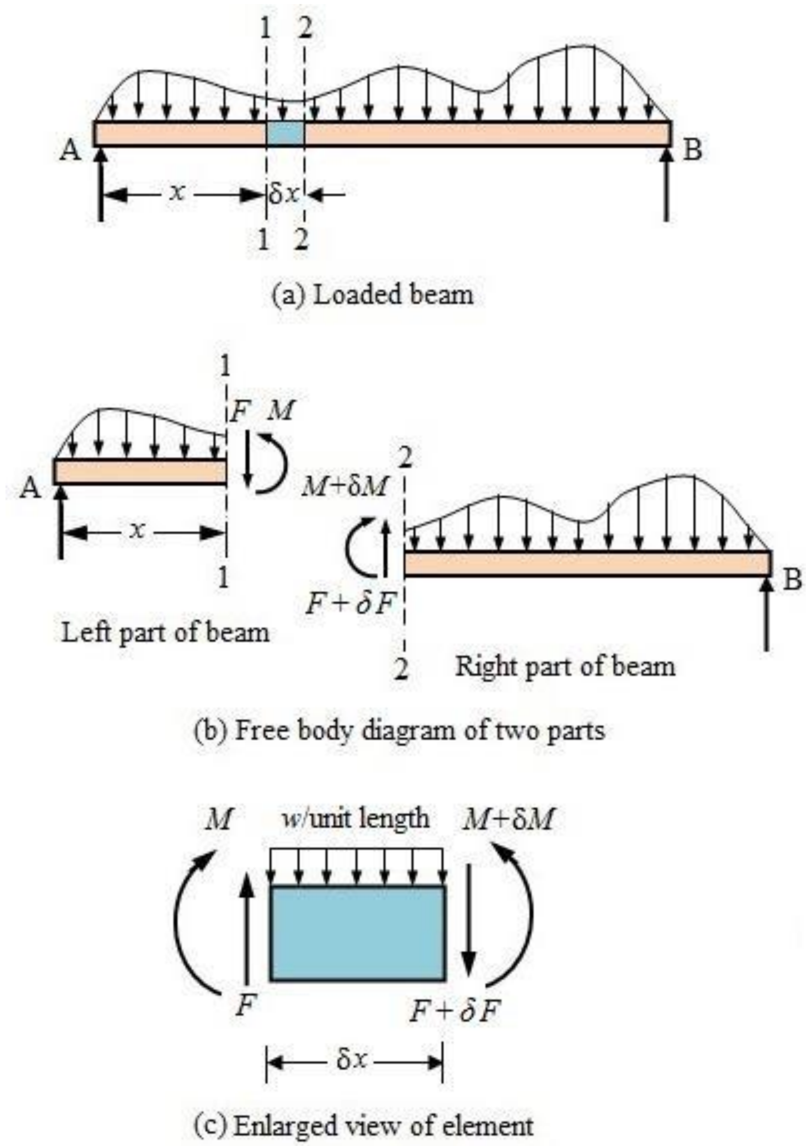


Fig. 14

Evaluation of Shear Force and Bending Moment:

Thus analysis of beam for shear force and bending moment is carried out by the following process.

1. Determine the reactions at the supports by considering the entire beam as a rigid body and applying equations of equilibrium.
2. Take sections at different points on the beam near supports and load application points.
3. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending moments.
4. Draw shear force and bending moment diagram.
5. Identify the maximum shear and bending-moment from plots of their distributions.
6. Find the position of *point of contaflexure* or *point of inflexion*.

Numerical

1. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying a point load W at its free end.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0,\ V_A = W\ (\uparrow)$

Taking moment about A, $\sum M_A = 0,\ W \times L + M_A = 0$

$$M_A = -WL \text{ (counter - clockwise)}$$

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and $A\left(0 \le x \le L\right)$.

Shear force at 1-1, $F_x = W$

Shear force at B, i.e., $x = 0\ \ F_B = W$

Shear force at A, $i.e, x = L,\ F_A = W$

Bending moment at 1-1, $M_x = -Wx$

Bending moment at B, $i.e, x = 0, M_B = 0$

Bending moment at A, $i.e, x = L, M_A = -WL$

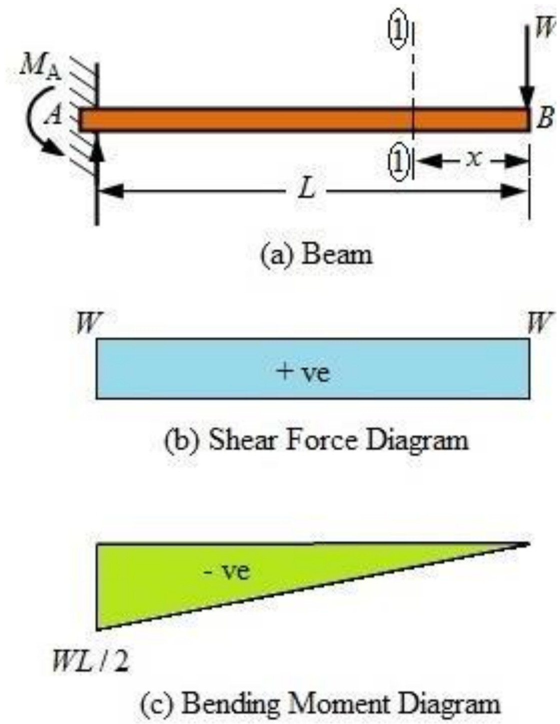


Fig.

2. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying uniformly distribute load of intensity w over the entire span.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0, V_A = wL$

Taking moment about $A, \sum M_A = 0$, $wL \times \frac{L}{2} + M_A = 0$

$$M_A = -\frac{wL^2}{2} \text{ (counter - clockwise)}$$

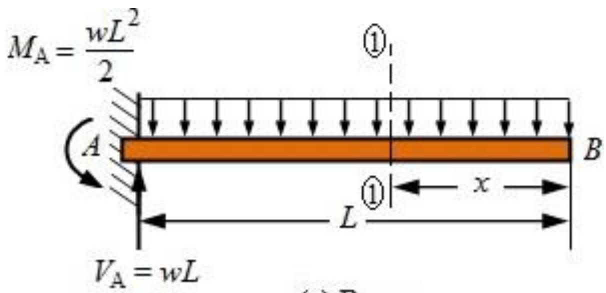
Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and $A\left(0 \le x \le L\right)$.

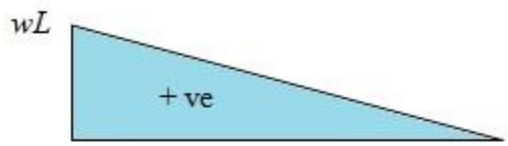
Shear force at 1-1, $F_x = wx$

Shear force at B, i.e., $x = 0$ $F_B = 0$

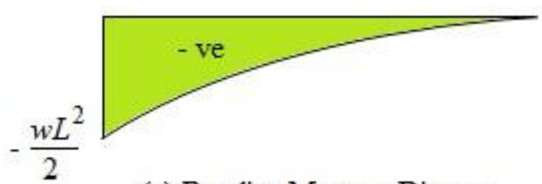
Shear force at A, $i.e, x = L,\ F_A = wL$



(a) Beam



(b) Shear Force Diagram



(c) Bending Moment Diagram

Bending moment at 1-1, $M_x = -wx \times \left(\frac{x}{2}\right) = -\frac{wx^2}{2}$

Bending moment at B, $i.e, x = 0, M_B = 0$

Bending moment at A, $i.e, x = L, M_A = -\frac{wL^2}{2}$

3. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying uniformly distribute load of intensity w per unit length from the fixed support to the centre of the beam.

Solution:

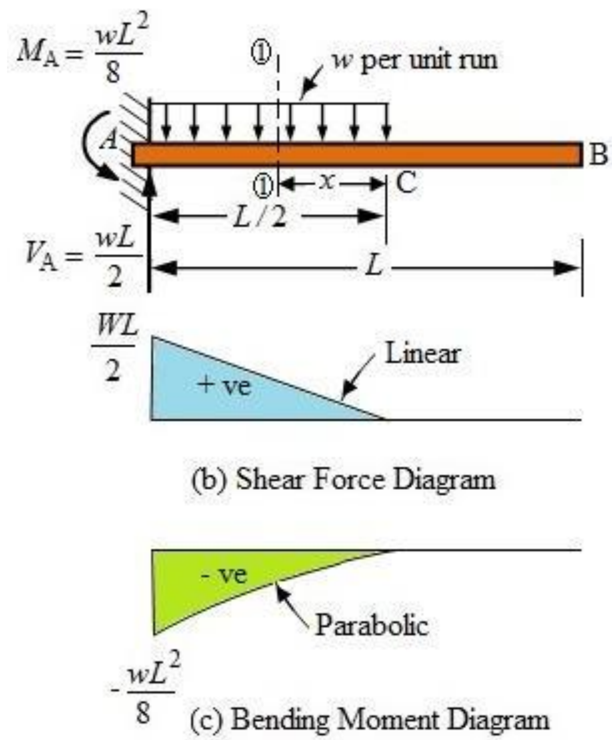
Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0,\ V_A = \dfrac{wL}{2}$

Taking moment about A, $\sum M_A = 0,\ \dfrac{wL}{2} \times \dfrac{L}{4} + M_A = 0$

$$M_A = -\frac{wL^2}{8} \text{ (counter - clockwise)}$$



(b) Shear Force Diagram

(c) Bending Moment Diagram

Fig.

Calculation of Shear force and bending moments:

Shear force and bending moment at the free end B, $F_B = 0$; $M_B = 0$

Shear force and bending moment anywhere between B and C is zero since there is no load on the beam in this portion when considered from right side.

Now, considering C as the origin, take a section 1-1 at a distance of x from C between C and A $\left(0 \le x \le \frac{L}{2}\right)$.

Shear force at 1-1, $F_x = wx$

Shear force at A, $i.e, x = L,\ F_A = w\frac{L}{2} = \frac{wL}{2}$

Bending moment at 1-1, $M_x = -wx \times \left(\frac{x}{2}\right) = -\frac{wx^2}{2}$

Bending moment at A, $i.e, x = \frac{L}{2}, M_A = -\frac{w\left(\frac{L}{2}\right)^2}{2}$

$$= -\frac{wL^2}{8}$$

4. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying uniformly distribute load of intensity w per unit length from the free end up to a distance of a.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0, V_A = wa$

Taking moment about A, $\sum M_A = 0,\ wa \times \left(L - \frac{a}{2}\right) + M_A = 0$

$$M_A = -\frac{wa}{2}(L - 2a) \quad \text{(counter - clockwise)}$$

Calculation of Shear force and bending moments:

Shear force and bending moment at the free end B, $F_B = 0$; $M_B = 0$

Now, considering B as the origin, take a section 1-1 at a distance of x from B between B and C $(0 \le x \le a)$.

Shear force at 1-1, $F_x = wx$

Shear force at B, $i.e, x = 0,\ F_C = w \times 0 = 0$

Shear force at C, $i.e, x = a,\ F_C = wa$

Bending moment at 1-1, $M_x = -wx \times \left(\frac{x}{2}\right) = -\frac{wx^2}{2}$

Bending moment at C, $i.e, x = a, M_A = -\frac{wa^2}{2}$

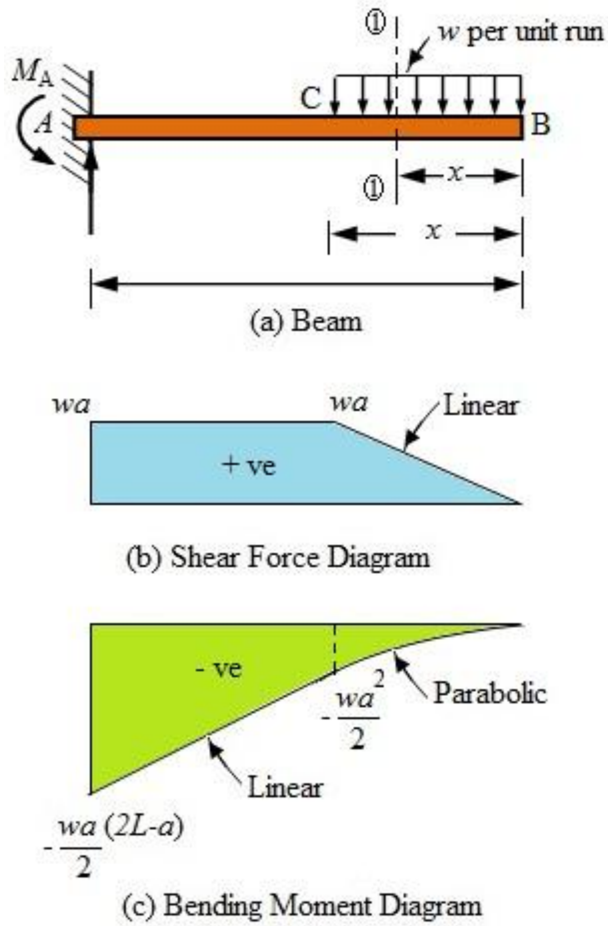


Fig.

Now, take a section 2-2 at a distance of x from B between C and $A\ (a \le x \le L)$.

Shear force at 2-2, $F_x = F_C = wa$

Shear force will remain same as wa from C to A.

Bending moment at 2-2, $M_x = -wa \times \left(x - \frac{a}{2}\right)$

Bending moment at A, *i.e.*, $x = L$, $M_x = -wa\left(L - \frac{a}{2}\right) = -\frac{wa}{2}(2L - a)$

5. A cantilever of 3.5 m long carries point loads of 15 kN, 15 kN and 7.5 kN at 1 m, 1 m and 1.5 m respectively from the fixed end. Draw the Shear force and bending moment diagram for the beam.

Solution: Calculation of Shear force and bending moments:

Portion *BD*: At section 1-1 at a distance x from B between B and D $(0 \le x \le 1.5m)$

Shear force at 1-1, $F_x = 7.5\,kN$ (constant from B to just right of D)

Shear force at B, $F_B = 7.5\ kN$

Shear force just right of D, $F_{DL} = 7.5\ kN$

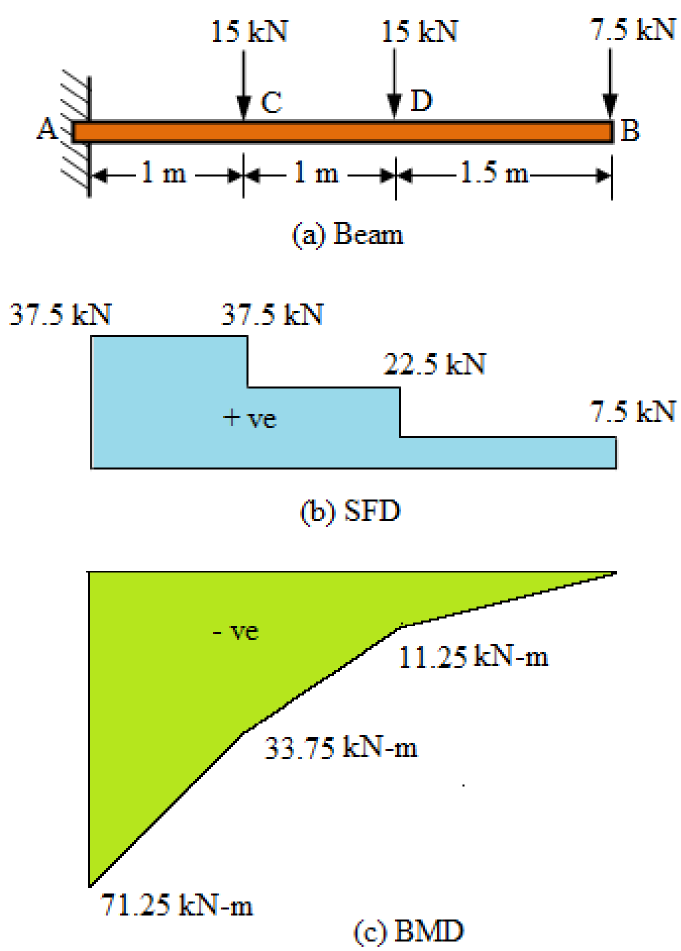


Fig.

Bending moment at 1-1, $M_x = -7.5x$

Bending moment at B, $i.e., at\ x = 0, M_B = -7.5 \times 0 = 0$

Bending moment at D, $i.e., at\ x = 1.5, M_D = -7.5 \times 1.5 = 11.25 kN - m$

Portion *DC*: At section 2-2 at a distance x from B between D and C $(1.5 \leq x \leq 2.5m)$

Shear force at 2-2, $F_x = 7.5 + 15$ (constant from D to just right of C)

Shear force at D, $F_D = 22.5\ kN$

Shear force just right of C, $F_{CL} = 22.5\ kN$

Bending moment at 2-2, $M_x = -7.5x - 15(x - 1.5)$

$$= -22.5x + 22.5$$

Bending moment at C, $i.e., at\ x = 2.5, M_B = -22.5 \times 2.5 + 22.5 = -33.75 kN - m$

Portion *CA*: At section 3-3 at a distance x from B between C and A $(2.5 \leq x \leq 3.5m)$

Shear force at 3-3, $F_x = 7.5 + 15 + 15$ (constant from C to A)

Shear force at C, $F_C = 37.5\ kN$

Shear force at A, $F_A = 37.5\ kN$

Bending moment at 3-3, $M_x = -7.5x - 15(x - 1.5) - 15(x - 2.5)$

$$= -37.5x + 60$$

Bending moment at A, $i.e., at\ x = 3.5m, M_B = -37.5 \times 3.5 + 60 = -71.25 kN - m$

6. A cantilever of 1.6 *m* long carries a uniformly distributed load of intensity 1.5 *kN/m* over the entire span and a point load of 2.5 *kN* at the free end. Draw the Shear force and bending moment diagram for the beam.

Solution:

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and A $(0 \leq x \leq 1.6m)$.

Shear force at 1-1, $F_x = 2.5 + 1.5x$

Shear force at B, $i.e., x = 0, F_B = 2.5 + 1.5 \times 0 = 2.5 kN$

Shear force at A, $i.e., x = 1.6m, F_x = 2.5 + 1.5 \times 1.6 = 4.9 kN$

Bending moment at 1-1, $M_x = -2.5x - 1.5x\left(\frac{x}{2}\right)$

$$= -2.5x - 0.75x^2$$

Bending moment at B, $i.e., x = 0, M_B = 0$

Bending moment at A, $i.e., x = 1.6, M_B = -2.5 \times 1.6 - 0.75 \times 1.6^2$

$$= -5.92\, kN - m$$

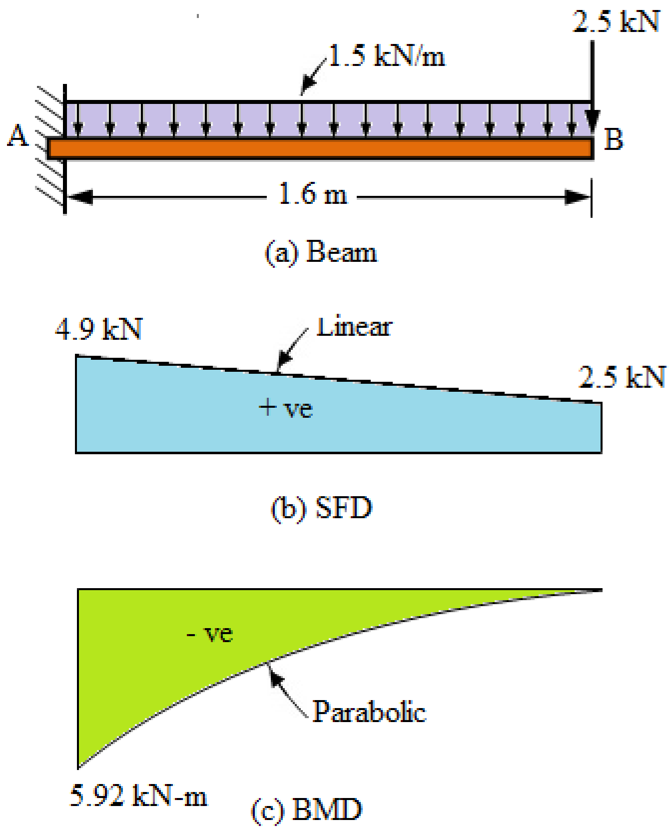


Fig.

7. A cantilever of 1.5 *m* long is loaded with a uniformly distributed load of intensity 2 *kN/m* and a point load of 2.5 *kN* as shown in the figure. Draw the Shear force and bending moment diagram for the cantilever.

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and $D\left(0 \le x \le 0.25m\right)$.

Shear force at 1-1, $F_x = 2x$

Shear force at B, $i.e., x = 0, F_B = 2 \times 0 = 0$

Shear force just right of D, $i.e., x = 0.25m, F_D = 2 \times 0.25 = 0.5kN$

Shear force at D, $i.e., x = 0.25m, F_D = 0.5 + 2.5 = 3kN$

Bending moment at 1-1, $M_x = -2x\left(\frac{x}{2}\right) = -x^2$

Bending moment at D, i.e., $x = 0.25$, $M_D = -(0.25)^2 = -0.0625\,kN - m$

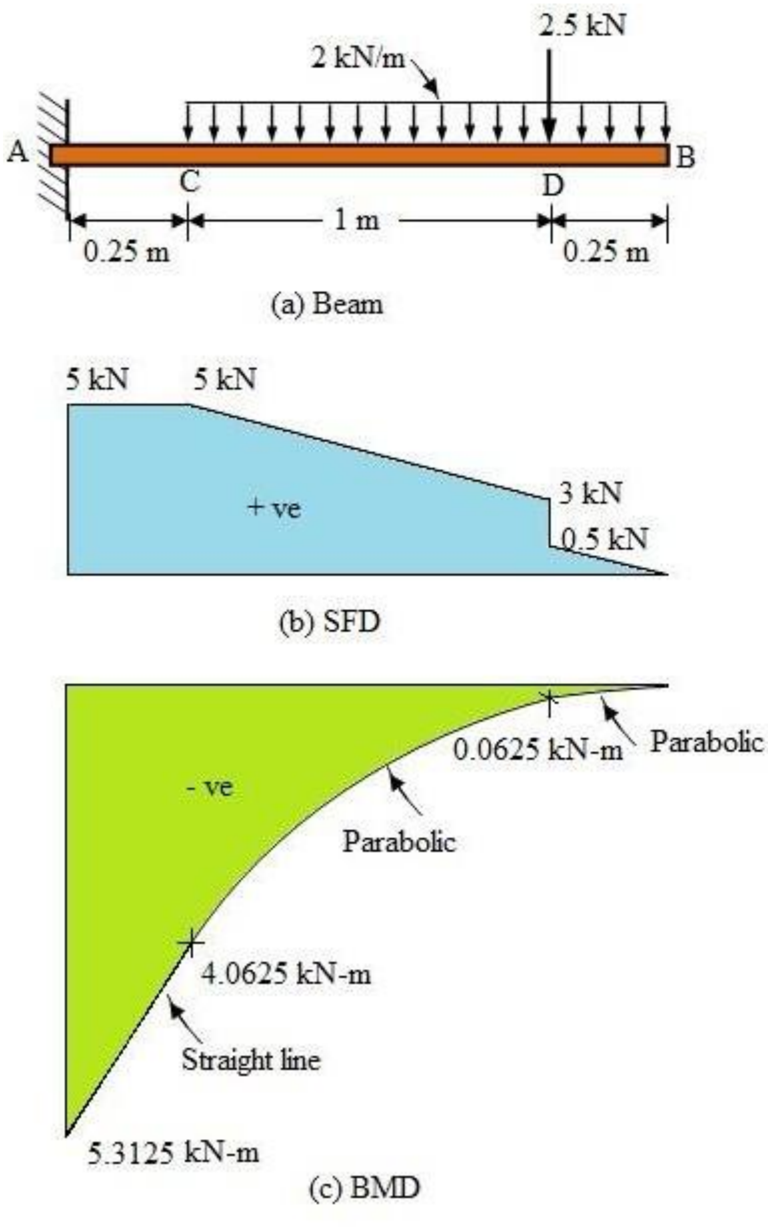


Fig.

Now, take a section 2-2 at a distance of x from B between D and $C\left(0.25 \le x \le 1.25m\right)$.

Shear force at 1-1, $F_x = 2x + 2.5$

Shear force at D, $i.e., x = 0.25, F_B = 2 \times 0.25 + 2.5 = 3kN$

Shear force C, $i.e., x = 1.25m, F_D = 2 \times 1.25 + 2.5 = 5kN$

Bending moment at 1-1, $M_x = -2x\left(\frac{x}{2}\right) - 2.5\left(x - 0.25\right)$

$$= -x^2 - 2.5x + 0.625$$

Bending moment at C, i.e., $x = 1.25$, $M_C = -1.25^2 - 2.5 \times 1.25 + 0.625$

$$= -4.0625\, kN - m$$

8. Calculate the shear force and bending moment for the beam subjected to a concentrated load of W as shown in the figure. Draw the shear force diagram (SFD) and bending moment diagram (BMD).

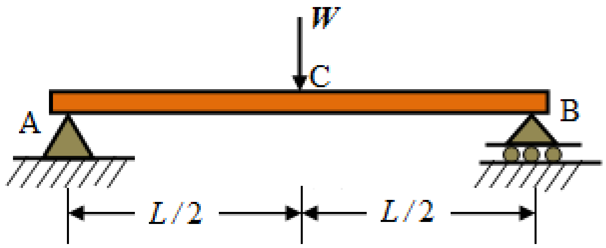


Fig.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about B, $\sum M_B = 0$, $V_A \times L = W \times \frac{L}{2}$

$$V_A = \frac{W}{2}$$

Sum of the vertical forces, $\sum V = 0$, $V_A + V_B = W$

Hence, $$V_B = \frac{W}{2}$$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $\left(0 < x < L/2\right)$.

Shear force at 1-1, $F_x = V_A = \dfrac{W}{2}$

Shear force at A, i.e., $x = 0$ $F_A = \dfrac{W}{2}$

Shear force just left of C, i.e., $x = 0$ $i.e., x = \dfrac{L}{2}$, $F_{LC} = \dfrac{W}{2}$

Bending moment at 1-1, $M_x = V_A \times x = \dfrac{W}{2}x$

Bending moment at A, i.e., $x = 0$, $M_A = \dfrac{Wx}{2} = 0$

Bending moment at C, i.e., $x = \dfrac{L}{2}$, $M_C = \dfrac{W}{2} \times \dfrac{L}{2} = \dfrac{WL}{4}$

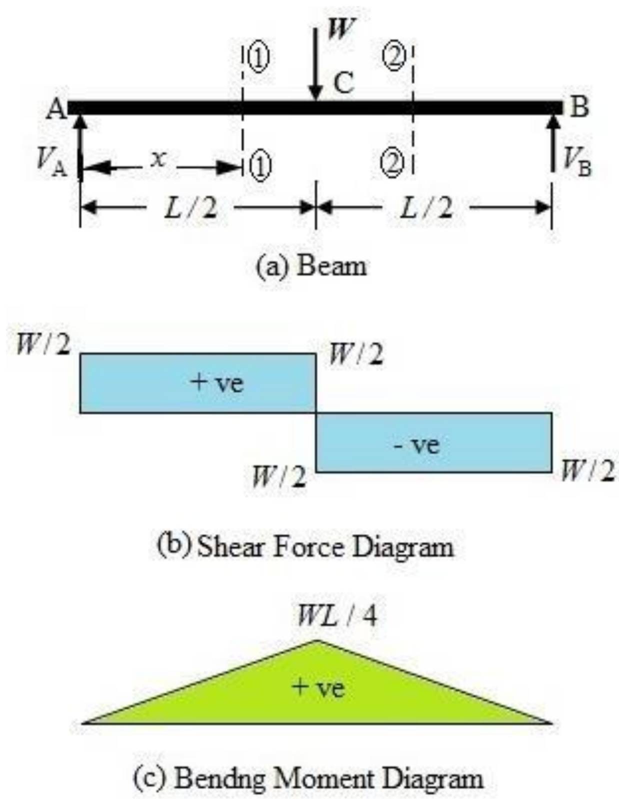


Fig.

Take a section 2-2 at a distance of x from A between C and $B\left(\frac{L}{2} \le x \le L\right)$.

Shear force at 2-2, $F_x = \frac{W}{2} - W = -\frac{W}{2}$

Shear force at $C\left(x = \frac{L}{2}\right)$, $F_C = \frac{-W}{2}$

Shear force at B, $(x = L)$, $F_B = \frac{-W}{2}$

Bending moment at 2-2, $M_x = \frac{W}{2}x - W\left(x - \frac{L}{2}\right)$

$$= \frac{Wx}{2} - Wx + \frac{WL}{2}$$

$$= -\frac{Wx}{2} + \frac{WL}{2}$$

Bending moment at B, i.e., $x = L$, $M_B = -\frac{Wl}{2} + \frac{WL}{2} = 0$

Bending moment at C, i.e., $x = \frac{L}{2}$, $M_C = -\frac{W}{2}\left(\frac{L}{2}\right) + \frac{WL}{2} = \frac{WL}{4}$

9. Draw the Shear force and bending moment diagram for a simply supported beam of length L carrying uniformly distribute load of intensity w per unit length over the entire span.

Solution:

Evaluation of support reactions:

The simply supported beam with uniformly distributed load over the entire span is symmetrically loaded symmetric beam. Hence, reactions at both supports are equal.

$$R_A = R_B = \frac{wL}{2}$$

Calculation of Shear force and bending moments:

In a symmetric beam, we need only to analyze half of the beam for shear force and bending moment. The other half will just be the mirror-image of the first half.

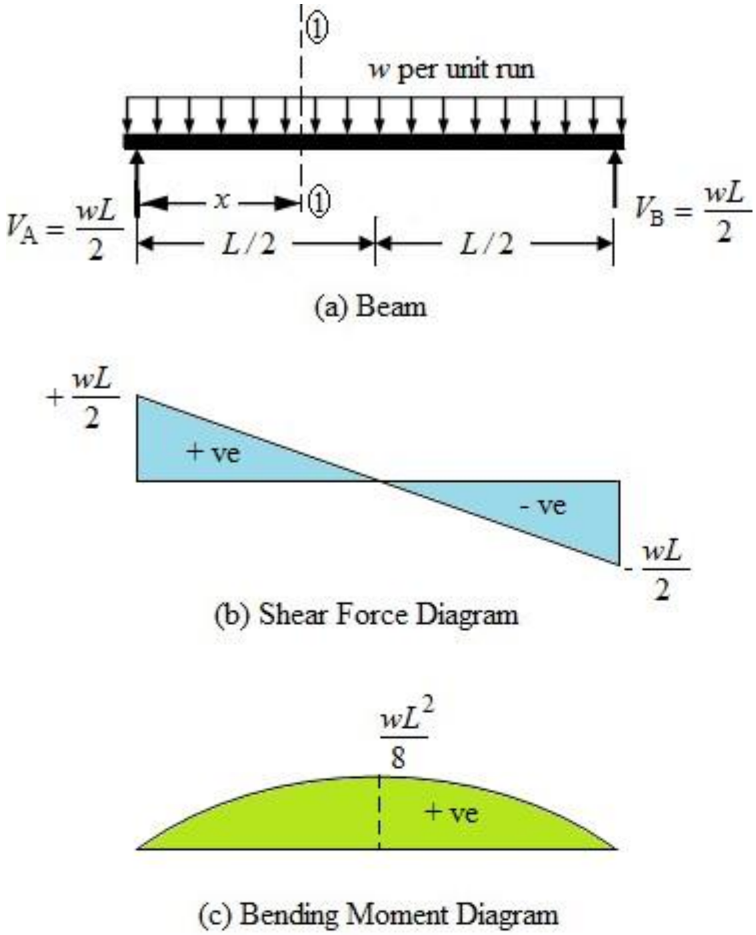


Fig.

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $\left(0 \le x \le L/2\right)$.

Shear force at 1-1, $V_x = V_A - wx = \frac{wL}{2} - wx$

Shear force at A, i.e., $x = 0$ $F_A = \frac{wl}{2}$

Shear force at C, i.e., $F_C = \frac{wL}{2} - w \times \frac{L}{2} = 0$

Bending moment at 1-1, $M_x = V_A \times x - wx\left(\frac{x}{2}\right) = \frac{wL}{2}x - wx\left(\frac{x}{2}\right)$

$$= \frac{wL}{2}x - \frac{wx^2}{2}$$

Bending moment at A, i.e., $x = 0$ $M_x = 0$

Bending moment at C, $M_C = \frac{wL}{2}\left(\frac{L}{2}\right) - \frac{w\left(\frac{L}{2}\right)^2}{2} = \frac{wL^2}{4} - \frac{wL^2}{8}$

$$= \frac{wL^2}{8}$$

Bending moment equation is a quadratic in form, hence the bending moment diagram will be parabolic between A and B.

Due to symmetry, the bending moment and shear force for the other half at respective point of symmetry will be same as the first half AB.

10. A simply supported beam shown in the figure carries two concentrated loads and a uniformly distribute load. Analyze the beam for shear force and bending moment, and draw the SFD and BMD.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about B, $\sum M_B = 0$, $V_A \times 8 = 25 \times 6 + 15 \times 4 + 7.5 \times 4 \times 2$

$$V_A = 33.75kN$$

Sum of the vertical forces, $\sum V = 0$, $33.75 + V_B = 25 + 15 + 7.5 \times 4$

Hence, $V_B = 70 - 33.75 = 36.25kN$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $(0 \le x \le 2)$.

Shear force at 1-1, $F_x = V_A = 33.75kN$

Shear force at A, i.e., $x = 0$ $F_A = 33.75kN$

Shear force just left of C, $i.e., x = 2$, $F_{LC} = 33.75kN$

Shear force at C, $i.e., x = 2$, $F_C = 33.75 - 25 = 8.75kN$

Bending moment at 1-1, $M_x = V_A \times x = 33.75x$

Bending moment at C, $i.e., x = 2$, $M_C = 33.75 \times 2 = 67.5\,kN - m$

Take a section 2-2 at a distance of x from A between C and D $(2 \le x \le 4)$.

Shear force at 2-2, $F_x = 33.75 - 25 = 8.75\,kN$

Shear force at C, i.e., $x = 2$, $F_C = 8.75\,kN$

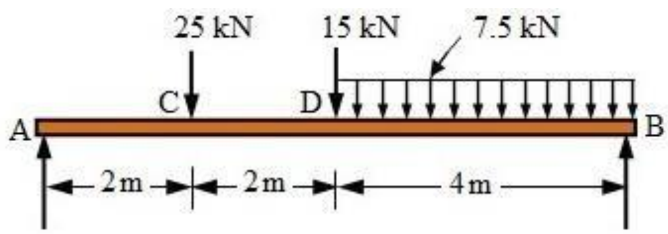
Shear force just left of D, $i.e., x = 4$, $F_{LD} = 8.75\,kN$

Shear force at D, $i.e., x = 4$, $F_D = 8.75 - 15 = -6.25\,kN$

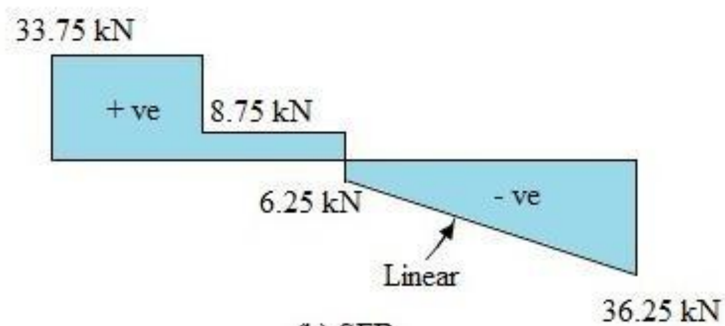
Bending moment at 2-2, $M_x = 33.75x - 25(x-2)$

$$M_x = 8.75x + 50$$

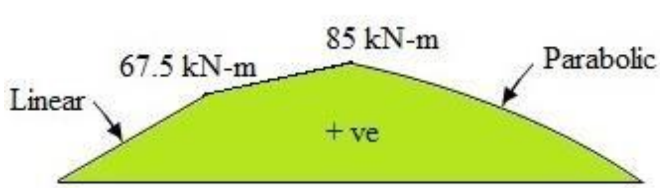
Bending moment at C, $i.e., x = 4$, $M_D = 8.75 \times 4 + 50 = 85\,kN - m$



(a) Beam



(b) SFD



(c) BMD

Fig.

Now, considering from the right side and taking B as the origin, take a section 3-3 at a distance of x from B between B and D $\left(0 \le x \le 4\right)$.

Shear force at 3-3, $F_x = -36.25 + 7.5x$

Shear force at B, i.e., $x = 0$ $F_B = 36.25kN$

Shear force just right of D, i.e., at $x = 4$, $F_x = -36.25 + 7.5 \times 4 = -6.25\,kN$

Shear force at D, $i.e., x = 4$, $F_D = -6.25 + 15 = 8.75kN$

Bending moment at 3-3, $M_x = 36.25x - \dfrac{7.5}{2}x^2 = 36.25x - 3.75x^2$

Bending moment at D, i.e., $x = 4$, $M_D = 36.25 \times 4 - 3.75 \times 4^2 = 85\,kN - m$

11. Draw the shear force and bending moment diagram for the overhanging beam shown in the figure.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about A, $\sum M_A = 0$, $V_D \times 4 = 20 \times 5 + 50 \times 2 + 20 \times 2 \times 1$

$V_D = 60kN$

Sum of the vertical forces, $\sum V = 0$, $V_A + 60 = 20 \times 2 + 50 + 20$

Hence, $V_A = 110 - 60 = 50kN$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $\left(0 \le x \le 2\right)$.

Shear force at 1-1, $F_x = 50 - 20x$

Shear force at A, i.e., $x = 0$ $F_A = 50\,kN$

Shear force just left of C, $i.e., x = 2$, $F_{LC} = 50 - 20 \times 2$

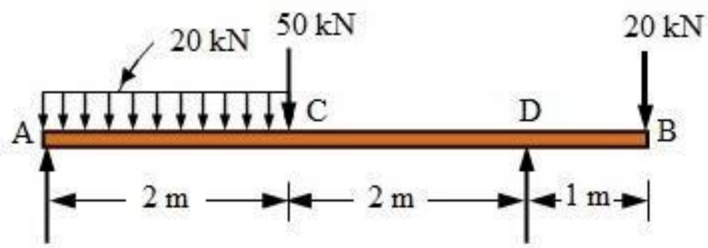
$F_{LC} = 10\,kN$

Shear force at C, $i.e., x = 2$, $F_C = 10 - 50 = -40kN$

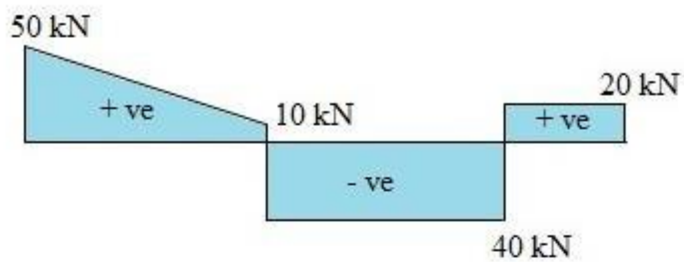
Bending moment at 1-1, $M_x = V_A \times x - 20 \times \frac{x^2}{2}$

$$= 50x - 10x^2$$

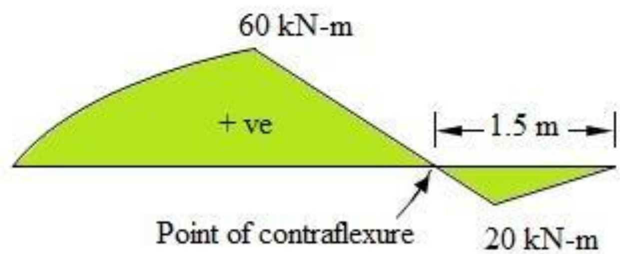
Bending moment at C, *i.e.*, $x = 2$, $M_C = 50 \times 2 - 10 \times 2^2 = 60\,kN - m$



(a) Beam



(b) SFD



(c) BMD

Fig.

Now, considering from the right side and taking B as the origin, take a section 2-2 at a distance of x from B between B and D $(0 \le x \le 4)$.

Shear force at 2-2, $F_x = 20\,kN$

Shear force at B, i.e., $x = 0$ $F_B = 20\,kN$

Shear force just right of D, i.e., at $x = 1$, $F_D = 20kN$

Shear force at D, $i.e., x = 1$, $F_D = 20 - 60 = -40\,kN$

Bending moment at 2-2, $M_x = -20x$

Bending moment at D, i.e., $x = 1$, $M_B = 0$

Bending moment at D, $i.e., M_D = -20 \times 1 = -20\,kN - m$

Take a section 3-3 at a distance of x from B between D and C $(1 \leq x \leq 3)$.

Shear force at 3-3, $F_x = 20 - 60 = -40\,kN$

Shear force at D, i.e., $x = 1$, $F_D = -40\,kN$

Bending moment at 3-3, $M_x = -20x + 60(x-1)$

$$= 40x - 60$$

Bending moment at C, i.e., $x = 3$, $M_x = 40 \times 3 - 60 = 60\,kN - m$

It is observed that bending moment changes sign between D and C. So, point of contraflexure exists between D and C.

Equating bending moment equation to zero, we get

$$40x - 60 = 0$$

$$x = 1.5\,m$$

Point of contraflexure:

A point of contraflexure is a point where the curvature of the beam changes signs. It is sometimes referred to as a ***point of inflexion***. In other words, point of contraflexure is a point where bending moment changes its sign from positive to negative or from negative to positive through zero. This means, bending moment is zero at point of contraflexure.

CHAPTER-6

COLUMNS AND STRUTS

Introduction:-

A structural member, subjected to an axial compressive force, is called a strut. As per definition, a strut may be horizontal, inclined or even vertical. But a vertical strut, used in buildings or frames, is called a column.

Definition of Column

A long slender bar subjected to axial compression is called a column.

The term is frequently used to describe a vertical member. Sometimes direct stresses dominate and sometimes flexural or bending stresses dominate.

Axial Compression means the compressive forces act at the two ends of the member in the opposite direction and are along the same axis.

Difference between columnandstrut

The difference between column and strut is that former is used to describe a vertical member whereas latter is used for the inclined members.

Short Column

The failure initiates due to crushing of material and direct stresses are dominant. For short column, if

$L < 4d$ and $kL/r_{min} < 30$

Where

d = least lateral dimension.

L = Unbraced length of the column.

k = effective length factor depends upon the end conditions of the column.

r_{min} = least radius of gyration.

Slender or long Column

In these, failure initiates due to lateral buckling and flexural stresses are dominant. If

$L > 30d$

or

kL/r_{min} > critical slenderness ratio.

Slenderness Ratio

The tendency of the column to buckle (fail) with ease under the action of axial compressive load is measured by a parameter known as slenderness ratio which is usually defined as the ratio of equivalent (or unsupported) length of column to the least radius of gyration of the column section. It is obviously unit less.

Failure of a Column or Strut:-

It has been observed, that when a column or a strut is subjected to some compressive force, then the compressive stress induced,

$$\sigma = \frac{P}{A}$$

Where P = Compressive force and

A= Cross-sectional area of the column.

A little consideration will show that if the force or load is gradually increased the column will reach a stage, when it will be subjected to the ultimate crushing stress. Beyond this stage, the column will fail by crushing. The load corresponding to the crushing stress, is called crushing load.

It has also been experienced that sometimes, a compression member does not fail entirely by crushing, but also by bending *i.e.*, buckling. This happens in the case of long columns. It has also been observed that all the short columns fail due to their crushing. But, if a long column is subjected to a compressive load, it is subjected to a compressive stress. If the load is gradually increased, the column will reach a stage, when it will start buckling. The load, at which the column is said to have developed an elastic instability, is called buckling load or crippling load. A little consideration will show that for a long column, the value of buckling load will be less than the crushing load. Moreover, the value of buckling load is low for long columns and relatively high for short columns.

Euler's Column Theory:-

The first rational attempt, to study the stability of long columns, was made by Mr. Euler. He derived an equation, for the buckling load of long columns based on the bending stress. While deriving this equation, the effect of direct stress is neglected. This may be justified with the statement that the direct stress induced in a long column is negligible as compared to the bending stress. It may be noted that the

Euler's formula cannot be used in the case of short columns, because the direct stress is considerable and hence cannot be neglected.

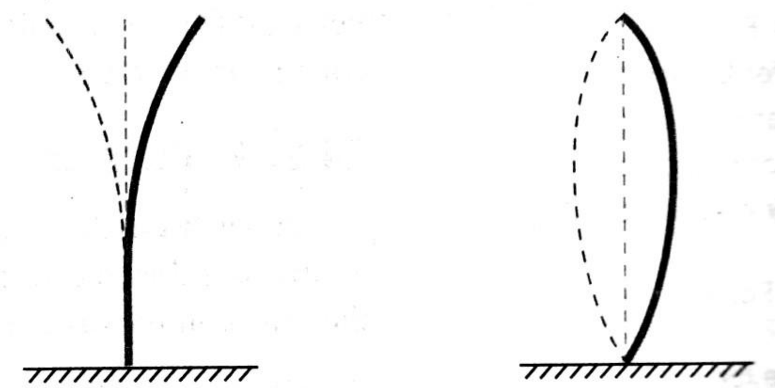
Assumptions in the Euler's Column Theory:-

The following simplifying assumptions are made in the Euler's column theory:-

1. Initially the column is perfectly straight and the load applied is truly axial.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and thus obeys Hooke's law.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.

Sign Conventions:-

Though there are different signs used for the bending of columns in different books, yet we shall follow the following sign conventions which are commonly used and internationally recognized.



(*a*) Positive (*b*) Negative

1. A moment, which tends to bend the column with convexity towards its initial central line as shown in (a) is taken as positive.
2. A moment, which tends to bend the column with convexity towards its initial central line as shown in (b) is taken as negative.

Types of end Conditions of Columns:-

In actual practice there are a number of end conditions for columns. But usually four types are important from subject point of view. They are as follows:

- Both ends hinged

- Both ends fixed
- One end is fixed and other end is hinged, and
- One end is fixed and other end is free.

Columns with both ends hinged (Derivation of expression for Critical load)

Consider a column *AB*of length *l*hinged at both of its ends *A* and *B* and carrying a critical load at *B.* As a result of loading, let the column deflect into a curved form AX_1B as shown in Figure below.

Now consider any section X, at a distance x from A.

Let *P* = Critical load on the column,

Y = Deflection of the column at *X.*

∴ Moment due to the critical load P,

M = -P.y

∴ $$EI\frac{d^2y}{dx^2} = -P.y \quad \ldots$$ (Minus sign due to Concavity towards initial Centre line)

∴ $$EI\frac{d^2y}{dx^2}\, p.y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{9}{EI}.y = 0$$

The general solution of the above differential equation is

$$y = A.\cos\left(x\sqrt{\frac{p}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right)$$

Where *A* and*B* are the constants of integration. We know that when *x=0, y=0.* Therefore *A=0,* Similarly when *x=l,* then *y=0.* Therefore

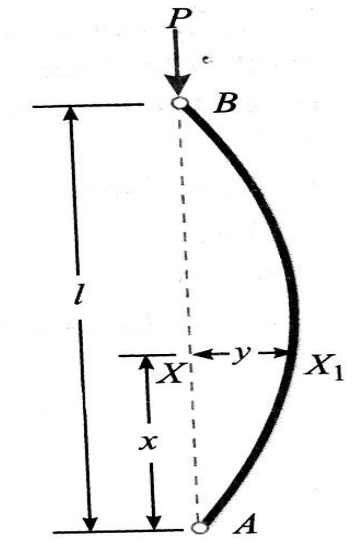
$$0 = B\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

A little consideration will show that either *B* is equal to zero or $\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$. Now if we consider *B* equal to zero, then it indicates that the column has not bent at all. But if $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

∴ $$\left(l\sqrt{\frac{P}{EI}}\right) = 0 = \pi = 2\pi = 3\pi = ...$$

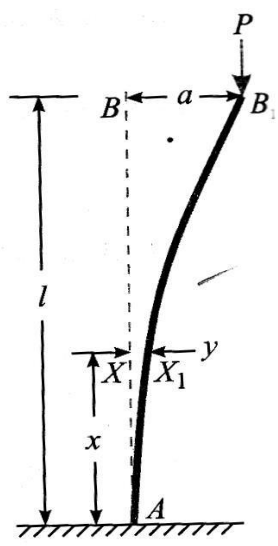
Now taking the least significant value,

$$\left(l\sqrt{\frac{P}{EI}} \right) = \pi$$



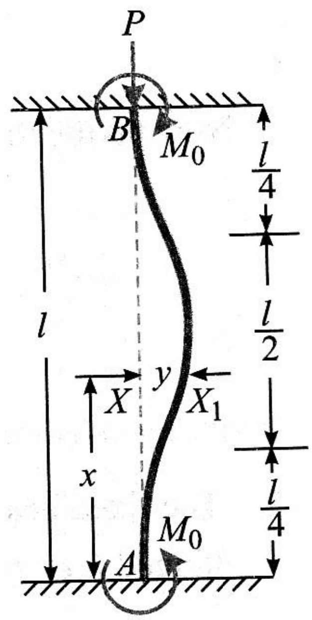
Or $$p = \frac{\pi^2 EI}{l^2}$$

Columns with One End Fixed and the Other Free:-



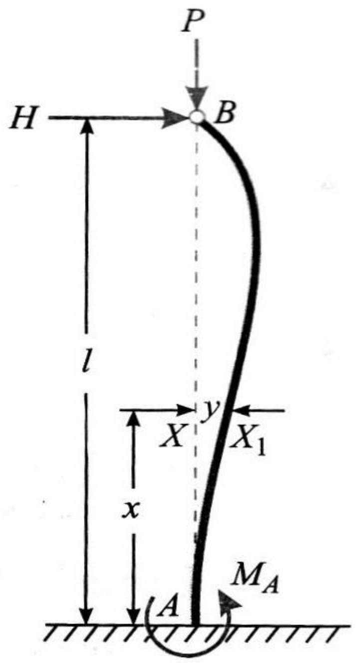
$$p = \frac{\pi^2 EI}{4l^2}$$

Columns with Both Ends Fixed:-



$$p = \frac{4\pi^2 EI}{l^2}$$

Columns with One End Fixed and the Other Hinged:-



$$p = \frac{2\pi^2 EI}{l^2}$$

Equivalent length/Effective length of a column

The equivalent length of a given column with given end conditions, is the length of an equivalent column of the same material and cross-section with both ends hinged and having the value of the crippling load equal to that of the given column.

EXAMPLE 1:- *A steel rod 5 m long and of 40 mm diameter is used as a column, with one end fixed and the other free. Determine the crippling load by Euler's formula. Take E as 200 GPa.*

SOLUTION. Given:-Length (l) = 5 × 10^3 mm ; Diameter of column (d) = 40 mm and modulus of elasticity (E)= 200 GPa = 200×10^3 N/mm^2.

We know that moment of inertia of the column section,

$$I = \frac{\pi}{64} \times (d)^4 = \frac{\pi}{64} \times (40)^4 = 40000\pi \text{ mm}^4$$

Since the column is fixed at one end and free at the other, therefore equivalent length of the column,

$$L_e = 2l = 2 \times (5 \times 10^3) = 10 \times 10^3 \text{ mm}$$

∴ Euler's crippling load, $$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times (200 \times 10^3) \times (40000\pi)}{(10 \times 10^3)^2} = 2480 \text{ N}$$

= 2.48 kN **Ans.**

EXAMPLE 2:- *A hollow alloy tube 4 m long external and internal diameters of 40 mm and 25 mm respectively was found to extend 4.8 mm under a tensile load of 60 kN. Find the buckling load for the tube with both ends pinned. Also find the safe load on the tube, taking a factor of safety as 5.*

SOLUTION Given:-Length *l*, = 4 m ; External diameter of column (D) = 40 mm ; Internal diameter of column (d) = 25 mm; Deflection (δl)= 4.8 mm ; Tensile load = 60 kN= 60 × 10^3 N and factor of safety = 5.

Buckling load for the tube

We know that area of the tube,

$$A = \frac{\pi}{4} \times \left[D^2 - d^2\right] = \frac{\pi}{4}\left[(40)^2 - (25)^2\right] = 765.8 \text{ mm}^2$$

And moment of inertia of the tube,

$$I = \frac{\pi}{64} \times \left[D^4 - d^4\right] = \frac{\pi}{64}\left[(40)^4 - (25)^4\right] = 106\ 500 \text{ mm}^4$$

We also know that strain in the alloy tube,

$$e = \frac{\delta l}{l} = \frac{4.8}{4 \times 10^3} = 0.0012$$

And modulus of elasticity for the alloy,

$$E = \frac{Load}{Area \times Strain} = \frac{60 \times 10^3}{765.8 \times 0.0012} = 65\ 290 \text{ N/mm}^2$$

Since the column is pinned at its both ends, therefore equivalent length of the column,

$$L_e = l = 4 \times 10^3 \text{ mm}$$

∴ Euler's buckling load, $P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 65290 \times 106500}{(4\times 10^3)^2} = 4290$ N

= 4.29 kN **Ans.**

Safe load for the tube

We also know that safe load for the tube

$$= \frac{Bucklingload}{Factorofsafety} = \frac{4.29}{5} = 0.858 \text{ kN}$$ **Ans.**

EXAMPLE 3:- *Comparethe ratio of the strength of a solid steel column to that of a hollow of the same cross-sectional area. The internal diameter of the hollow column is ¾ of the external diameter. Both the columns have the same length and are pinned at both ends.*

SOLUTION. Given:-Area of solid steel column A_S= A_H (where A_H= Area of hollow column); internal diameter of hollow column (d) is 3*D*/4 (where *D* = External diameter) and length of solid column (l_s) = l_H =(where l_H =Length of hollow column).

Let D_1 = Diameter of the solid column,

k_H = Radious of gyration for hollow column and

k_S = Radious of gyration for solid column.

Since both the columns are pinned at their both ends, therefore equivalent lengths of the solid column and hollow column,

$$L_S = l_S = L_H = l_H = L$$

We know that Euler's crippling load for the solid column,

$$P_s = \frac{\pi^2 EI}{L_S^2} = \frac{\pi^2 E.A_S.k_S^2}{L^2}$$

Similarly Euler's crippling load for the hollow column

$$P_H = \frac{\pi^2 EI}{L_H^2} = \frac{\pi^2 E.A_H.k_H^2}{L^2}$$

Dividing equation (ii) by (i),

$$\frac{P_H}{P_S} = \left(\frac{k_H}{k_S}\right)^2 = \frac{\frac{D^2+d^2}{16}}{\frac{D_1^2}{16}} = \frac{D^2+d^2}{D_1^2} = \frac{D^2 = \left(\frac{3D}{4}\right)^2}{D_1^2}$$

$$= \frac{25D^2}{16D_1^2}$$

Since the cross-sectional areas of the columns is equal, therefore

$$\frac{\pi}{4}\times D_1^2 = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[D^2 - \left(\frac{3D}{4}\right)^2\right] = \frac{\pi}{4}\times\frac{7D^2}{16}$$

∴ $D_1^2 = \frac{7D^2}{16}$

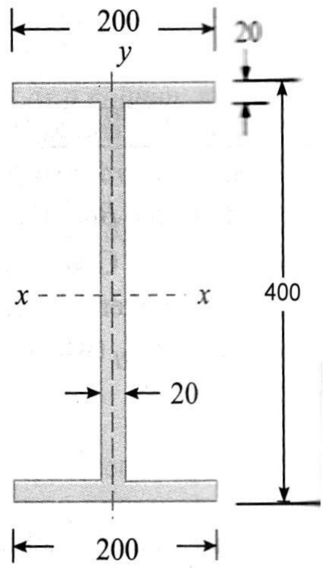
Now substituting the value of D_1^2 in equation (iii),

$$\frac{P_H}{P_S} = \frac{25D^2}{16 \times \frac{7D^2}{16}} = \frac{25}{7}$$ **Ans.**

EXAMPLE 4:- *An I section joist 400 mm ×200 mm × 20 mm and 6 m long is used as a strut with both ends fixed. What is Euler's crippling load for the column? Take Young's modulus for the joist as 200 GPa.*

SOLUTION. Given:-Outer depth (D) = 400 mm ; Outer width (B) = 200 mm ; Length (l) = 6 m = 6 ×10^3 mm and modulus of elasticity (E) = 200 GPa = 200 ×10^3 N/mm^2 .

From the geometry of the figure, we find that inner depth,



d = 400 – (2 ×20)= 360 mm

and inner width, b = 200-20 = 180 mm

We know that moment of inertia of the joist section about X-X a

$$I_{XX} = \frac{1}{12}\left[BD^3 - ba^3\right]$$

$$= \frac{1}{12}\left[200 \times (400)^3 - 180 \times (360)^3\right] \text{mm}^4$$

$$= 366.8 \times 10^6 \text{ mm}^4 \qquad \ldots\text{(i)}$$

Similarly I_{YY} $$= \left[2 \times \frac{2 \times (200)^3}{12}\right] + \frac{360 \times (20)^3}{12} \text{ mm}^4$$

$$= 2.91 \times 10^6 \text{ mm}^4$$

Since I_{YY} is less than $I_{XX,}$ therefore the joist will tend to buckle in *Y-Y* direction. Thus, we shall take the value of *I* as I_{YY}= 2.91 × 10^6 mm^4. Moreover, as the column is fixed at its both ends, therefore equivalent length of the column,

$$L_e = \frac{l}{2} = \frac{(6 \times 10^3)}{2} = 3 \times 10^3 \, mm$$

∴ Euler's crippling load for the column,

$$P_E = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times (200 \times 10^3) \times (2.91 \times 10^6)}{(3 \times 10^3)^2} = 638.2 \times 10^3 \text{ N}$$

= 638.2 kN **Ans.**

Mohr's circle:

Mohr's circle is a graphical representation of stress transformation equations. The equations of stress transformation describe a circle if normal stress and shear stress are represented as abscissa and ordinate respectively. Each point on the circumference of Mohr's circle represents a plane through the centre of the circle and the coordinates (σ, τ) of the point represents the normal stress (σ) and shear stress (τ) on the given plane.

Mohr's circle can be drawn from a given state of stress at a point in a structural member. Consider a stress element representing the state of stress at a point as shown in the figure.

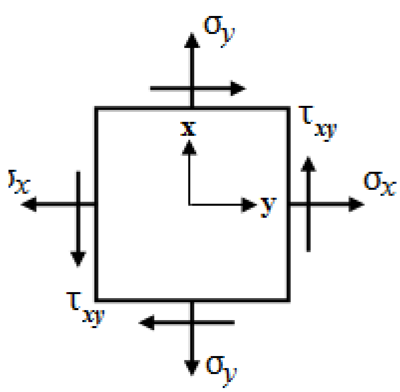


Fig.

Stress transformation equations for normal and tangential components on a plane are given by

Normal stress on the plane, $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (12)$$

Shear stress on the plane, $$\tau_\theta = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (13)$$

Rearranging the equation (12), we have

$$\sigma_\theta - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14)$$

Squaring both sides of equation (13) and (14) and adding them together, we have

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)^2$$

This is the equation of a circle with centre $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and this circle is known as Mohr's circle named after the German Civil Engineer Otto Mohr (1835-1918). It provides a simple and clear picture of an otherwise complicated analysis.

Procedure for drawing Mohr's circle:

1. Draw coordinates axes in Cartesian coordinate system with O as origin, normal stress (σ) as abscissa (positive to the right) and shear stress (τ) as ordinate (positive upward).
2. Locate the centre C of the circle at the point having coordinates $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.
3. Locate point *A*, representing the state of stress on the vertical plane, i.e., face *x* of the element by plotting its coordinates σ_x and τ. Point *A* on the circle corresponds to $\theta = 0^0$ and represents the vertical plane.
4. Locate point *B*, representing the state of stress on the horizontal plane, i.e., face y of the element by plotting its coordinates σ_y and $-\tau$. Point *A* on the circle corresponds to $\theta = 90^0$ and represents the horizontal plane.
5. Join AB so as to intersect the normal stress axis at *C*.
6. With the point *C* as the centre and *CA* (= *CB*) as radius, draw Mohr's circle through points *A* and *B*. This is the required Mohr's circle which has radius *R*.

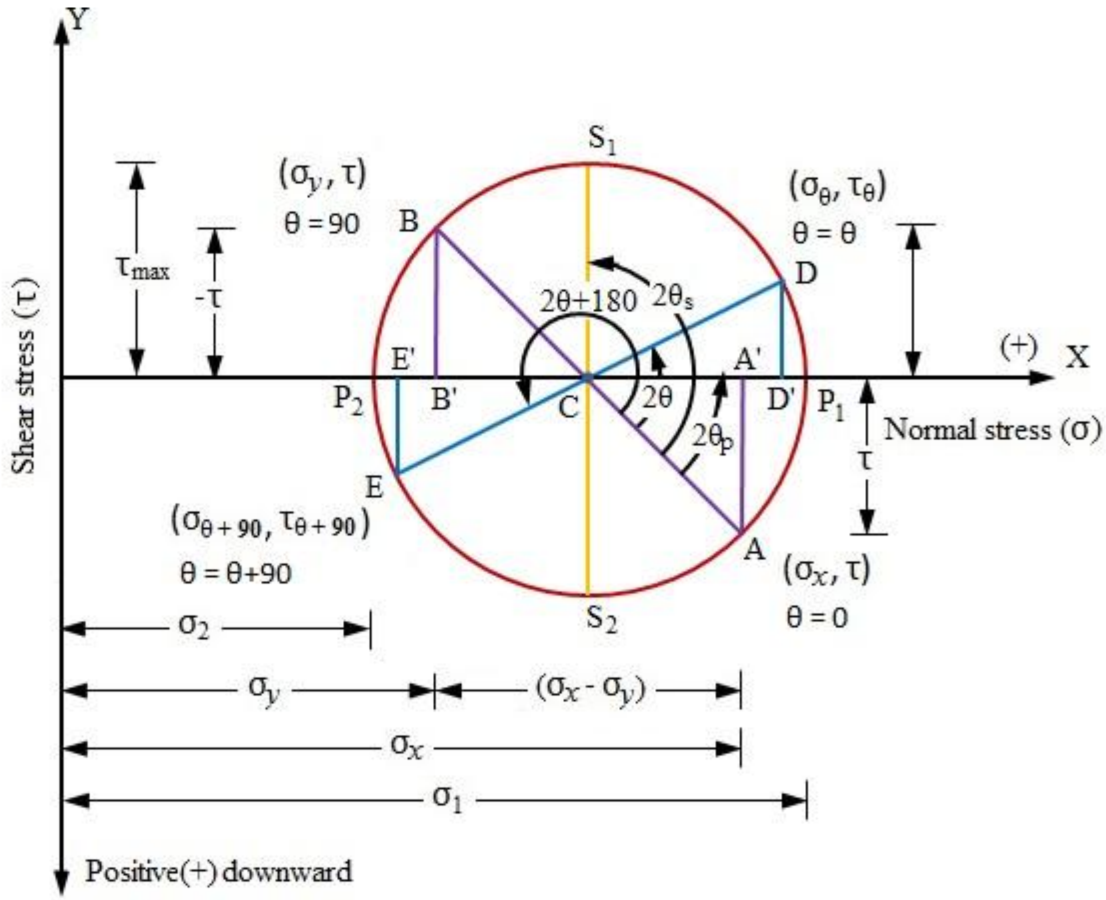


Fig. Mohr's circle

The stress state on an inclined plane with an angle θ is represented at point D on the Mohr's circle, which is measured an angle 2θ counter- clockwise from point A to show the coordinate at D.

Coordinates of D:

$$OD' = OC + CD' = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos\left(2\theta - 2\theta_p\right)$$

$$= \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos 2\theta \cos 2\theta_p + R\sin 2\theta \sin 2\theta_p$$

But $$R\cos 2\theta_p = \frac{1}{2}\left(\sigma_x - \sigma_y\right) \text{ and } R\sin 2\theta_p = \tau_{xy}$$

Therefore, $$OD' = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta$$

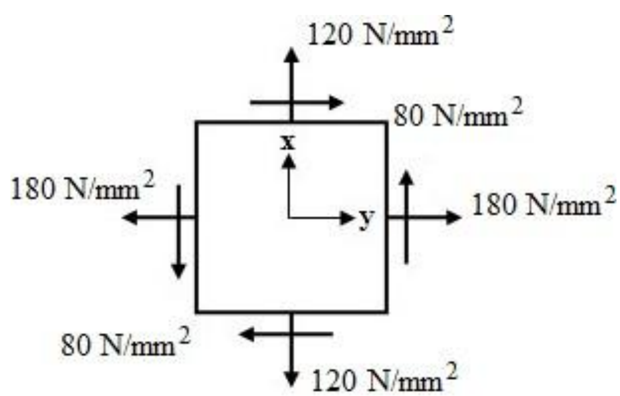
Similarly, $$DD' = R\sin(2\theta - 2\theta_p)$$

$$DD' = R\sin 2\theta \cos 2\theta_p - R\cos 2\theta \sin 2\theta_p$$

$$DD' = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy} \cos 2\theta$$

Problem. The state of stress in a material is as shown in the figure. Determine

a. the magnitude and directions of principal stresses, and
b. the magnitude of maximum shear stresses and its direction.



Shear Stress Distribution

Except for the case of uniform bending or pure bending as discussed in the section of simple bending, for general transverse loading condition, a beam is subjected to varying shear force as well as bending moment along the beam. The bending moment at a section cause normal stresses (σ) called bending stresses to occur through the depth of the beam. The strength of beam and hence its design often dominated by bending stress. However, as the beams become short and thick, traverse shear become dominant.

The shear force at any section of a beam produces vertical (transverse) shear stresses (τ) varying through the depth of the beam. This vertical shear stress is accompanied by complimentary longitudinal (horizontal) shear stress of equal magnitude acting perpendicular to the transverse section in the direction of length.

Shear stress distribution through the depth of the section is as shown in the Fig.

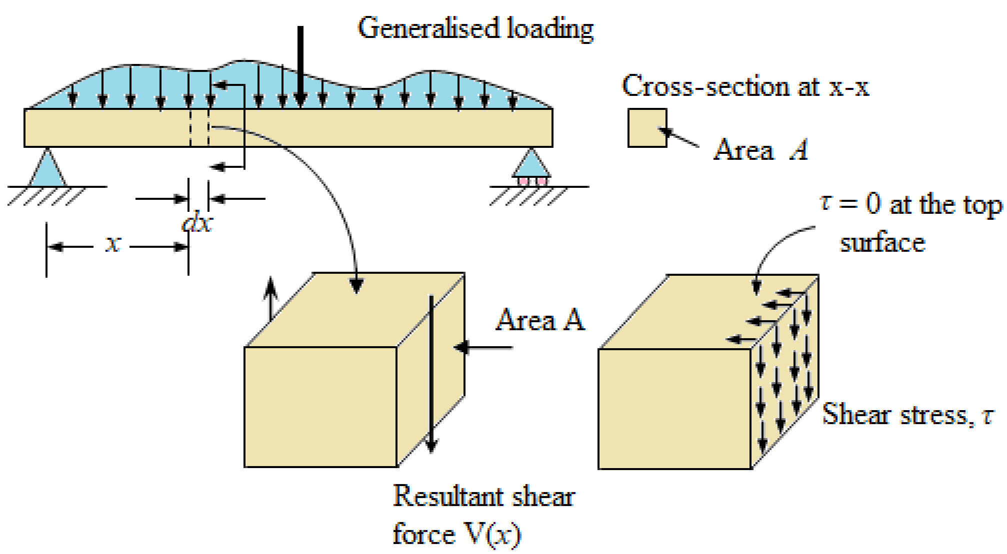
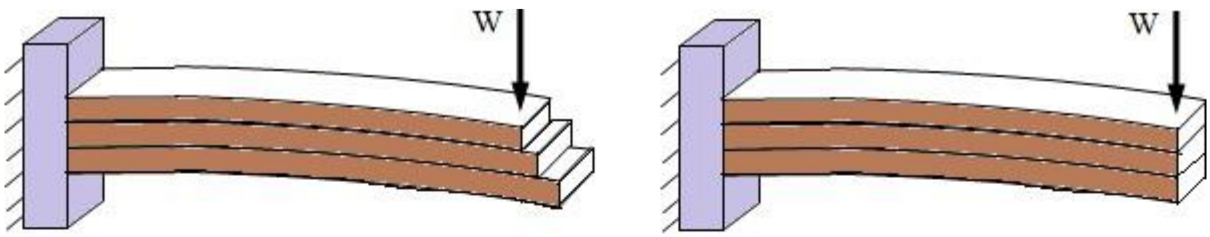


Fig. Shear stress distribution through the depth of beam beam section

Transverse shear in bending:

Transverse shear can often be difficult to understand. To visualize, let us consider a cantilever beam, comprising of several wooden planks as shown in the Fig., carries a point load at its free end. Imagine two cases, (i) planks are unglued and bend independent of each other (ii) the planks are bonded together with the help of glue or anchor bolts.

In the first case Fig.(a), the planks will bend independently and slide past each other. In the second case Fig. (b), the glue or anchor bolts will prevent the planks to slide past each other. This resistance to sliding or the resistance to horizontal force parallel to the beam's surface generates the shear stress within the material.



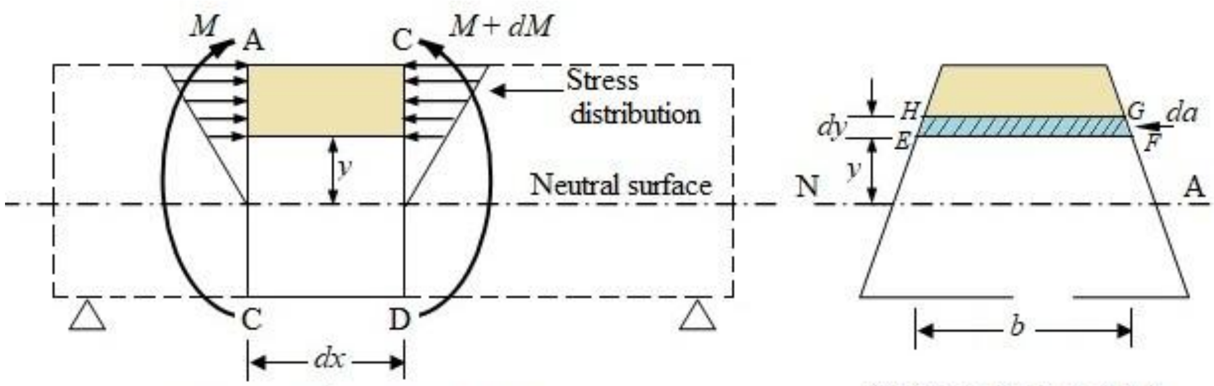
(a) Unglued wooden planks

(b) Glued wooden planks

Fig. 2 Cantilevered planks

Distribution of shear stress in beam section:

Consider an elemental length, *dx* of a beam between two sections *AB* and *CD* in a beam under generalized transverse loading as shown in the Fig. 3. The 3*D* view of beam element is also shown in Fig. 4 for better clarity.



(a) Elementary beam ABCD

(b) Cross Section at AA

Fig. 3

Let M and $M + dM$ be the bending moment at sections AB and CD respectively as shown in the Fig. Let, it is required to find the shear stress intensity on section AB at the level of EF at a distance of y from the neutral axis.

Consider a rectangular element $EFGH$, of width b and thickness dy. Let da be the area of the element. Let σ_1 and σ_2 be the bending stress on the elementary area at the level y at section AB and CD respectively.

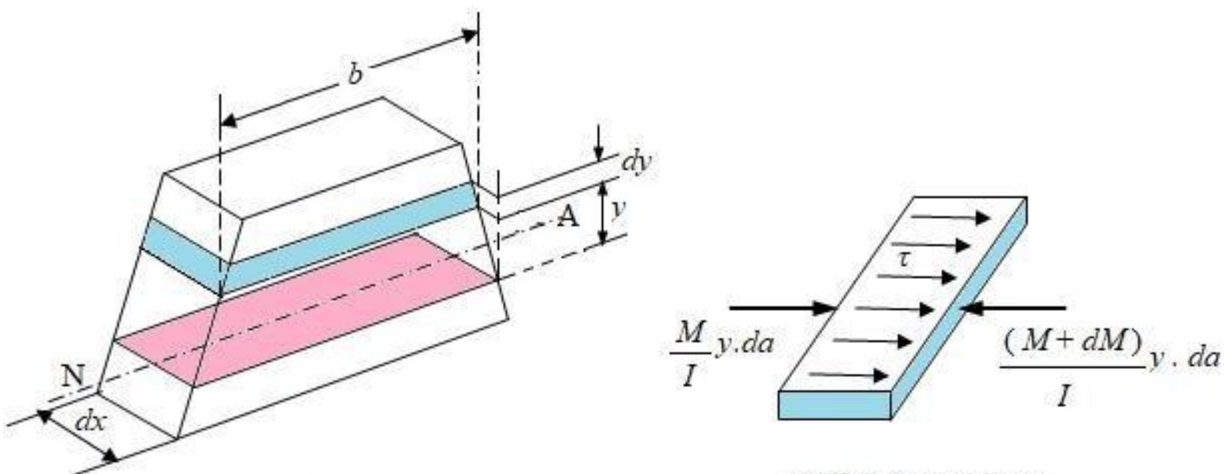
Bending stress on the elementary area at the level y at section AB, $\sigma_1 = \frac{M}{I}y$

Bending stress on the elementary area at the level y at section CD, $\sigma_2 = \frac{M + dM}{I}y$

Force on the elementary area at the level y at section $AB = \frac{M}{I}y.da$

Force on the elementary area at the level y at section $CD = \frac{M + dM}{I}y.da$

The unbalanced force on the elementary area $= \frac{dM}{I}y.da = \frac{dM}{I}y.(b.dy)$



(a) Elementary length (dx) of beam

(c) Unbalanced force

Fig. 4

Total unbalanced force on the portion of the beam above EF and between sections AB and CD

$$= \int_{y_1}^{y_2} \frac{dM}{I} y.bdy = \frac{dM}{I} \int_{y_1}^{y_2} yda$$

$$= \frac{dM}{I} a\bar{y} ,$$

where $a\bar{y}$ is the moment of area above level EF about the neutral axis.

This unbalanced horizontal force tries to shear off the portion of beam above *EF* and between the section *AB* and *CD* from the rest of the portion below *EF*. The equilibrium of the portion above *EF* is ensured by the shearing resistance of the material of beam on the plane at level *EF*.

Horizontal shear stress at level *EF*, $\tau = \frac{\text{Unbalanced force}}{\text{shear area}} = \frac{dM}{I} \frac{a\bar{y}}{bdx}$

$$= \frac{dM}{dx} \frac{a\bar{y}}{Ib}$$

We know, shear force at the section, $F = \frac{dM}{dx}$

Hence, $\tau = \frac{F}{Ib} a\bar{y}$

Where A = area of cross section of beam above the level EF

$\bar{y} =$ Distance of the centroid of the area above the level *EF*, from the neutral axis

For a given cross section of beam, F/I remains constant any point along the depth of the beam. Therefore, the shear stress at any point on a cross section is proportional to $\frac{a\bar{y}}{Ib}$. For sections of uniform width, the shear stress will have maximum value at neutral axis, since $a\bar{y}$ is maximum at neutral axis. The shear stress will have zero value at top and bottom layers of cross sections, since $a\bar{y}$ is zero at these layers.

Shear stress distribution for beams of standard sections:

a) Rectangular section:

Consider a rectangular section *ABCD* of width *b* and depth *d* is subjected to a shear force of *F*. Let τ be the shear stress intensity at any level *EF* at a distance of *y* from the neutral axis as shown in the Fig.

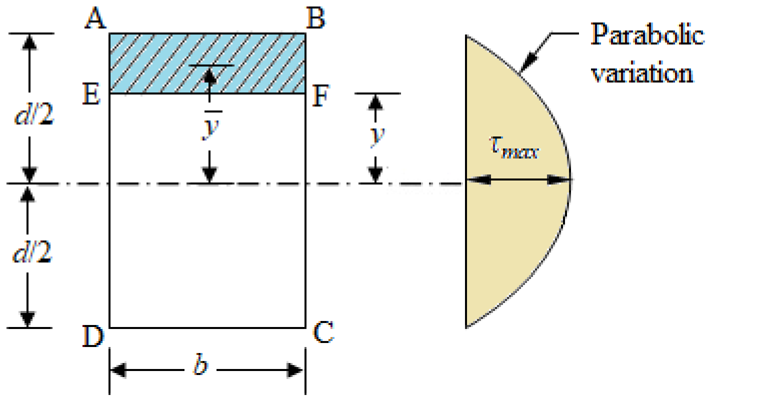


Fig. 5

We know, $\tau = \frac{Fa\bar{y}}{Ib}$

Here, $a\bar{y}$ is the moment of the shaded area above *EF*.

Area *ABFE*, $a = b\left(\frac{d}{2} - y\right)$

Centroid of the area *ABFE* from neutral axis, $\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right)$

$$= \frac{1}{2}\left(\frac{d}{2} + y\right)$$

$\therefore$ Shear stress on *EF*

$$\tau = \frac{F}{Ib}a\bar{y} = \frac{F}{Ib}b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

Shear stress variation along the depth is parabolic in nature.

The maximum shear stress occurs when $y = 0$, i.e., at neutral axis.

$$\tau_{\max} = \frac{F}{2I}\left(\frac{d^2}{4} - 0\right) = \frac{Fd^2}{8I}$$

$$= \frac{Fd^2}{8} \times \frac{12}{bd^3}$$

$$= \frac{3}{2} \times \frac{F}{bd}$$

Minimum shear stress, $\tau_{\min} = 0$ occurs at the top and bottom edge, i.e., at $y = \frac{d}{2}$.

Average shear stress, $\tau_{avge} = \frac{\text{Shear force}}{\text{Cross - sectional area}}$

$$= \frac{F}{bd}$$

Maximum shear stress, $\tau_{\max} = \frac{3}{2}\frac{F}{bd} = 1.5\tau_{avge}$

Thus, maximum shear stress is 1.5 times the average shear stress in rectangular section and occurs at neutral axis.

***b) Circular section*:**

Consider circular section of radius, r subjected to a shear force of F. Let τ be the shear stress intensity at any level EF at a vertical distance of y and angular distance of θ from the neutral axis as shown in the Fig.

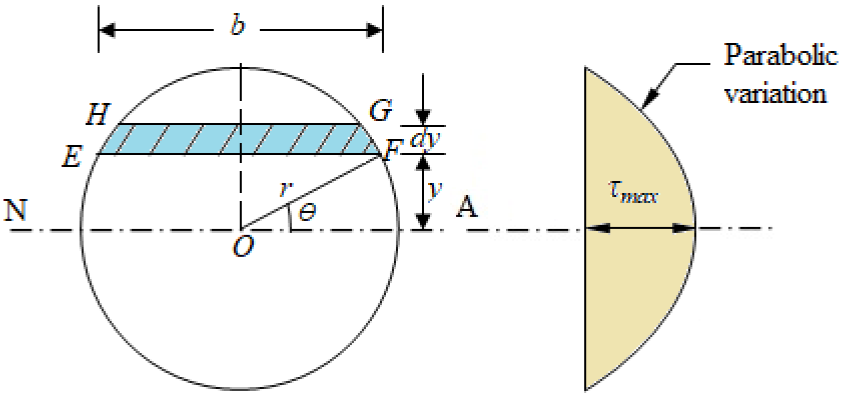
Consider an elementary rectangular strip $EFGH$ of thickness dy at a distance of y from the neutral axis.

Width of the strip, $b = 2r\cos\theta$

Vertical distance, $y = r\sin\theta$

Area of the elementary strip, $da = bdy = 2r\cos\theta . r\cos\theta . d\theta$

$$= 2r\cos^2\theta.d\theta$$



(a) Beam cross section (b) Shear stress distribution

Fig. 6

Consider an elementary rectangular strip *EFGH* of thickness *dy* at a distance of *y* from the neutral axis.

Width of the strip, $b = 2r\cos\theta$

Vertical distance, $y = r\sin\theta$

Area of the elementary strip, $da = bdy = 2r\cos\theta.r\cos\theta.d\theta$

$$= 2r^2\cos^2\theta.d\theta$$

Moment of elementary strip about neutral axis, $= (da)y = 2r^2\cos^2\theta.d\theta.r\sin\theta$

$$= 2r^3\cos^2\theta\sin\theta.d\theta$$

Moment of the area above *EF* about neutral axis, $a\bar{y} = \int_{\theta}^{\pi/2} 2r^3\cos^2\theta\sin\theta.d\theta$

$$= 2r^3\int_{\theta}^{\pi/2}\cos^2\theta\sin\theta.d\theta$$

Let $\cos\theta = t \Rightarrow \sin\theta.d\theta = -dt$

When $\theta = \theta, t = \cos\theta; \quad \theta = \frac{\pi}{2}, t = \cos\frac{\pi}{2} = 0$

Hence,
$$a\bar{y} = -2r^3 \int_{\theta}^{0} t^2 dt$$
$$= -2r^3 \left[\frac{t^3}{3}\right]_{\theta}^{0} = -2r^3 \left(\frac{-\cos^3\theta}{3}\right)$$
$$= \frac{2}{3} r^3 \cos^3\theta$$

Moment of inertia of circular section about neutral axis, $I = \frac{1}{4}\pi r^4$

∴ Shear stress
$$\tau = \frac{F}{Ib} a\bar{y} = \frac{F}{I \times 2r\cos\theta} \times \frac{2}{3} r^3 \cos^3\theta$$
$$= \frac{F}{3I} r^2 \cos^2\theta = \frac{F}{3I}\left(r^2 - r^2 \sin^2\theta\right)$$
$$= \frac{F}{3I}\left(r^2 - y^2\right)$$

Hence, variation of shear stress along the depth is parabolic in nature.

Shear stress is maximum when $y = 0$, at neutral axis.

$$\therefore \quad \tau_{\max} = \frac{F}{3I} r^2$$
$$= \frac{Fr^2}{3} \times \frac{4}{\pi r^4} = \frac{4}{3}\frac{F}{\pi r^2}$$
$$= \frac{4}{3}\tau_{avg}$$

Shear stress is zero at top and bottom layer, i.e., $y = r$

***c) Isosceles triangular*:**

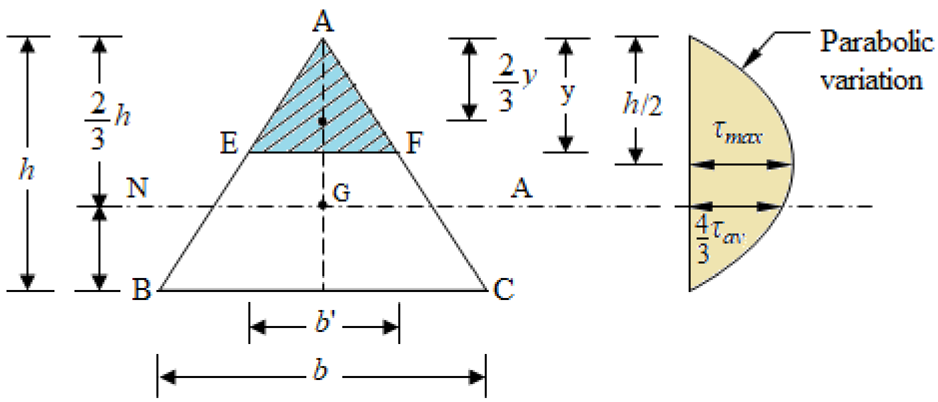
Consider a isosceles triangular section *ABC* of base *b* and height *h* as subjected to a shear force of *F*. Let *τ* be the shear stress intensity at any level *EF* at a vertical distance of *y* from the apex, *A* as shown in the Fig. 7.

Moment of inertia of triangular section about neutral axis, $I = \dfrac{bh^3}{36}$

Let *b*' be the width of the triangular portion, *AEF* above *EF*.

From similar triangles *AEF* and *ABC*, we have

$$b' = \frac{b}{h} y$$



(a) Beam cross section (b) Shear stress distribution

Fig. 7

Centroid of triangle *AEF* from the neutral axis, $\bar{y} = \dfrac{2h}{3} - \dfrac{2y}{3} = \dfrac{2}{3}(h-y)$

Area of the triangular portion *AEF*, $a = \dfrac{1}{2} b'y = \dfrac{1}{2}\dfrac{b}{h} y^2$

$\therefore$ Shear stress

$$\tau = \frac{F}{Ib'} a\bar{y} = \frac{F}{I\frac{b}{h}y} \times \frac{1}{2}\frac{b}{h} y^2 \times \frac{2}{3}(h-y)$$

$$= \frac{F}{3I} y(h-y)$$

At, $y = 0$, $\tau = 0$

At, $y = h$ $\tau = 0$

At neutral axis, $y = \frac{2h}{3}$, $\quad \tau_{na} = \frac{F}{3I} \times \frac{2h}{3}\left(h - \frac{2h}{3}\right) = \frac{2Fh^2}{27I}$

$$= \frac{2Fh^2}{27 \times \frac{bh^3}{36}} = \frac{8}{3}\left(\frac{F}{bh}\right)$$

For maximum shear stress, $\quad \frac{d\tau}{dy} = 0$

$$\frac{F}{3I}(h - 2y) = 0$$

$$y = \frac{h}{2}$$

Hence, maximum shear stress, $\tau_{\max} = \frac{F}{3I}\frac{h}{2}\left(h - \frac{h}{2}\right) = \frac{Fh^2}{12I}$

$$= \frac{Fh^2}{12 \times \frac{bh^3}{36}} = \frac{3F}{bh}$$

Average shear stress, $\quad \tau_{avg} = \frac{F}{\frac{1}{2}bh} = \frac{2F}{bh}$

$$\frac{\text{Maximum shear stress}}{\text{Average shear stress}} = \frac{\tau_{\max}}{\tau_{avg}} = \frac{\frac{3F}{bh}}{\frac{2F}{bh}} = 1.5$$

$$\tau_{\max} = 1.5\tau_{avg}$$

$$\frac{\text{Shear stress at Neutral axis}}{\text{Average shear stress}} = \frac{\tau_{na}}{\tau_{avg}} = \frac{\frac{8F}{3bh}}{\frac{2F}{bh}} = \frac{4}{3}$$

$$\tau_{na} = \frac{4}{3}\tau_{avg}$$

d) I-section

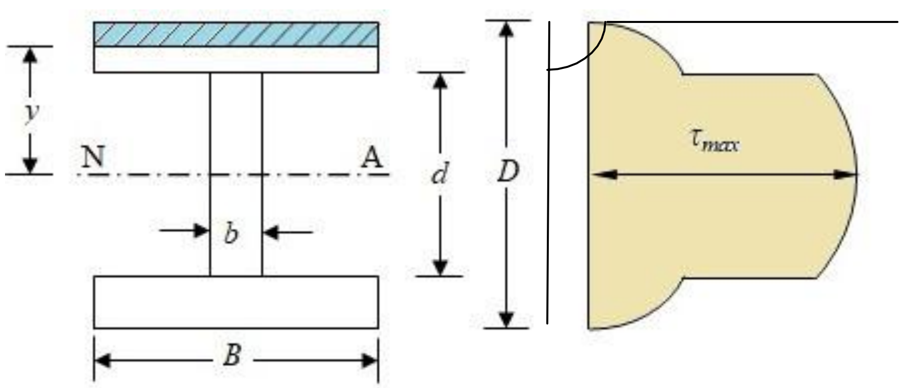
Consider an *I*-section of flange width *B* and overall depth *D*. Let *b* and *d* be the thickness and depth of web. The section is subjected to a shear force of *F*.

Shear stress distribution in the flange

Let τ be the shear stress intensity at any level *EF* on flange at a vertical distance of *y* from the neutral axis as shown in the Fig. 8.

Area above $EF = B\left(\frac{D}{2} - y\right)$

Centriodal distance of this area from the neutral axis $= y + \frac{1}{2}\left(\frac{D}{2} - y\right) = \frac{1}{2}\left(\frac{D}{2} + y\right)$



(a) Beam section (b) Shear stress distribution

Fig. 8

Moment of the area above the plane *EF* about the neutral axis,

$$a\bar{y} = B\left(\frac{D}{2} - y\right)\frac{1}{2}\left(\frac{D}{2} + y\right)$$

$$= \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)$$

$\therefore$ Shear stress on *EF*

$$\tau = \frac{F}{IB}a\bar{y} = \frac{F}{IB} \times \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)$$

$$= \frac{F}{2I}\left(\frac{D^2}{4} - y^2\right)$$

Thus, the shear stress distribution in flange is parabolic in nature.

At the top of the flange, i.e., at $y = \frac{D}{2}$, shear stress intensity, $\tau = 0$.

At the junction of flange and web, i.e., at $y = \frac{d}{2}$,

$$\tau = \frac{F}{2I}\left(\frac{D^2}{4} - \frac{d^2}{4}\right) = \frac{F}{8I}\left(D^2 - d^2\right)$$

Shear stress distribution in the flange

Let τ be the shear stress intensity at any level EF on flange at a vertical distance of y from the neutral axis as shown in the Fig. 9.

Area above the plane EF = Area of the flange + area of the web above EF

$$a = B\left(\frac{D-d}{2}\right) + b\left(\frac{d}{2} - y\right)$$

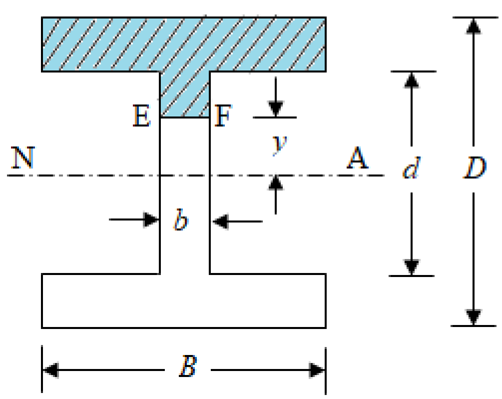


Fig. 9

Centriodal distance of flange area from the neutral axis $y_1 = \frac{d}{2} + \frac{1}{2}\left(\frac{D}{2} - \frac{d}{2}\right) = \frac{1}{2}\left(\frac{D+d}{2}\right)$

Centriodal distance of web area from the neutral axis $y_2 = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = \frac{1}{2}\left(\frac{d}{2} + y\right)$

Moment of the total area above *EF* about neutral axis,

$$a\bar{y} = a_1 y_1 + a_2 y_2 = B\left(\frac{D-d}{2}\right)\times\frac{1}{2}\left(\frac{D+d}{2}\right) + b\left(\frac{d}{2}-y\right)\times\frac{1}{2}\left(\frac{d}{2}+y\right)$$

$$= \frac{B}{8}\left(D^2 - d^2\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\therefore \quad \text{Shear stress on } EF, \tau = \frac{F}{Ib}a\bar{y}$$

$$= \frac{F}{Ib}\left[\frac{B}{8}\left(D^2 - d^2\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

Hence, shear stress distribution in the web is also parabolic in nature.

At the junction of flange and web, i.e., at $y = \frac{d}{2}$,

$$\therefore \quad \text{Shear stress on } EF, \tau = \frac{F}{Ib}\times\frac{B}{8}\left(D^2 - d^2\right)$$

It can be observed that at the junction of flange and web, there is a sudden jump in the shear stress intensity.

At neutral axis, i.e., at $y = 0$, the shear stress is maximum.

$$\tau_{max} = \frac{F}{Ib}\left[\frac{B}{8}\left(D^2 - d^2\right) + \frac{bd^2}{8}\right]$$

From the shear stress distribution diagram, it is evident that most of the shear stress on the section is carried by the web.

Numerical

1. A 2.5 *m* long rectangular timber beam of 125 *mm* wide and 250 *mm* deep is carrying a uniformly distributed load of 60 *kN/m*. Determine the maximum shear stress intensity and draw the variation of shear stress along the depth of the beam.

Solution:

Given: Length of beam, $l = 2.5\ m$, Width, $b = 125\ mm$, Depth, $d = 250\ mm$

Uniformly distributed load, $w = 50\ kN/m$

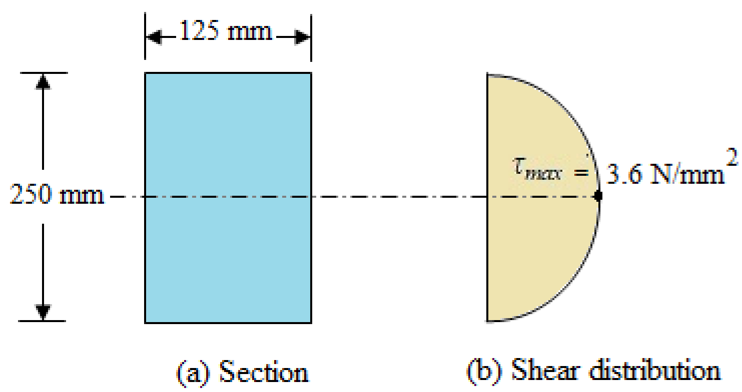


Fig. 10

Maximum shear force in the beam, $F = \dfrac{wl}{2} = \dfrac{60 \times 2.5}{2} = 75\,kN$

$$= 75 \times 10^3\ N$$

Average shear stress, $\tau_{avg} = \dfrac{F}{bd} = \dfrac{75 \times 1000}{125 \times 250} = 2.4\ N/mm^2$

Maximum shear stress, $\tau_{max} = 1.5 \times 2.4 = 3.6 N/mm^2$

2. A rectangular beam of 100 *mm* wide is subjected to a maximum shear force of 45 *kN*. If the maximum shear stress in the beam is 3 N/mm^2, calculate the depth of the beam.

Solution:

Given: Width of beam section, $b = 100\ mm$, Maximum shear force, $F = 45\ kN$, and

Maximum shear stress in the beam, $\tau_{max} = 3\ N/mm^2$

Let the depth of the beam be *d mm*.

Average shear stress, $\tau_{avg} = \frac{F}{bd} = \frac{45\times1000}{100d} = \frac{450}{d}\ N/mm^2$

Maximum shear stress, $\tau_{max} = 1.5\tau_{avg}$

$$3 = 1.5\times\frac{450}{d}$$

$$\therefore \quad d = 225\,mm$$

3. A circular beam section of 150 *mm* diameter is subjected to a maximum shear force of 50 *kN*. Evaluate the maximum shear stress and plot the shear stress distribution diagram across the depth of the section.

Solution:

Given: Diameter of circular section, d = 150 *mm*,

Maximum shear force, F = 50 *kN*

Area of the section, $a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times150^2 = 17671.458\,mm^2$

Average shear stress, $\tau_{avg} = \frac{F}{a} = \frac{50\times1000}{17671.458} = 2.83\,N/mm^2$

Maximum shear stress, $\tau_{max} = 1.5\times2.83 = 4.245\,N/mm^2$

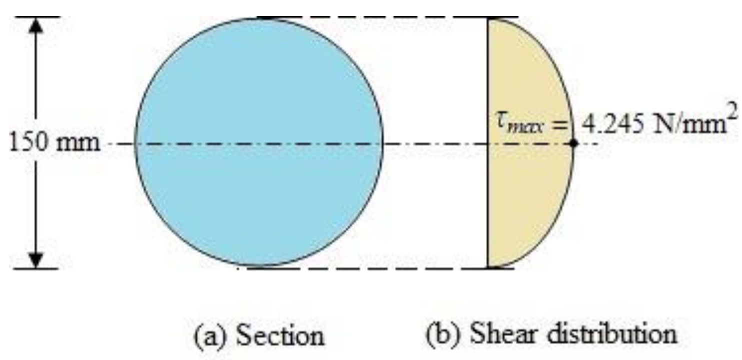


Fig. 11

4. A beam of isosceles triangular section of base width 120 *mm* and height 150 *mm* is subjected to a shear force of 15 *kN*, Find the maximum shear stress and the shear stress at neutral axis. Draw the shear stress variation diagram along the depth of the section.

Given: Base width, b = 120 *mm*, Height, h = 150 mm

Maximum shear force, F = 15 *kN*

Maximum shear stress is at a height of $\frac{h}{2}$ from the base of the triangle.

Maximum shear stress $\tau_{\max} = \frac{3F}{bh} = \frac{3\times 15\times 1000}{120\times 150} = 2.5\,N/mm^2$

Shear stress at the neutral axis, $\tau_{na} = \frac{8F}{3bh} = \frac{8\times 15\times 1000}{3\times 120\times 150} = 2.222\,N/mm^2$

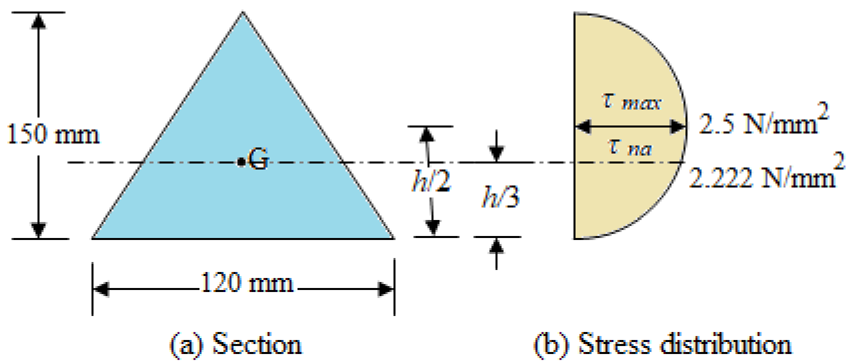


Fig. 12

Alternatively

Area of the section, $a = \frac{1}{2}\times bh = \frac{1}{2}\times 120\times 150 = 9000\,mm^2$

Average shear stress $\tau_{avg} = \frac{F}{a} = \frac{15\times 1000}{9000} = 1.667\,N/mm^2$

Maximum shear stress, $\tau_{\max} = 1.5\times 1.667 = 2.5\,N/mm^2$

Shear stress at the neutral axis, $\tau_{na} = \frac{4}{3}\tau_{avg} = \frac{4}{3}\times 1.667 = 2.222\,N/mm^2$

5. An I-section has an overall depth of 240 *mm* with horizontal flanges each measuring 120 *mm* x 20 *mm* and a vertical web 200 *mm* x 20 *mm*. It is subjected to a vertical shear force of 200 *kN*. Find the maximum shear stress and its position. Draw the shear stress distribution diagram.

Solution:

Shear force on the section, $F = 200\,kN$

Moment of inertia of the *I*-section about the neutral axis,

$$I = \frac{120 \times 240^3}{12} - \frac{100 \times 200^3}{12} = 71.573333 \times 10^6\ mm^4$$

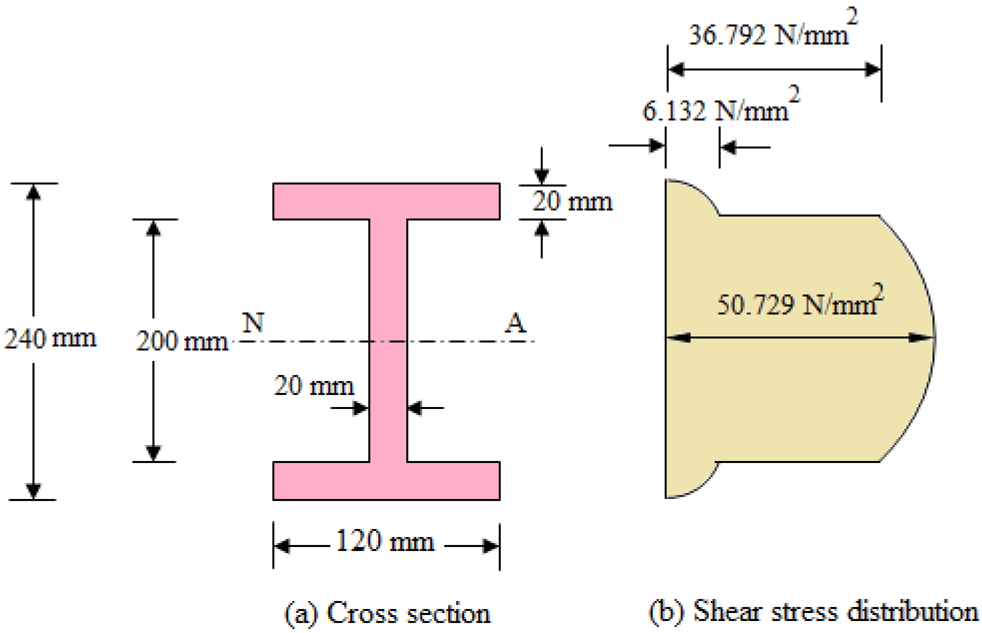


Fig. 13

Shear stress in the flange at the junction of flange and web:

$$a\bar{y} = 120 \times 20 \times 110 = 264000\,mm^3$$

$$\text{Shear stress, } = \frac{Fa\bar{y}}{Ib} = \frac{200 \times 10^3 \times 264000}{71.753333333 \times 10^6 \times 120} = 6.132\,N/mm^2$$

$$\text{Stress in the web at the junction of flange and web } = \frac{200 \times 10^3 \times 264000}{71.753333 \times 10^6 \times 20} = 36.792\,N/mm^2$$

Shear stress at neutral axis

$$a\bar{y} = 120\times 20\times 110 + 20\times 100\times 50 = 364000\,mm^3$$

$$\text{Shear stress, } = \frac{Fa\bar{y}}{Ib} = \frac{200\times 10^3\times 364000}{71.753333333\times 10^6\times 20} = 50.729\,N/mm^2$$

6. The unsymmetrical I-section shown in the Fig. is subjected to a shear force of 40 kN. Draw the shear stress distribution across the depth showing the salient points.

Solution:

Distance of the centroid from the bottom fibre,

$$y_b = \frac{150\times 20\times 10 + 20\times 200\times (20 + \frac{200}{2}) + 100\times 20\times \left(20 + 200 + \frac{20}{2}\right)}{150\times 20 + 20\times 200 + 100\times 20}$$

$$= 107.78\,mm$$

Distance of the centroid from the top fibre, $y_t = 240 - y_b = 240 - 107.777$

$$= 132.22\ mm$$

Moment of inertia about the neutral axis,

$$I = \frac{150\times 20^3}{12} + (150\times 20)\left(107.78 - \frac{20}{2}\right)^2 + \frac{20\times 200^3}{12} + (20\times 200)\times\left(20 + \frac{200}{2} - 107.78\right)^2$$

$$+ \frac{100\times 20^3}{12} + (100\times 20)\times\left(132.22 - \frac{20}{2}\right)^2$$

$$= 72655555.60\,mm^4$$

Shear stress at the junction of top flange and web:

Area above the bottom of top flange, $a = 100\times 20 = 2000\ mm^2$

C.G of this area from the neutral axis, $\bar{y} = y_t - \frac{20}{2} = 132.22 - \frac{20}{2}$

$$= 122.22\,mm$$

Stress in the flange at the junction of top flange and web $= \frac{F}{Ib_f} \times a\bar{y}$

$$= \frac{40 \times 10^3}{72655555.60 \times 100} \times (100 \times 20) \times 122.22$$

$$= 1.346\, N/mm^2$$

Stress in the web at the junction of top flange and web

$$= \frac{40 \times 10^3}{72655555.60 \times 20} \times (100 \times 20) \times 122.22$$

$$= 6.728\, N/mm^2$$

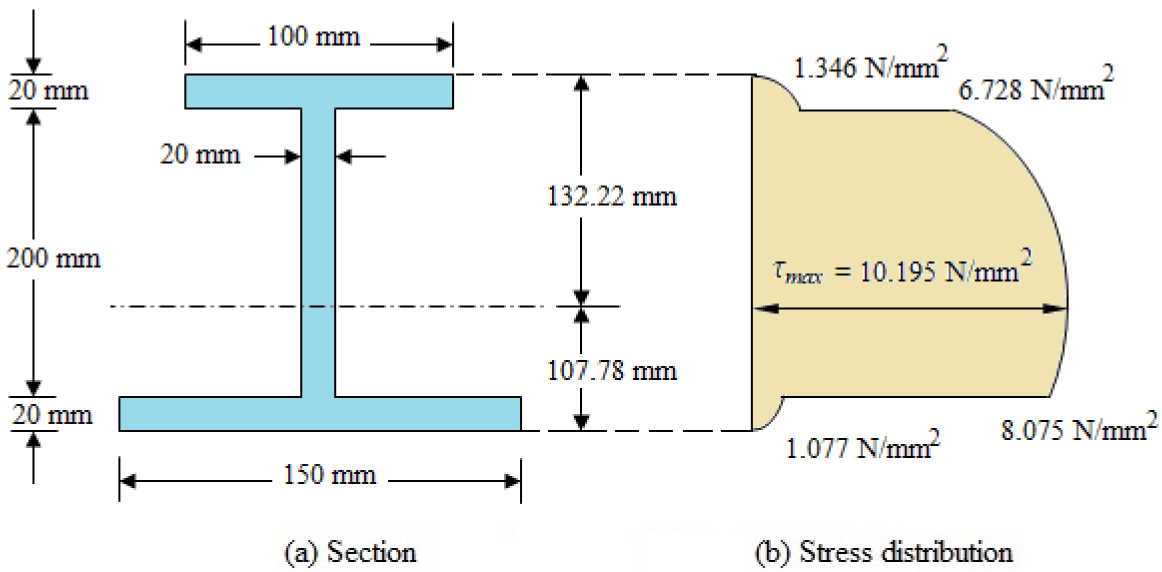


Fig. 14

Shear stress at neutral axis:

$a\bar{y}$ of area above neutral axis $= a\bar{y}$ of flange about N.A $+ a\bar{y}$ of web about N.A

$$= (100 \times 20) \times \left(132.22 - \frac{20}{2}\right) + 20 \times (132.22 - 20) \times \frac{(132.22 - 20)}{2}$$

$$= 370373.284$$

Maximum shear stress at neutral axis, $\tau_{max} = \frac{F}{Ib} \times a\bar{y} = \frac{40 \times 10^3}{72655555.6 \times 20} \times 370373.284$

$$= 10.195\ N/mm^2$$

Shear stress at the junction of bottom flange and web:

Area of the bottom flange, $a = 150 \times 20 = 3000\ mm^2$

C.G of this area from the neutral axis, $\bar{y} = y_b - \frac{20}{2} = 107.78 - \frac{20}{2} = 97.78$

Stress in the web at the junction of web and bottom flange

$$= \frac{40 \times 10^3}{72655555.60 \times 20} \times 3000 \times 97.78$$

$$= 8.075\ N/mm^2$$

Stress in the bottom flange at the junction of web and bottom flange

$$= \frac{40 \times 10^3}{72655555.60 \times 150} \times 3000 \times 97.78$$

$$= 1.077\ N/mm^2$$

7. A beam of *I*-section 500 *mm* deep and 200 *mm* wide has flanges 25 *mm* thick and web 20 *mm* thick. It carries a shearing force of 425 *KN* at a section. Calculate the maximum intensity of shear stress in the section. Also calculate the total shear force carried by the web and draw the shear stress distribution across the depth of the section.

Solution:

Moment of inertia about neutral axis, $I = \frac{200 \times 500^3}{12} - \frac{180 \times 450^3}{12}$

$$= 716458333.333 mm^4$$

Maximum shear stress intensity will occur at the neutral axis.

$$\tau_{max} = \frac{F}{Ib} a\bar{y}$$

Where F = maximum shear force

$a\bar{y}$ = moment of the area above the neutral axis about the neutral axis.

I = Moment of inertia of the section about neutral axis.

b = breath of the web

The section is symmetrical about both horizontal as well as vertical axis.

Hence centroid is horizontal centroidal axis is half-way, i.e., 250 mm from the top as well as bottom edge.

$$a\underline{y} = 200 \times 25 \times \left(225 + \frac{25}{2}\right) - 225 \times 20 \times \frac{225}{2}$$

$$= 1693750 mm^3$$

$$\tau_{max} = \frac{F}{Ib} a\underline{y} = \frac{425000}{716458333.333 \times 20} \times 1693750$$

$$= 50.236\, N/mm^2$$

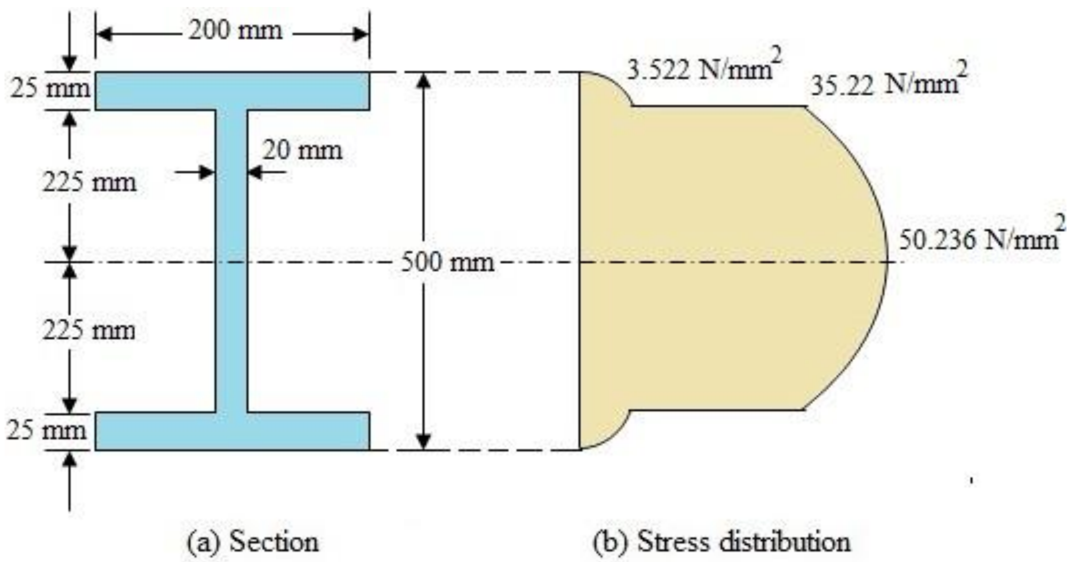


Fig. 15

Shear stress in the flange at a distance y from the neutral axis,

$$\tau = \frac{F}{IB} \times a\underline{y} = \frac{F}{IB} \times \left[B(250 - y)\frac{(250 + y)}{2} \right]$$

$$= \frac{F}{2I}\left(250^2 - y^2\right)$$

$$\therefore \qquad \tau = \frac{F}{2I}\left(62500 - y^2\right)$$

Shear resistance offered by elementary strip of the flange 200 *mm* wide and *dy mm* deep,

$$= \tau da = \frac{F}{2I}\left(62500 - y^2\right)200dy$$

$$= \frac{100F}{I}\left(62500 - y^2\right)dy$$

Shear resistance of one flange $= \frac{100F}{I}\int_{225}^{250}\left(62500 - y^2\right)dy$

$$= \frac{100F}{I}\left[62500y - \frac{y^3}{3}\right]_{225}^{250}$$

$$= \frac{100F}{I}\left[62500\left(250 - 225\right) - \left(\frac{250^3}{3} - \frac{225^3}{3}\right)\right]$$

$$= \frac{100 \times 425}{716458333.333} \times 151041.67 = 8.96\,kN$$

Total shear resistance of two flanges $= 2 \times 8.96 = 17.62\,kN$

Total shear force carried by the web $= 425 - 17.62 = 407.08\,kN$

Shear stress distribution across the section:

Shear stress in the flange at junction of flange and web

$$= \frac{F}{IB} \times \text{Moment of the area about N.A}$$

$$= \frac{425000}{716458333.333 \times 200} \times \left(200 \times 25\right) \times \left(225 + \frac{25}{2}\right)$$

$$= 3.522\,N/mm^2$$

Shear stress in the web at junction of flange and web $= \frac{200}{20} \times 3.522 = 35.22\,N/mm^2$

8. A simply supported beam of 4 *m* length carries a uniformly distributed load of intensity 50 *kN*/*m* over the entire span. If a T-section of flange size 150 *mm* x 50 *mm* and web size 50 *mm* x 150 *mm* is used as beam section, find the maximum shear stress and draw the shear stress distribution diagram across the depth with values at important points.

Solution:

Given: Length of beam, l = 4 *m*, Uniformly distributed load, w = 50 *kN*/*m*

Maximum shear force at the support, $F = \frac{wl}{2} = \frac{50 \times 4}{2} = 100\ kN$

Distance of the centroid from the bottom fibre, $y_b = \frac{50 \times 150 \times 75 + 150 \times 50 \times (150 + 25)}{50 \times 150 + 150 \times 50}$

$$= 125\, mm$$

Distance of the centroid from the top fibre, $y_t = (150 + 50) - y_b = 200 - 125$

$$= 75\ mm$$

Moment of inertia about the neutral axis,

$$I = \frac{50 \times 150^3}{12} + (50 \times 150) \times (75 - 25)^2 + \frac{150 \times 50^3}{12} + (50 \times 150) \times (125 - 75)^2$$

$$= 53125000 mm^4$$

Shear stress at the junction of flange and web:

Area above the bottom of flange $a = 150 \times 50 = 7500\ mm^2$

C.G of this area from the neutral axis, $\bar{y} = y_t - \frac{50}{2} = 75 - 25$

$$= 50\ mm$$

Stress in the flange at the junction of flange and web $= \frac{F}{Ib_f} \times a\bar{y}$

$$= \frac{100 \times 10^3}{53125000 \times 150} \times 7500 \times 50$$

$$= 4.706\, N/mm^2$$

Stress in the web at the junction of flange and web $= \dfrac{100 \times 10^3}{53125000 \times 50} \times 7500 \times 50$

$$= 14.12\, N/mm^2$$

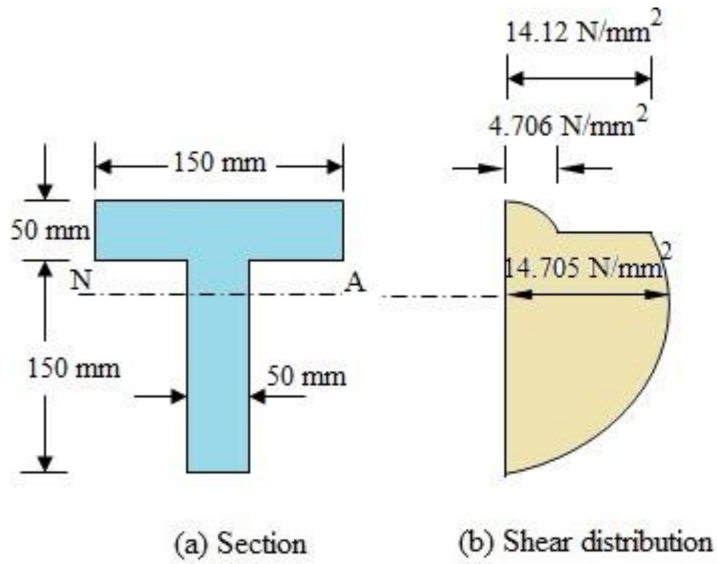


Fig. 16

Shear stress at neutral axis:

$$a\bar{y} \text{ of area above neutral axis} = a\bar{y} \text{ of flange about N.A} + a\bar{y} \text{ of web about N.A}$$

$$= (150 \times 50)\left(75 - \frac{50}{2}\right) + (50 \times 25) \times \frac{25}{2}$$

$$= 390625$$

Maximum shear stress at neutral axis, $\tau_{max} = \dfrac{F}{Ib} \times a\bar{y} = \dfrac{100 \times 10^3}{53125000 \times 50} \times 390625$

$$= 14.705\ N/mm^2$$

Shear stress is zero at the top and bottom fibres.

9. A 2.5 *m* long simply supported wooden beam, rectangular section of 100 *mm* wide and 150 *mm* deep carries a uniformly distributed load over entire span. If the safe stresses are 12

N/mm^2 and 1.5 N/mm^2 in bending and shear respectively, find the safe load that the beam can carry.

Solution:

Let the safe uniformly distributed load = w kN/m.

Case I: Bending stress consideration

Maximum bending moment $M = \frac{wl^2}{8} = \frac{w \times 2.5^2}{8} = 0.78125w \ kN.m$

$$= 0.78125 \times 10^6 \, w \, N.mm$$

Moment of resistance of the beam = Maximum bending moment

$$\frac{1}{6}\sigma_{allow} bd^2 = 0.78125 \times 10^6 \, w$$

$$0.78125 \times 10^6 \, w = \frac{1}{6} \times 12 \times 100 \times 150^2$$

$$w = 5.76 \ kN/m$$

Case II: Shear stress consideration

Maximum shear force $F = \frac{wl}{2} = \frac{w \times 2.5}{2} = 1.125w \ kN$

$$= 1125w \, N$$

Maximum shear stress $= \frac{3}{2} \times$ Average shear stress $= \frac{3}{2} \times \frac{F}{bd}$

$$= \frac{3}{2} \times \frac{1125w}{100 \times 150} = 0.1125w \ N/mm^2$$

For safe load, maximum shear stress on the beam shall at best be equal to safe shear stress of the material.

$$0.1125w = 1.5$$

$$w = 13.33kN$$

The beam can carry safe uniformly distributed load of 5.76 *kN*/*m*.

10. A simply supported rectangular timber beam of length l carries a concentrated load of *W* at a distance of *l*/4 from the left support. The maximum allowable stresses in bending and shear are 12 N/mm^2 and 1.5 N/mm^2 respectively. If the depth of the beam is 250 *mm*, find the length such that both bending and shear stresses simultaneously reach their maximum allowable limits.

Solution:

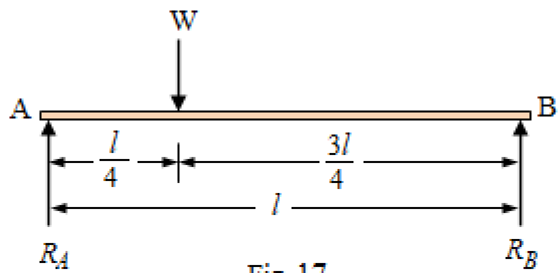


Fig. 17

Consider the equilibrium of the simply supported beam in Fig.

$$R_A \times l = W \times \frac{3l}{4}$$

$$\therefore \qquad R_A = \frac{3W}{4}$$

Maximum shear force, $F = R_A = \frac{3W}{4}$

Maximum bending moment, $M = R_A \times \frac{l}{4} = \frac{3W}{4} \times \frac{l}{4} = \frac{3Wl}{16}$

Maximum bending stress in the beam, $\sigma = \frac{M}{Z} = \frac{M}{\frac{1}{6}bd^2} = \frac{6M}{bd^2}$

$$= \frac{6 \times \frac{3Wl}{16}}{bd^2} = \frac{9Wl}{8bd^2} \qquad (1)$$

Maximum shear stress, $\tau = 1.5 \times Average shear\ stress$

$$= 1.5 \times \frac{F}{bd} = 1.5 \times \frac{\frac{3W}{4}}{bd} = \frac{4.5W}{4bd} \qquad (2)$$

From Eqn. (1) and (2), we have

$$\frac{\sigma}{\tau} = \frac{\frac{9Wl}{8bd^2}}{\frac{4.5W}{4bd}} = \frac{l}{d}$$

$$l = \frac{\sigma}{\tau} \times d = \frac{12}{1.5} \times 250 = 2000\,mm$$

$$= 2\,m$$

CHAPTER-2

SLOPE AND DEFLECTION OF BEAMS

Introduction

When a beam or for that matter any part of a structure is subjected to the action of applied loads, it undergoes deformation due to which the axis of the member is deflected from its original position. The deflections also occur due to temperature variations and lack-of-fit of members. Accurate values for these deflections are sought in many practical cases. The deflections of structures are important for ensuring that the designed structure is not excessively flexible. The large deformations in the structures can cause damage or cracking of non-structural elements. The computation of deflections in structures is also required for solving the statically indeterminate structures.

The deflection of beam depends on four general factors:

1. Stiffness of the material that the beam is made of,
2. Dimension of the beam,
3. Applied loads, and
4. Support conditions

Elastic curve

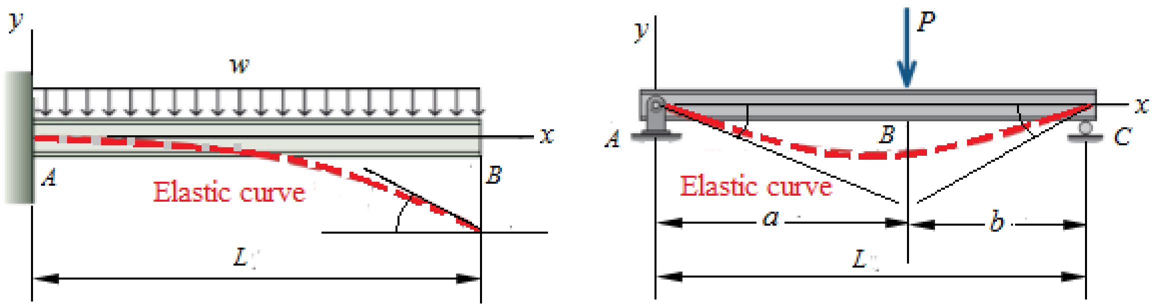
The curve that is formed by plotting the position of the neutral axis of the beam under loading along the longitudinal axis is known as the ***elastic curve***. The curve into which the axis of the beam is transformed under the given loading is called the ***elastic curve***. The nature of the elastic curve depends on the support conditions of the beam and the nature and type of loadings. The slope at a given point may be clockwise or anticlockwise measured from the original axis of the beam. Figure 1 shows the elastic curves for cantilever and simply supported beams. Sagging or positive bending moment produces an elastic curve with curvature of concave upward whereas a hogging or negative bending moment gives rise to an elastic curve with curvature of concave downward.

Deflection

The vertical displacement of a point on elastic curve of a beam with respect to the original position of the point on the longitudinal axis of the beam is called the ***deflection***.

Slope

The angular displacement or rotation of the tangent drawn at a point on the elastic curve of a beam with respect to the longitudinal axis of the original beam without loading is known as the ***slope*** at a given point.



(a) Cantilever beam (b) Simply supported beam

Figure 1

Importance of slope and deflection

Accurate values for these beam deflections are sought in many practical cases. The deflection of a beam must be limited in order to: (a) provide integrity and stability of structure or machine, (b) minimize or prevent brittle-finish materials from cracking The computation of deflections at specific points in structures is also required for analyzing a statically indeterminate structures.

Equation of elastic curve

The following assumptions are made to derive the equation of the elastic curve of a beam.

Assumptions:

1. The deflection is very small compared to the length of the beam.
2. The slope at any point is very small.
3. The beam deflection due to shearing stresses is negligible, i.e., plane sections remain plane after bending.
4. The values of E and I remain constant along the beam. If they are constant and can be expressed as functions of x, then the solution using the equation of elastic curve is possible.

Let us consider an elemental length $PQ = ds$ of the elastic curve of a beam under loading as shown in the Figure 1. The tangents drawn at the points P and Q make angles θ and $\theta + d\theta$ with x-axis. Let the coordinates of P and Q be (x, y) *and* $(x + dx, y + dy)$ respectively. The normals at P and Q meet at C. C denote the centre of curvature and ρ the radius of curvature of the part of the elastic curve between P and Q.

From the geometry of the curve, it is obvious that $ds = \rho d\theta$

or $$\rho = \frac{ds}{d\theta}$$ sass

and $\frac{dy}{dx} = \tan\theta$, $\frac{dy}{ds} = \sin\theta$, and $\frac{dx}{ds} = \cos\theta$

$$\rho = \frac{ds}{d\theta} = \frac{ds}{dx}\frac{dx}{d\theta} = \frac{\frac{ds}{dx}}{\frac{d\theta}{dx}}$$

$$\rho = \frac{\sec\theta}{\frac{d\theta}{dx}} \qquad (1)$$

Further, $\tan\theta = \frac{dy}{dx}$

Differentiating with respect to *x*, one can get

Asaa $\sec^2\theta \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$

$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{\sec^2\theta} \qquad (2)$$

Substituting the value of $\frac{d\theta}{dx}$ in Eq.(1), one gets

$$\rho = \frac{\sec^3\theta}{\frac{d^2y}{dx^2}}$$

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\sec^3\theta} = \frac{\frac{d^2y}{dx^2}}{\left(\sec^2\theta\right)^{3/2}} = \frac{\frac{dy}{dx}}{\left(1+\tan^2\theta\right)^{3/2}}$$

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}$$

For real life actual, the slope *dy*/*dx* is very small and its square is even smaller and hence the term $\left(\frac{dy}{dx}\right)^2$ can be neglected as compared to unit. The above expression thus becomes

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \qquad (3)$$

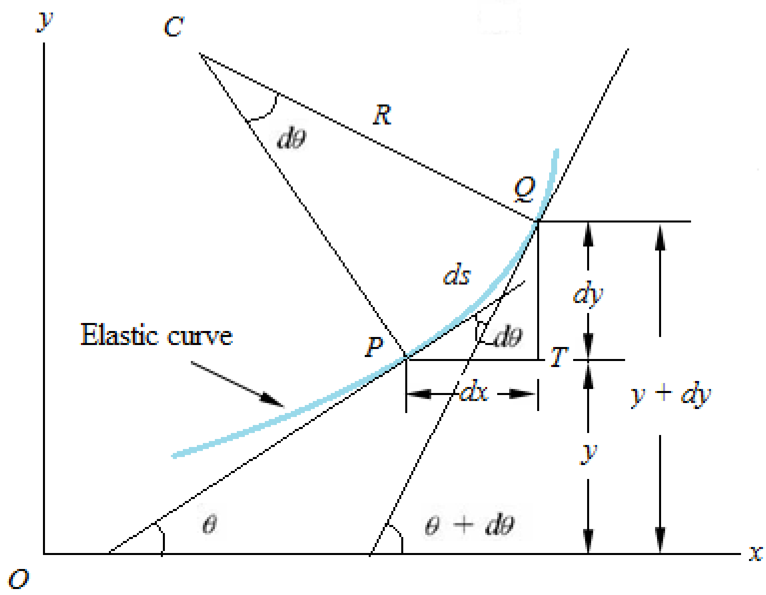


Figure 2

From theory of pure bending, it is known that

$$\frac{M}{I} = \frac{E}{\rho}$$

$$\frac{1}{\rho} = \frac{M}{EI} \quad (4)$$

From Eq, (3) and (4) we get

$$\boxed{EI\frac{d^2y}{dx^2} = M} \quad (5)$$

Equation (5) is the governing equation of deflection of beam, also known as equation of elastic curve.

Boundary condition

The equation of elastic curve or the governing equation for deflection of the beam is a second order differential equation; hence we need to know two boundary conditions to find out two constants of integration for complete solution of the problem. The boundary conditions generally come from the support conditions, where either the slope or the deflection is known. Sometimes, due to symmetry of the beam, as in the case of a simply supported beam with point load at the centre of the beam or uniformly distributed load throughout the beam, an intermediate point representing the point of symmetry may give a boundary condition.

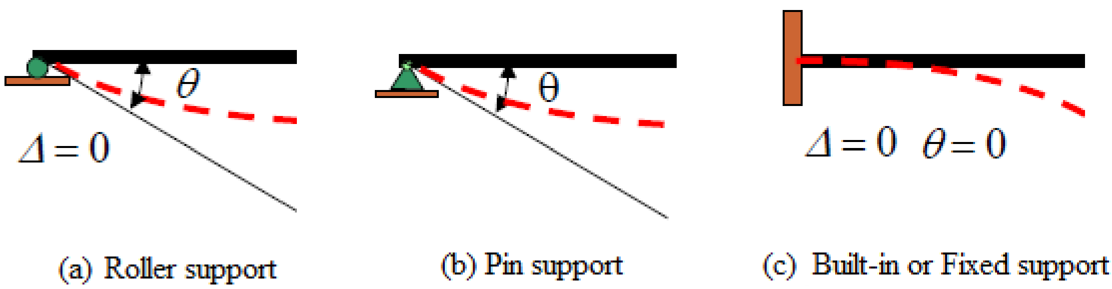


Figure 3

General procedure for computing deflection by integration

1. Select the interval or intervals of the beam to be used and place a set of coordinate axis on the beam with the origin at one end of an interval and then indicate the range of values of x in each interval.
2. List the variable boundary and continuity or matching conditions for each interval.
3. Express the bending moment M as a function of x for each interval selected and equate it to $EI\left(d^2y/dx^2\right)$.
4. Solve the differential equation from step 3 and evaluate all constants of integration. Calculate slope (dy/dx) and deflection (y) at the specific points.

Numerical Problems
Problem 1.
Derive the equation of elastic curve and find the slope and deflection at the free end of the cantilever beam shown in the Figure 4.

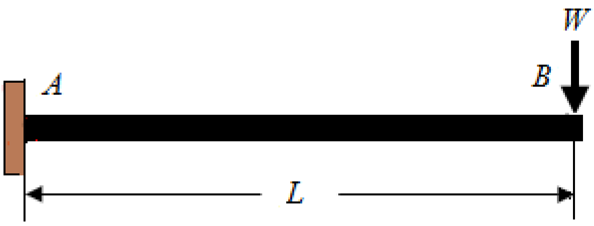


Figure 4

Solution.

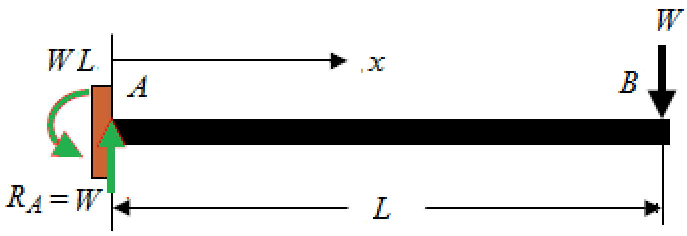


Figure 5

Determine the support reactions
Sum of the vertical forces , $\sum V = 0, \quad R_A = W$

Sum of the vertical forces , $\sum M_A = 0, \qquad M_A = W\,L$
Taking moment about any section between A and B over the entire length of the cantilever, we have

$$M(x) = -WL + Wx$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -WL + Wx$$

Integrating with respect to *x*, we get

$$EI\theta = EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \quad (6)$$

Integrating again with respect to *x*, we get

$$EIy = -\frac{WLx^2}{2} + \frac{Wx^3}{6} + C_1x + C_2 \quad (7)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0, \theta = 0 \ and \ x = 0, \ y = 0$$

Substituting $x = 0, \theta = 0$ in Eq. ( ), we get $C_1 = 0$

Substituting $x = 0, \ y = 0$ in Eq. ( ), we get $C_2 = 0$

Substituting the values of $C_1 = 0$ and $C_2 = 0$ in Eq. ( ) and Eq. ( ), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} \quad (8)$$

General equation for deflection $$EIy = -\frac{WLx^2}{2} + \frac{Wx^3}{6} \quad (9)$$

Slope at free end $(x = L)$ $$EI\theta_B = -WL^2 + \frac{WL^2}{2}$$

$$\theta_B = -\frac{WL^2}{2EI}$$

Slope at free end $(x = L)$ $$EIy_B = -\frac{WL^3}{2} + \frac{Wx^3}{6}$$

$$y_B = -\frac{WL^3}{3EI}$$

Problem 2.

A cantilever beam of length L carries a uniformly distributed load of w per unit length over its entire length. Determine the slope and deflection at the free end of the beam.

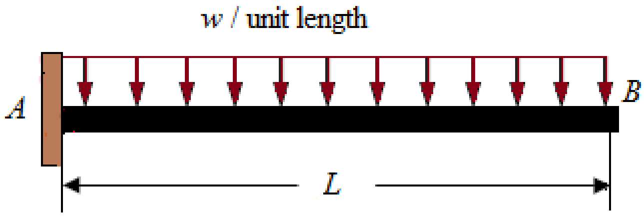


Figure 6

Solution.

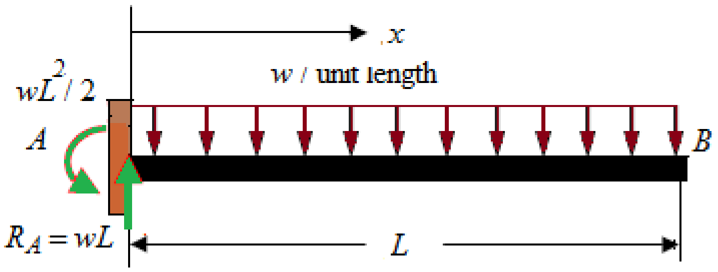


Figure 7

Determine the support reactions

Sum of the vertical forces , $\sum V = 0,\ \ R_A = wL$

Sum of the vertical forces , $\sum M_A = 0,\qquad M_A = -\frac{wL^2}{2}$

Taking moment about any section between A and B over the entire length of the cantilever, we have

$$M(x) = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

Integrating with respect to x, we get

$$EI\theta = EI\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} + C_1 \qquad (10)$$

Integrating again with respect to x, we get

$$EIy = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} + C_1x + C_2 \qquad (11)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0,\ \theta = 0 \text{ and } x = 0,\ y = 0$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

$$x = 0, \theta = 0 \ and \ x = 0, \ y = 0$$

Substituting $x = 0, \theta = 0$ in Eq. ( ), we get $C_1 = 0$

Substituting $x = 0, \ y = 0$ in Eq. ( ), we get $C_2 = 0$

Substituting the values of $C_1 = 0$ and $C_2 = 0$ in Eq. ( ) and Eq. ( ), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -\frac{wL^2x}{2} - \frac{wx^3}{6} + \frac{wLx^2}{2} \qquad (12)$$

General equation for deflection $$EIy = -\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6} \qquad (13)$$

Slope at free end $(x = L)$ $$EI\theta_B = -\frac{wL^3}{2} - \frac{wx^3}{6} + \frac{wL^3}{2}$$

$$\theta_B = -\frac{WL^3}{6EI}$$

Slope at free end $(x = L)$ $$EIy_B = -\frac{wL^4}{4} - \frac{wx^4}{24} + \frac{wL^4}{6}$$

$$y_B = -\frac{WL^3}{8EI}$$

Problem 3.

Determine the slope at the end supports and deflection at centre of a prismatic simply supported beam of length L carrying a point of W at the mid span.

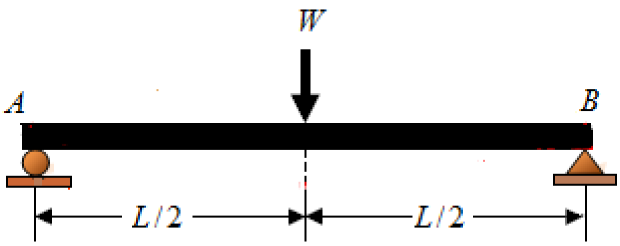


Figure 8

Solution.

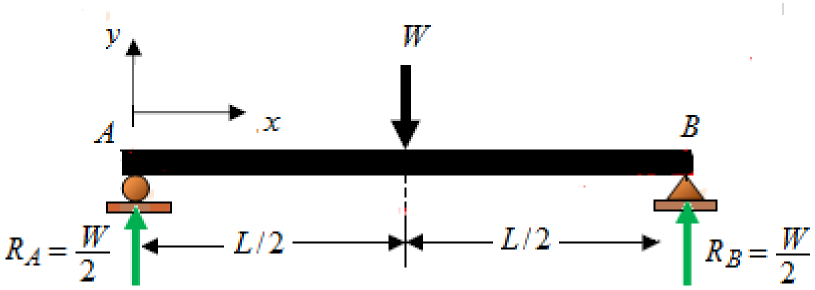


Figure 9

The beam is symmetrical, so the reactions at both ends are $\frac{W}{2}$, The bending moment equation will change beyond the centre position but because the bending will be symmetrical on each side of the centre we need to only to solve for the left hand side.

Taking moment about any section between the left hand support*A* and the centre of the beam, we have

$$M(x) = -\frac{W}{2}x$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = -\frac{Wx}{2}$$

Integrating with respect to *x*, we get

$$EI\theta = EI\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1 \qquad (14)$$

Integrating again with respect to *x*, we get

$$EIy = -\frac{Wx^3}{12} + C_1x + C_2 \qquad (15)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

At A $x = 0,\ y = 0$ (No deflection at roller supported or hinged ends)

At C $x = \frac{L}{2},\ \theta = 0$ (Tangent to the elastic curve is horizontal at the centre)

Substituting $x = \frac{L}{2},\ \theta = 0$ in Eq. (14), we get $C_1 = \frac{WL^2}{16}$

Substituting $x = 0,\ y = 0$ in Eq. (15), we get $C_2 = 0$

Substituting the values of $C_1 = \frac{WL^2}{16}$ and $C_2 = 0$ in Eq. (14) and Eq. (15), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{WL^2}{16} \qquad (16)$$

General equation for deflection $$EIy = -\frac{Wx^3}{12} + \frac{WL^2 x}{16} \qquad (17)$$

Slope at end $A\,(x=0)$ $$EI\theta_A = -\frac{W(0)^2}{4} + \frac{WL^2}{16}$$

$$\theta_A = \frac{WL^2}{16EI}$$

Deflection at the centre $\left(x = \frac{L}{2}\right)$ $$EIy_C = -\frac{W}{12}\left(\frac{L}{2}\right)^3 + \frac{WL^2}{16}\left(\frac{L}{2}\right)$$

$$EIy_C = -\frac{WL^3}{96} + \frac{WL^3}{32}$$

$$y_C = -\frac{WL^3}{48EI}$$

Problem 4.
Determine the slope at the end supports and deflection at the centre of a prismatic simply supported beam shown in the Figure 10 carrying uniformly distributed load of w per unit length over the entire span of the beam.

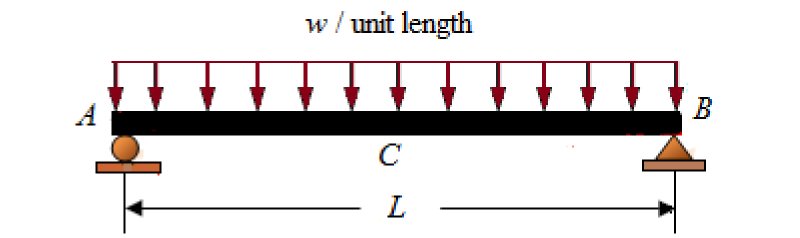


Figure 10

Solution.

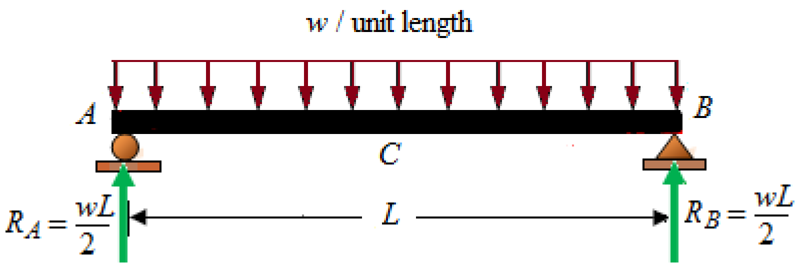


Figure 11

The beam is symmetrical, so the reactions at both ends are $\frac{wL}{2}$, The bending moment equation will change beyond the centre position but because the bending will be symmetrical on each side of the centre we need to only solve for the left hand side.

Taking moment about any section between *A* and *B* over the entire length of the cantilever, we have

$$M(x) = \frac{wLx}{2} - \frac{wx^2}{2}$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

Integrating with respect to *x*, we get

$$EI\theta = EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1 \qquad (18)$$

Integrating again with respect to *x*, we get

$$EIy = \frac{wLx^2}{12} - \frac{wx^4}{24} + C_1x + C_2 \qquad (19)$$

The constants integration C_1 and C_2 may be determined from the boundary conditions.

At *A* $x = 0,\ y = 0$ (No deflection at roller supported or hinged ends)

At *C* $x = \frac{L}{2}, \theta = 0$ (Tangent to the elastic curve is horizontal at the centre)

Substituting $x = \frac{L}{2}, \theta = 0$ in Eq. (18), we get

$$EI(0) = \frac{wL}{4}\left(\frac{L}{2}\right)^2 - \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

$$C_1 = -\frac{wL^3}{16} + \frac{wL^3}{48}$$

$$C_1 = -\frac{wL^3}{24}$$

Substituting $x = 0,\ y = 0$ in Eq. (15), we get $C_2 = 0$

Substituting the values of $C_1 = -\frac{wL^3}{24}$ and $C_2 = 0$ in Eq. (18) and Eq. (19), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24} \qquad (20)$$

General equation for deflection $$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x \qquad (21)$$

Slope at end A $(x = 0)$ $$EI\theta_A = \frac{wL}{4}(0)^2 - \frac{w}{6}(0)^3 - \frac{wL^3}{24}$$

$$\theta_A = -\frac{wL^3}{24EI}$$

Deflection at the centre $\left(x = \frac{L}{2}\right)$ $$EIy_C = \frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)$$

$$EIy_C = \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}$$

$$y_C = -\frac{wL^4}{384EI}$$

Problem 5.
Determine the slope and deflection of the prismatic simply supported beam under the point load.

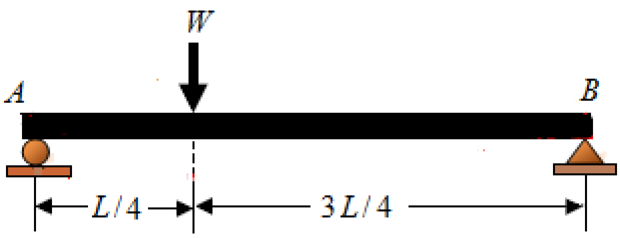


Figure 12

Solution.

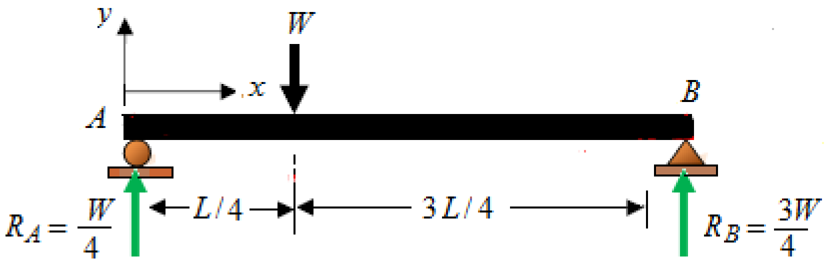


Figure 13

Determine the support reactions

Sum of the vertical forces , $\sum V = 0$, $\qquad R_A + R_B = W$

Sum of the vertical forces , $\sum M_A = 0$, $\qquad R_A \text{ x } L = W \text{ x } \frac{3L}{4}$

$$R_A = \frac{3WL}{4}$$

Bending moment over the portion *AC* and *CB* of the beam may be expressed by two different functions and hence the equations for elastic curves.

For portion A to *C* $(x < \frac{L}{4})$

Taking moment about any section between *A* and *C*, we have

$$M_1(x) = \frac{3Wx}{4}$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y_1}{dx^2} = \frac{3Wx}{4}$$

where $y_1(x)$ is function which defines the elastic curve for portion *AC* of the beam. Integrating the equation we get,

$$EI\theta_1 = EI\frac{dy_1}{dx} = \frac{3Wx^2}{8} + C_1 \qquad (22)$$

$$EIy_1 = \frac{Wx^3}{8} + C_1x + C_2 \qquad (23)$$

For portion C to *B* $\left(x \geq \frac{L}{4}\right)$

Taking moment about any section between *A* and *C*, we have

$$M_2(x) = \frac{3Wx}{4} - W\left(x - \frac{L}{4}\right)$$

The equation of the elastic curve may be written as

$$EI\frac{d^2y_2}{dx^2} = \frac{3Wx}{4} - W\left(x - \frac{L}{4}\right)$$

On rearrangement of terms, we get

$$EI\frac{d^2y_2}{dx^2} = -\frac{Wx}{4} + \frac{WL}{4}$$

where $y_2(x)$ is the function which defines the elastic curve for portion *CB* of the beam. Integrating the equation we get,

$$EI\theta_2 = \frac{dy_2}{dx} = -\frac{Wx^2}{8} + \frac{WLx}{4} + C_3 \qquad (24)$$

$$EIy_2 = -\frac{Wx^3}{24} + \frac{WLx^2}{8} + C_3x + C_4 \qquad (25)$$

Determination of constants of integration from boundary conditions and continuity conditions

Boundary conditions: At support $A, x = 0, y_1 = 0$ and at support $B\ x = L,\ y_2 = 0$

Continuity conditions: There can be no sudden change in the slope and deflection at *C* which requires that at $x = \frac{L}{4}, \theta_1 = \theta_2$ and $y_1 = y_2$

Substituting $x = 0, y_1 = 0$ in Eq. (23), we get

$$C_2 = 0$$

Substituting $x = L, y_2 = 0$ in Eq. (25), we get

$$0 = \frac{WL^3}{12} + C_3 L + C_4$$

Substituting $x = \frac{L}{4}, \theta_1 = \theta_2$ into the Eq. (22) and (24) and equating the slopes at the point *C*, the boundary of two segments *AC* and *CB*, we get

$$\frac{3WL^2}{128} + C_1 = \frac{7WL^2}{128} + C_3$$

Substituting $x = \frac{L}{4}$, $y_1 = y_2$ into the Eq. (23) and (25) and equating the deflections at the point *C*, the boundary of two segments *AC* and *CB*, we get

$$\frac{WL^3}{512} + \frac{C_1 L}{4} = \frac{11WL^3}{1536} + \frac{C_3 L}{4} + C_4$$

Solving these equations simultaneously, we get

$$C_1 = -\frac{7WL^2}{128},\ C_2 = 0,\ C_3 = -\frac{11WL^2}{128} \text{ and } C_4 = \frac{WL^3}{384}$$

Substituting C_1 *and* C_2 into Eq. (22) and (23) and for $x \le \frac{L}{4}$

$$EI\theta_1 = \frac{3Wx^2}{8} - \frac{7WL^2}{128} \qquad (26)$$

$$EIy_1 = \frac{Wx^3}{8} - \frac{7WL^2 x}{128} \qquad (27)$$

Substituting $x = \frac{L}{4}$ into Eq. (26) and (27), we get

$$\theta_C = -\frac{WL^2}{32EI} \text{ and } y_C = -\frac{3WL^3}{256EI}$$

Macaulay's method

Double integration method is a convenient and effective way for solving the slope and deflection of prismatic beam as long as the bending moment can be represented by a single function of *M*(*x*). However, it is not always the case. When the loading of the beam is such that two or more functions are needed to represent the bending moment over the entire length of the beam, as was the case in the previous problem. In such cases, additional constants of integration and as many numbers of equations become necessary to express continuity conditions at the points of load change-over in addition to the boundary conditions. Thus the process becomes lengthy and cumbersome. To overcome this difficulty, British engineer W. H. Macaulay proposed an innovative approach of solving such problems by using *singularity function* to express the bending moment over the entire length.

The execution of Macaulay's method is explained by way of solution to Problem 5.

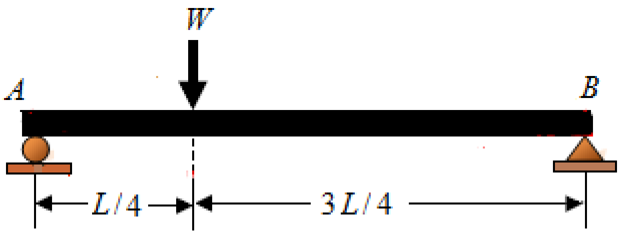


Figure 14

Solution.

Determine the support reactions

Sum of the vertical forces, $\sum V = 0,$ $\quad R_A + R_B = W$

Sum of the vertical forces, $\sum M_B = 0,$ $\quad R_A \text{ x } L = W \text{ x } \frac{3L}{4}$

$$R_A = \frac{3WL}{4}$$

Bending moment over the portion *AC* and *CB* of the beam may be expressed by two different functions as

$$M_1(x) = \frac{3Wx}{4} \qquad \left(0 \le x \le \frac{L}{4}\right)$$

$$M_2(x) = \frac{3Wx}{4} - W\left(x - \frac{L}{4}\right) \qquad \left(\frac{L}{4} \le x \le L\right)$$

where*x* is the distance measured from end *A*. The functions $M_1(x)$ and $M_2(x)$ may be represented by single expression as

$$M(x) = \frac{3Wx}{4} - W\left\langle x - \frac{L}{4}\right\rangle$$

If we want to compute slope and deflection in the portion *CB* i.e., when $x \ge \frac{L}{4}$, the brackets $\langle \; \rangle$ should be replaced by ordinary parentheses $(\;)$. Similarly if we want to compute slope and deflection when $x < \frac{L}{4}$, the brackets $\langle \; \rangle$ should be replaced by zero.

Thus the equation of elastic curve over the entire length of the beam may be written as

$$EI\frac{d^2y}{dx^2} = \frac{3Wx}{4} - W\left\langle x - \frac{L}{4}\right\rangle$$

Integrate with respect to *x* considering the bracket $\langle \; \rangle$ as a single variable.

$$EI\theta = EI\frac{dy}{dx} = \frac{3Wx^2}{8} - \frac{W}{2}\left\langle x - \frac{L}{4}\right\rangle^2 + C_1 \qquad (28)$$

Follow the same rule and integrate again with respect to *x*.

$$EIy = \frac{Wx^3}{8} - \frac{W}{6}\left\langle x - \frac{L}{4} \right\rangle^3 + C_1 x + C_2 \qquad (29)$$

The constants C_1 and C_2 may be determined from the boundary conditions.
At $x = 0$, $y = 0$ and at $x = L$, $y = 0$

For $x = 0 < \frac{L}{4}$, the brackets are equal to zero, hence $C_2 = 0$

For $x = L \geq \frac{L}{4}$, the brackets may be replaced by parentheses,

$$0 = \frac{WL^3}{8} - \frac{W}{6}\left(L - \frac{L}{4}\right)^3 + C_1 L$$

$$0 = \frac{WL^3}{8} - \frac{9WL^3}{128} + C_1 L$$

$$C_1 = -\frac{7WL^2}{128}$$

Substituting the value of C_1 in Eq. (28) and C_1 and C_2 in Eq. (29), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = \frac{3Wx^2}{8} - \frac{W}{2}\left\langle x - \frac{L}{4} \right\rangle^2 - \frac{7WL^2}{128} \qquad (30)$$

General equation for deflection $$EIy = \frac{Wx^3}{8} - \frac{W}{6}\left\langle x - \frac{L}{4} \right\rangle^3 - \frac{7WL^2}{128}x \qquad (31)$$

The need for additional constants C_3 and C_4 as in Problem 5 has been eliminated and hence need for writing additional equations of continuity for slope deflection.

Substituting the value of $x = \frac{L}{4}$ in each of the above equations, we get

$$\theta_C = -\frac{WL^2}{32EI} \text{ and}$$

$$y_C = -\frac{3WL^3}{256EI}$$

Problem 6.

Determine the slope and deflection at points *B* of the beam shown in the Figure. 15. Take E = 200 GPa and $I = 250(10^6)$ mm^4.

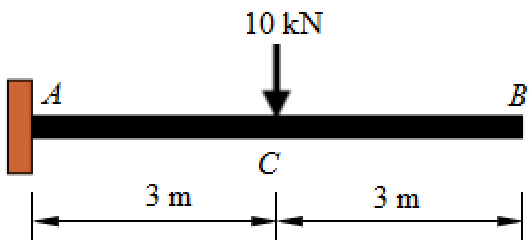


Figure 15

Solution.

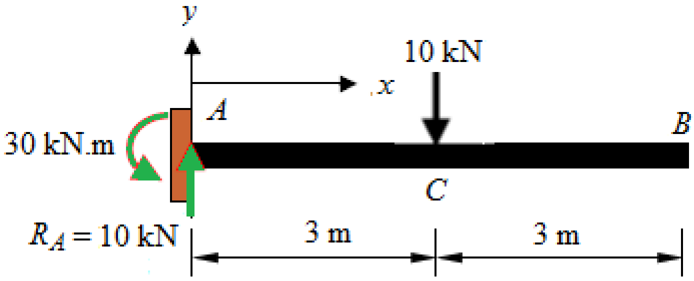


Figure 16

Determine the support reactions

Sum of the vertical forces , $\sum V = 0, \ \ R_A = 10\,kN$

Sum of the vertical forces , $\sum M_A = 0,$ $\qquad M_A = 10 \text{ x } 3 = 30\,kN.m$

Considering from the left hand side and taking moment about any section between *C* and *B*, we have

$$M(x) = 10\,x - 30 - 10(x-3)$$

Do not simplify. On simplification the moment becomes zero between *B* and *C* which is obvious.

Thus the equation of elastic curve over the entire length of the beam may be written as

$$EI\frac{d^2y}{dx^2} = 10\,x - 30 - 10\langle x-3\rangle$$

Integrate with respect to *x* considering the bracket $\langle \ \rangle$ as a single variable.

$$EI\theta = EI\frac{dy}{dx} = 5\,x^2 - 30x - 5\langle x-3\rangle^2 + C_1 \qquad (32)$$

Follow the same rule and integrate again with respect to *x*.

$$EIy = \frac{5}{3}x^3 - 15x^2 - \frac{5}{3}\langle x-3\rangle^3 + C_1 x + C_2 \qquad (33)$$

The constants C_1 and C_2 may be determined from the boundary conditions.

At $x = 0, \theta = 0$ and $x = 0, \ y = 0$

For x = 0< 3 m, the brackets are equal to zero, hence from Eq. (32) C_1 = 0 and from Eq. (33) C_2 = 0

Substituting the values of C_1 and C_2 in Eq. (32) and (33), we get

General equation for slope $\qquad EI\theta = EI\frac{dy}{dx} = 5\,x^2 - 30x - 5\langle x-3\rangle^2 \qquad (34)$

General equation for deflection $\qquad EIy = \frac{5}{3}x^3 - 15x^2 - \frac{5}{3}\langle x-3\rangle^3 \qquad (35)$

Substituting the value of $x = 6$ in each of the above equations, we get

From Eq. (34) the slope, $EI\theta_B = 5(6)^2 - 30 \text{ x } 6 - 5(6-3)^2$

$$\theta_B = -\frac{45}{EI}$$

$$\theta_B = -\frac{45\, kN.m^2}{\left(200 \text{x} 10^6 \frac{kN}{m^2}\right) \text{ x } \left(250 \text{ x } 10^{-6}\ m^4\right)}$$

$$\theta_B = -0.0009\, radian$$

From Eq. (35), the deflection $EIy_B = \frac{5}{3}(6)^3 - 15(6)^2 - \frac{5}{3}(6-3)^3$

$$EIy_B = 360 - 540 - 45$$

$$y_B = -\frac{225}{EI}$$

$$y_B = -\frac{225\, kN.m^3}{\left(200 \text{ x } 10^6 \frac{kN}{m^2}\right) \text{ x } \left(250 \text{ x } 10^{-6}\ m^4\right)}$$

$$y_B = -0.0045\, m = -4.5 mm$$

For the equation of elastic curve between *A* and *C*, neglecting the bracketed term in Eq. (35), we get

$$EIy = \frac{5}{3} x^3 - 15x^2 \text{ which is cubic}$$

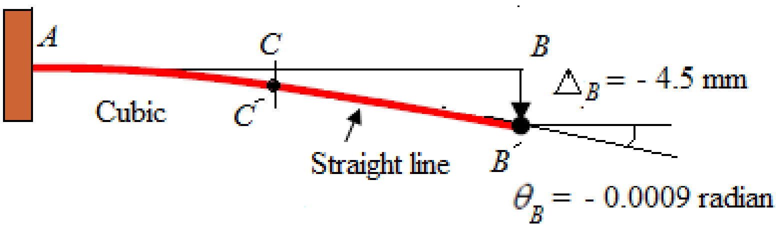
For the equation of elastic curve between *C* and *B*, considering the bracketed term in Eq. (35) and replacing with parentheses, we get

$$EIy = \frac{5}{3} x^3 - 15x^2 - \frac{5}{3}(x-3)^3$$

$$EIy = \frac{5}{3} x^3 - 15x^2 - \frac{5}{3}(x^3 - 9x^2 - 27x + 27)$$

$$EIy = 45x - 45 \text{ which is linear}$$

The elastic curve of the beam with the salient points is shown in the figure



Problem 7.

The cantilevered beam shown is subjected to a uniformly distributed load *w* per unit length. Determine the slope and deflection at point *C* and *B*. Also draw the elastic curve. *EI* is constant.

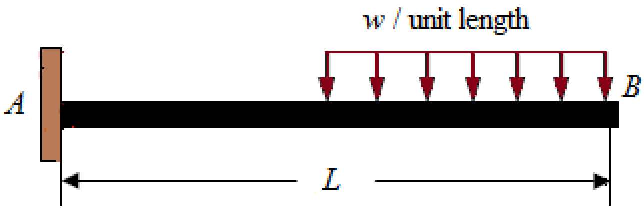


Figure 17

Solution.

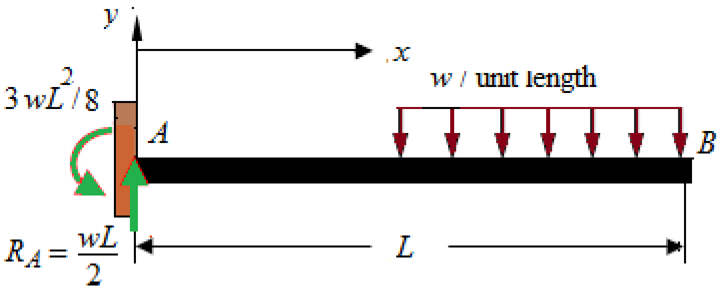


Figure 18

Determine the support reactions

Sum of the vertical forces , $\sum V = 0, \quad R_A = \frac{wL}{2}$

Sum of the vertical forces , $\sum M_A = 0, \qquad M_A = -\frac{wL}{2} \text{x} \frac{3L}{4} = -\frac{3wL^2}{8}$

Considering from the left hand side and taking moment about any section between *C* and *B*, we have

$$M(x) = -\frac{3wL^2}{8} + \frac{wL}{2}x - \frac{w}{2}\left(x - \frac{L}{2}\right)^2$$

Following Macaulay‟s method, the equation of elastic curve over the entire length of the beam may be written as

$$EI\frac{d^2y}{dx^2} = -\frac{3wL^2}{8} + \frac{wL}{2}x - \frac{w}{2}\left\langle x - \frac{L}{2}\right\rangle^2$$

Integrate with respect to x considering the bracket $\langle\ \rangle$ as a single variable.

$$EI\theta = EI\frac{dy}{dx} = -\frac{3wL^2x}{8} + \frac{wLx^2}{4} - \frac{w}{6}\left\langle x - \frac{L}{2}\right\rangle^3 + C_1 \qquad (36)$$

Follow the same rule and integrate again with respect to x.

$$EIy = -\frac{3wL^2x^2}{16} + \frac{wLx^3}{12} - \frac{w}{24}\left\langle x - \frac{L}{2}\right\rangle^4 + C_1x + C_2 \qquad (37)$$

The constants C_1 and C_2 may be determined from the boundary conditions.

At $x = 0, \theta = 0$ and $x = 0\,\text{m}, y = 0$

For $x = 0 < \frac{L}{2}$, the brackets are equal to zero, hence from Eq. (36) $C_1 = 0$ and from Eq. (37) $C_2 = 0$

Substituting the values of C_1 and C_2 in Eq. (36) and (37), we get

General equation for slope $$EI\theta = EI\frac{dy}{dx} = -\frac{3wL^2x}{8} + \frac{wLx^2}{4} - \frac{w}{6}\left\langle x - \frac{L}{2}\right\rangle^3 \qquad (38)$$

General equation for deflection $$EIy = -\frac{3wL^2x^2}{16} + \frac{wLx^3}{12} - \frac{w}{24}\left\langle x - \frac{L}{2}\right\rangle^4 \qquad (39)$$

Substituting the value of $x < \frac{L}{2}$ in each of the above equations and equating the bracketed term as zero, we get

From Eq. (37) the slope at C, $$EI\theta_C = -\frac{3wL^2}{8}\frac{L}{2} + \frac{wL}{4}\left(\frac{L}{2}\right)^2$$

$$EI\theta_C = -\frac{3wL^3}{16} + \frac{wL^3}{16}$$

$$\theta_C = -\frac{wL^3}{8EI}$$

From Eq. (39) the deflection at C,

$$EIy = -\frac{3wL^2}{16}\left(\frac{L}{2}\right)^2 + \frac{wL}{12}\left(\frac{L}{2}\right)^3$$

$$EIy = -\frac{3wL^4}{64} + \frac{wL^4}{96}$$

$$y_C = -\frac{7wL^4}{192EI}$$

Substituting the value of $x < L$ in Eq. (38) and (39) and replacing the brackets by parentheses, we get

From Eq. (38) the slope at B,

$$EI\theta_B = -\frac{3wL^2L}{8} + \frac{wL}{4}L^2 - \frac{w}{6}\left(L - \frac{L}{2}\right)^3$$

$$EI\theta_B = -\frac{3wL^3}{8} + \frac{wL^3}{4} - \frac{wL^3}{48}$$

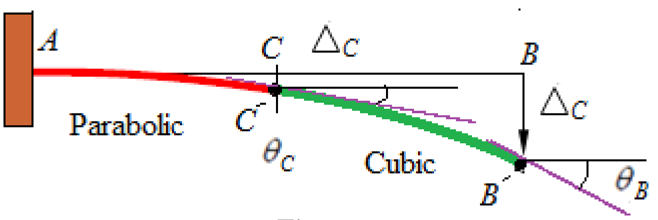
$$\theta_B = -\frac{7wL^3}{48EI}$$

From Eq. (39) the deflection at B,

$$EIy_B = -\frac{3wL^2L^2}{16} + \frac{wLL^3}{12} - \frac{w}{24}\left(L - \frac{L}{2}\right)^4$$

$$EIy_B = -\frac{3wL^4}{16} + \frac{wL^4}{12} - \frac{wL^4}{384}$$

$$y_B = -\frac{41wL^4}{384EI}$$



Figure

Problem 8.

Determine the maximum deflection, the slope and deflection at points C of the beam shown in the figure. Also, draw the elastic curve of the beam. Take E = 200 GPa and $I = 60(10^6)$ mm^4.

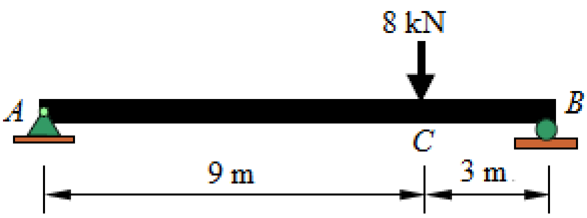


Figure 19

Solution.

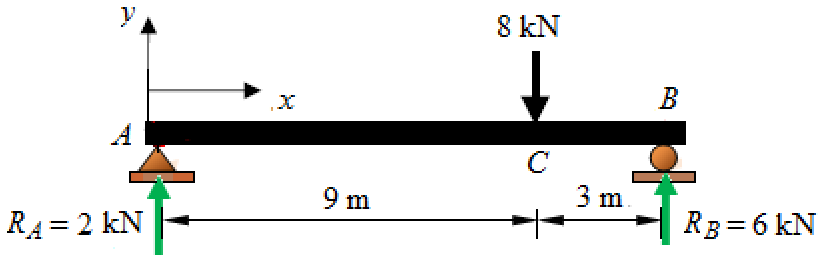


Figure 20

Determine the support reactions

Sum of the vertical forces, $\sum V = 0,$ $R_A + R_B = 8$

Sum of the vertical forces, $\sum M_B = 0,$ $R_A \text{ x } 12 = 8 \text{ x } 3$

$R_A = 2\, kN$

Considering from the left hand side and taking moment about any section between C and B, we have

$$M(x) = 2x - 8(x - 9)$$

Following Macaulay"s method, the equation of elastic curve over the entire length of the beam may be written as

$$EI\frac{d^2y}{dx^2} = 2x - 8\langle x - 9\rangle$$

Integrate with respect to x considering the bracket $\langle\ \rangle$ as a single variable.

$$EI\theta = EI\frac{dy}{dx} = x^2 - 4\langle x - 9\rangle^2 + C_1 \qquad (40)$$

Follow the same rule and integrate again with respect to x.

$$EIy = \frac{1}{3}x^3 - \frac{4}{3}\langle x - 9\rangle^3 + C_1x + C_2 \qquad (41)$$

The constants C_1 and C_2 may be determined from the boundary conditions.

At $x = 0,\ y = 0$ and $x = 12\,\text{m},\ y = 0$

For x = 0< 9 m, the brackets are equal to zero, hence from Eq. (41) $C_2 = 0$

and for x = 12 > 6, from Eq. (41)

$$EI(0) = \frac{1}{3}12^3 - \frac{4}{3}(12 - 9)^3 + C_1 \text{ x12}$$

$$12C_1 = -\frac{1}{3}12^3 + \frac{4}{3}(12-9)^3$$

$$C_1 = -45$$

Substituting the values of C_1 and C_2 in Eq. (40) and (41), we get

General equation for slope $\quad EI\theta = EI\frac{dy}{dx} = x^2 - 4\langle x-9\rangle^2 - 45 \quad (42)$

General equation for deflection $\quad EIy = \frac{1}{3}x^3 - \frac{4}{3}\langle x-9\rangle^3 - 45x \quad (43)$

Substituting the value of$x$< 4 in each of the above equations, we get

From Eq. (42) the slope, $\quad EI\theta_B = (9)^2 - 45$

$$\theta_B = \frac{36}{EI}$$

$$\theta_B = \frac{36\, kN.m^2}{\left(200\,x10^6 \frac{kN}{m^2}\right) \text{ x } \left(60 \text{ x } 10^{-6}\ m^4\right)}$$

$$\theta_B = 0.003 radian$$

From Eq. (43), the deflection $\quad EIy_B = \frac{1}{3}(9)^3 - 45 \text{ x } 9$

$$y_B = -\frac{162}{EI}$$

$$y_B = -\frac{162\, kN.m^3}{\left(200 \text{ x } 10^6 \frac{kN}{m^2}\right) \text{ x } \left(60 \text{ x } 10^{-6}\ m^4\right)}$$

$$y_B = -0.0135 m = -13.5 mm$$

For maximum deflection, the slope must be zero.

Let us assume that the maximum slope would occur in the portion AC, equating the slope equation in (42) without the bracketed term to zero

$$x^2 - 45 = 0 \text{ assa}$$

$$x = \pm\sqrt{45} = \pm 6.708\ m$$

Neglecting the – ve sign, the deflection would occur at $x = 6.708\ m$

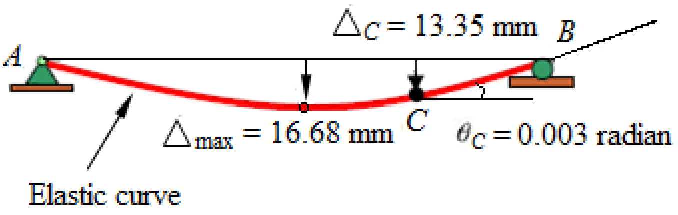
Maximum deflection, $EIy_{max} = \frac{1}{3}(6.708)^3 - 45 \text{ x } 6.708$

$$EIy_{max} = \frac{1}{3}(6.708)^3 - 45 \text{ x } 6.708$$

$$y_{max} = -\frac{200.246}{EI}$$

$$y_{max} = -\frac{200.246\, kN.m^3}{\left(200 \text{ x } 10^6 \frac{kN}{m^2}\right) \text{ x } \left(60 \text{ x } 10^{-6}\ m^4\right)}$$

$$y_{max} = -0.01668\, m = -16.68\, mm$$



Problem 9.

Determine the slope and deflection at points *C*of the beam shown in the Figure. 15. Take E = 200 GPa and $I = 250(10^6)$ mm^4.

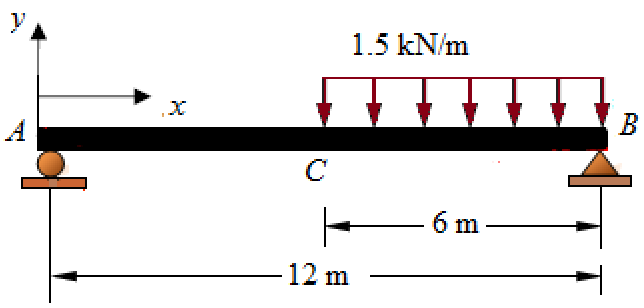
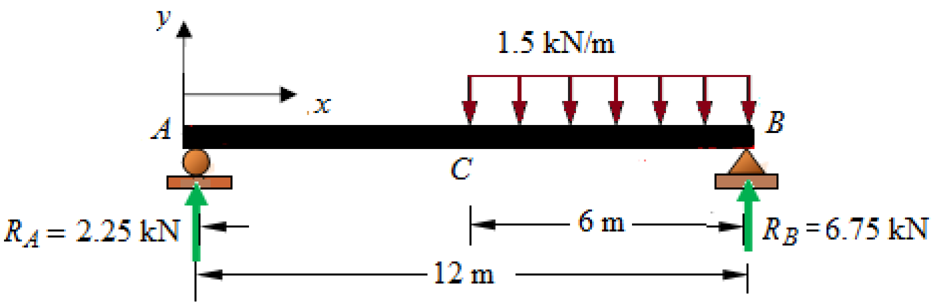


Figure 21

Solution.



Determine the support reactions

Sum of the vertical forces, $\sum V = 0,$ $\quad R_A + R_B = 1.5 \ x \ 6 = 9$

Sum of the vertical forces, $\sum M_B = 0,$ $\quad R_A \ x12 = 1.5 \ x \ 6 \ x \ \frac{6}{2}$

$R_A = \ 2.25\,kN$

Considering from the left hand side and taking moment about any section between C and B, we have

$$M\,(x) = \ 2.5x \ -1.5(x-6) \ \frac{(x-6)}{2}$$

$$= \ 2.5x - 0.75(x-6)^2$$

Following Macaulay"s method, the equation of elastic curve over the entire length of the beam may be written as

$$EI\frac{d^2y}{dx^2} = 2.5\,x - 0.75\langle x-6\rangle^2$$

Integrate with respect to x considering the bracket $\langle \ \rangle$ as a single variable.

$$EI\theta \ = EI\frac{dy}{dx} = 1.25\,x^2 \ -0.25\,\langle x-6\rangle^3 + C_1 \qquad ($$

Follow the same rule and integrate again with respect to x.

$$EIy = \frac{1.25}{3}x^3 - \frac{0.25}{4}\langle x-6\rangle^4 + C_1 x + C_2 \qquad (41)$$

The constants C_1 and C_2 may be determined from the boundary conditions.

At $x = 0,\ y = 0$ and $x = 12\,\text{m},\ y = 0$

For x = 0< 6 m, the brackets are equal to zero, hence from Eq. ( ) $C_2 = 0$

and$x$ = 12 > 6 from Eq. ( ) the brackets being replaced with parentheses

$$EI(0) = \frac{1.25}{3} \ x \ 12^3 - \frac{0.25}{4}(12-6)^4 + 12C_1$$

$$12C_1 = -720 + 81$$

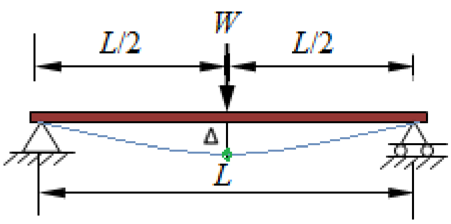
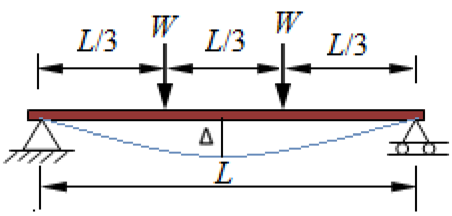
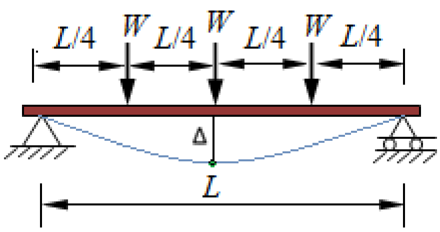
$$C_1 = -\frac{639}{12} = -53.25$$

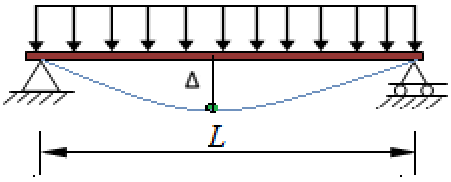
Substituting the values of C_1 and C_2 in Eq. (40) and (41), we get

$$EI\theta = EI\frac{dy}{dx} = 1.25x^2 - 0.25\langle x-6\rangle^3 - 53.25$$

$$EIy = \frac{1.25}{3}x^3 - \frac{0.25}{4}\langle x-6\rangle^4 - 53.25x$$

Slope and deflection of commonly loaded simply supported beam

Beam load and support	Deflection
 (a) Point load at mid-span	At centre $\Delta_{max} = \frac{WL^3}{48EI}$
 (b) Two equal point loads at third points	At centre $\Delta_{max} = \frac{23WL^3}{648EI}$
 (c) Three equal point loads at quarer points	At centre $\Delta_{max} = \frac{WL^3}{20.1EI}$

<table>
<tr>
<td>

(d) Unformly distributed load</td>
<td>At centre $\Delta_{max} = \frac{5WL^4}{384EI}$</td>
</tr>
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Principle of superimposition

For linear response structures, the structural responses such as slope and deflection due to several loads acting simultaneously may be obtained by superposing the effects of individual loads. This is called principle of superposition.

The principle of superposition is valid under the following conditions

1. Hooke's law holds for the material
2. the deflections and rotations are small
3. the presence of the deflection does not alter the actions of applied loads

These requirements ensure that the differential equations of the deflection curve are linear. A very useful application of the principle of superposition is to determine the deflection of statically indeterminate beams. In the present discourse we will restrict our study only to propped cantilever which falls within the scope of the syllabus.

MOMENT-AREA METHOD

Introduction

In this section we will discuss on the evaluation of slope and deflection of beams employing moment-area method. Unlike previous section, beams with non-uniform *EI* or flexural rigidity can be dealt with. Slope and deflection of non-prismatic beams with continuously varying moment of inertia can be conveniently determined.

Moment- Area Method

The moment-area method is one of the most effective methods for obtaining the bending displacement in beams and frames. For problems involving several changes in loading, the area-moment method is usually much faster than the double-integration method; consequently, it is widely used in practice. In this method, the area of the bending moment diagrams is utilized for computing the slope and or deflections at particular points along the

Stresses in shafts due to torsion

Concept of torsion:- The body is said to be in torsion, if they are subjected to twisting moments.Due to torsion the body is subjected to shear stress which is maximum at outer layer and minimum at the neutral axis through centre.

Basic assumptions of pure torsion:-

1. The shaft is homogeneous and isotropic.
2. The twist along the shaft is uniform.
3. Normal cross section of the shaft which were plane & circular before twist remains same after twist.
4. All diameter of normal cross section which were straight before twist remains straight with their magnitudes unchanged after twist.

Shear Stress developed in the shaft:-

Let us consider a shaft of length "l" subjected to torsion.
Let ɵ = the angle of twist
Let R = radius of shaft.
Let C = shear modulus of shaft
Let φ = angle of shear
Let ح = shear stress
Let T = twisting moment or Torque transmitted by shaft
Let J = **Polar Moment of Inertia**

For solid circular shaft $J = \frac{\pi d^4}{32}$

For hollow circular shaft $J = \frac{\pi(D^4 - d^4)}{32}$

Then we get $\frac{\tau}{R} = \frac{C.\theta}{l} = \frac{T}{J}$

The above equation is known as Equation of Torsion. This is similar to Bending equation which we know earlier.

Torsional Rigidity = C.J , which is defined as:- It is the twisting moment required to produce unit rotation in a shaft of one unit length.

Power Transmitted by a shaft P, $P = \frac{2\pi NT}{60}$in Watts

Where N = Speed or Rotation of shaft in RPM
T = Torque transmitted by shaft in (N.M)

Numericals

1. A circular shaft of 50 mm dia is required to transmit torque from one shaft to other. Find the safe torque the shaft can transmit, if the shear stress is not to exceed 40 MPa. (0.982 KN.M)

2. A solid shaft is to transmit a Torque of 10 KN.M. If the shear stress is not to exceed 45 MPa, find the minimum dia of the shaft. (104.2 mm)

3. A hollow shaft of external & internal dia of 80 mm & 50 mm respectively is required to transmit torque from one shaft to other. Find the safe torque it can transmit if the shear stress is not to exceed 40 MPa. (3.40 KN.M)

4. Calculate the maximum torque that a shaft of 125 mm dia can transmit, if the maximum angle of twist is 1^0 for a length of 1.5 m. Take shear modulus of the shaft as 70 GPa. (19.01 KN.M)

5. A circular shaft of dia 60 mm is running at a speed of 150 RPM.If the shear stress is not to exceed 50 MPa, find the power that can be transmitted by the shaft. (33.3 KW)

6. A solid circular shaft of 100 mm dia is transmitting 120 KW at 150 RPM. Find the intensity of shear stress in the shaft. (38.91 MPa)

7. A hollow shaft is to transmit 200 KW at 80 RPM. If the shear stress is not to exceed 60 MPa and internal dia of the shaft is 0.6 times the external dia, find out the diameters of the shaft. (132.75 mm , 79.65 mm)

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